THE CHARACTERISTIC FUNCTION OF A UNIFORMLY CONTINUOUS SEMIGROUP

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1. INTRODUCTION

Let $T(t)$ be a uniformly continuous one-parameter semigroup of operators on a separable Hilbert space $\mathcal{H}$. Thus, for each $t \geq 0$, $T(t)$ is a bounded operator on $\mathcal{H}$, $T(t_1)T(t_2) = T(t_1+t_2)$ for each $t_1, t_2 \geq 0$, $T(0) = I$, and $\|T(t) - I\| \to 0$ as $t \to 0^+$. Such a semigroup possesses a bounded infinitesimal generator A, defined as the limit (in norm) of $t^{-1}(T(t) - I)$, as $t \to 0^+$. We can then write $T(t) = \exp(At)$. (See, for example, [2], [4], [5], [7], [8], [9].)

As in [2], we define the following bounded operators on $\mathcal{H}$: $G = A + A^*$, $Q = |G|^{1/2}$, and $J = \operatorname{sgn}(-G)$ (this is the operator S in [2]). We have the relations

$$
\begin{aligned}
(1.1)\qquad & JQ^2 = -G, \\
& \frac{d}{dt}(T(t)T(t)^*) = T(t)GT(t)^*, \quad \text{and} \\
& \frac{d}{dt}(T(t)^*T(t)) = T(t)^*GT(t) .
\end{aligned}
$$

A Krein space $\mathcal{G}$ is defined by taking $\mathcal{G}$ to be the space $J\mathcal{H}$, equipped with the indefinite inner product

$$(1.2)\qquad [x,y] = (Jx,y) \quad x, y \in \mathcal{G}$$

where $(.,.)$ denotes the inner product on $\mathcal{H}$. (For the theory of Krein spaces, see [1].) The topology on $\mathcal{G}$ is that which it inherits as a subspace of $\mathcal{H}$. We also define the characteristic function $\Theta(\lambda): \mathcal{G} \to \mathcal{G}$ of the semigroup T by

$$(1.3) \qquad \Theta(\lambda) = I - Q(\lambda - A^*)^{-1} JQ$$

for all complex numbers λ for which $(\lambda - A^*)^{-1}$ is bounded. (Compare this with the characteristic function Θ_A given in [7, p. 358] for a dissipative operator.)

In this paper we consider semigroups for which $T(t)^*$ converges strongly to zero as $t \to \infty$. By the principle of uniform boundedness, this implies that there is a positive constant M such that

$$(1.4) \qquad \|T(t)\| = \|T(t)^*\| \leq M \quad \text{for all} \quad t \geq 0$$

(i.e., $T(t)$ is equi-bounded [9, p. 232]). Consequently [9, p. 240], we have the following integral representation of the resolvent of the infinitesimal generator A^* of $T(t)^*$, valid in the right half-plane:

$$(1.5) \qquad (\lambda - A^*)^{-1} = \int_0^\infty e^{-\lambda t}\, T(t)^*\, dt\,, \qquad \operatorname{Re} \lambda > 0.$$

It follows that the characteristic function (1.3) is defined in the right half-plane. We will be considering the case where the characteristic function is also bounded:

(1.6) $\quad \sup \{ \| \Theta(\lambda) \| : \mathrm{Re}\, \lambda > 0 \} = C < \infty$

and will prove the following theorem, analogous to [3].

THEOREM 1.1 *Suppose* $T(t)$ *is a uniformly continuous semigroup with bounded characteristic function, such that* $T(t)^*$ *converges strongly to zero as* $t \to \infty$. *Then* $T(t)$ *is similar to a contraction semigroup.* ■

As in [7], [3], and [6], the characteristic function is studied in the context of a unitary dilation, in this case, the dilation constructed by Davis [2]. Before a proof of the theorem can be given, it is necessary to develop a theory of Fourier transforms on $\mathcal{H}$ and to analyze the geometry of the dilation space (sections 2 and 3 below).

2. FOURIER TRANSFORMS

If $\mathcal{G}$ is a Krein space, and if $\mathbf{R}$ denotes the real numbers, then we denote by $L^p(\mathbf{R}, \mathcal{G})$ the Banach space of (equivalence classes of) functions $f : \mathbf{R} \to \mathcal{G}$ which are strongly measurable and for which

(2.1) $$\| f \|_p = \left(\int_{-\infty}^{\infty} \| f(t) \|^p \, dt \right)^{1/p} < \infty , \qquad p < \infty.$$

For $p = \infty$, (2.1) is replaced by

(2.2) $\quad \|f\|_\infty = \text{ess sup } \{\|f(t)\| : t \in \mathbb{R}\}.$

The subspace of $L^p(\mathbb{R}, \mathcal{G})$ consisting of all functions with support contained in the interval $[0,\infty)$ will be denoted by $L^p(\mathbb{R}^+, \mathcal{G})$. Likewise, $L^p(\mathbb{R}^-, \mathcal{G})$ and $L^p([0,s], \mathcal{G})$ will denote the subspaces of functions supported on $(-\infty,0]$ and $[0,s]$, respectively.

The space $L^2(\mathbb{R}, \mathcal{G})$ is a Krein space with indefinite inner product defined by

(2.3) $$[f,g] = \int_{-\infty}^{\infty} [f(t), g(t)]\, dt \qquad f, g \in L^2(\mathbb{R}, \mathcal{G}),$$

where $[f(t),g(t)]$ denotes the indefinite inner product of $\mathcal{G}$.

We will be needing vector-valued versions of some classic theorems, which are stated here without proof. First, a vector-valued Fubini theorem (see [4], Corollary III.11.15):

THEOREM 2.1. *If* f *is a strongly measurable function of two variables which satisfies*

(2.4) $$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \|f(u,v)\|\, du\right) dv < \infty,$$

then

(2.5) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\, du\, dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\, dv\, du. \qquad \blacksquare$$

When we need to interchange an order of integration, it will be done without explicit reference to Theorem 2.1; in such a case, verification of the condition (2.4) is left to the reader.

Consider the Fourier transform of a function $f \in L^1(\mathbf{R}, \mathcal{G})$, defined for $y \in \mathbf{R}$ by

$$(2.6) \qquad F(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyt} f(t)\, dt .$$

We have a vector-valued Plancherel theorem (see [8, p. 139]):

THEOREM 2.2 *Let* f *be a function in* $L^1(\mathbf{R}, \mathcal{G}) \cap L^2(\mathbf{R}, \mathcal{G})$, *and let* $F(y)$ *be the Fourier transform given by (2.6). Then* $F \in L^2(\mathbf{R}, \mathcal{G})$ *and* $\|F\|_2 = \|f\|_2$. ■

Consequently, the definition (2.6) can be extended by continuity from $L^1(\mathbf{R}, \mathcal{G}) \cap L^2(\mathbf{R}, \mathcal{G})$ to all $f \in L^2(\mathbf{R}, \mathcal{G})$.

Now let f be a function in $L^p(\mathbf{R}^+, \mathcal{G})$ $(1 \le p \le \infty)$. Then the function $e^{-\lambda t} f(t)$ is in $L^1(\mathbf{R}^+, \mathcal{G})$ for all complex numbers λ with $\mathrm{Re}\,\lambda > 0$, and we can define the holomorphic Fourier transform

$$(2.7) \qquad \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda t} f(t)\, dt , \qquad \mathrm{Re}\,\lambda > 0.$$

As above, when $f \in L^2(\mathbf{R}^+, \mathcal{G})$, this definition can be extended by continuity to $\mathrm{Re}\,\lambda = 0$.

We will be needing the following uniqueness theorem:

THEOREM 2.3 *Suppose* $f \in L^p(\mathbf{R}^+, \mathcal{G})$, $1 \le p \le \infty$, *and that* $\hat{f}(\lambda) = 0$ *for all complex numbers* λ *with* $\operatorname{Re} \lambda > 0$. *Then* $f = 0$. ■

Let $F(\lambda)$ denote a function taking values in $\mathcal{G}$ and holomorphic in the right half-plane $\operatorname{Re} \lambda > 0$, and for $x > 0$ define F_x by $F_x(y) = F(x+iy)$ $(y \in \mathbf{R})$. The Hardy-Lebesgue space $H^2(0, \mathcal{G})$ (cf. [9, p. 163]) is defined as the space of all such functions $F(\lambda)$, with $F_x \in L^2(\mathbf{R}, \mathcal{G})$ for all $x > 0$ and

$$(2.8) \qquad \|F\| = \sup\{\|F_x\|_2 : x > 0\} < \infty.$$

For this space, we have a vector-valued Paley-Wiener theorem:

THEOREM 2.4 *If* $f \in L^2(\mathbf{R}^+, \mathcal{G})$, *then* $\hat{f} \in H^2(0, \mathcal{G})$. *Conversely, if* $F \in H^2(0, \mathcal{G})$, *then there is a function* $f \in L^2(\mathbf{R}^+, \mathcal{G})$ *such that* $F = \hat{f}$. ■

From the above results, we can get:

THEOREM 2.5 *The holomorphic Fourier transform (2.7) is a unitary operator from* $L^2(\mathbf{R}^+, \mathcal{G})$ *onto* $H^2(0, \mathcal{G})$. ■

The above theorem shows that $H^2(0, \mathcal{G})$ is a Hilbert space. Since $L^2(\mathbf{R}^+, \mathcal{G})$ is also a Krein space, with indefinite inner product given by (2.3), we can make $H^2(0, \mathcal{G})$ into a Krein space too, by defining

$$(2.9) \qquad [\hat{f},\hat{g}] = [f,g] \ , \quad f,\ g \in L^2(\mathbf{R}^+, \mathcal{G}).$$

3. THE DILATION SPACE

Let us now return to the study of the semigroup $T(t)$ by introducing its unitary dilation, as constructed by Davis [2].

Define a Krein space $\mathcal{K}$ by

$$(3.1) \qquad \mathcal{K} = \mathcal{H} \oplus L^2(\mathbf{R}, \mathcal{G}),$$

where $\mathcal{G}$ is the Krein space introduced in section 1, with indefinite inner product given by (1.2). A vector k in $\mathcal{K}$ will be denoted by $k = \langle h,f\rangle$, where $h \in \mathcal{H}$ and $f \in L^2(\mathbf{R}, \mathcal{G})$; the indefinite inner product on $\mathcal{K}$ is given by

$$(3.2) \qquad [k,k'] = (h,h') + [f,f'], \quad k = \langle h,f\rangle, \quad k' = \langle h',f'\rangle,$$

where (h,h') is the Hilbert space inner product on $\mathcal{H}$ and $[f,f']$ is the indefinite inner product (2.3) on $L^2(\mathbf{R}, \mathcal{G})$.

It will be convenient to consider the dilation of $T(t)$ as a system obtained by adding inputs and outputs to a state space. Specifically, consider $\mathcal{H}$ as the state space of a system, with $T(t)h$ representing the state of the system t units of time after being in state h. Functions in $L^2(\mathbf{R}^-, \mathcal{G})$ can be considered inputs to the state space, and functions in $L^2(\mathbf{R}^+, \mathcal{G})$

can be considered outputs from the state space.

The dilation semigroup $U(s)$ has two components. One is its action as a shift on $L^2(\mathbf{R}, \mathcal{G})$, and the other is its interaction with $\mathcal{H}$. The second part can be roughly described by specifying that $U(s)$ acts as $T(s)$ on $\mathcal{H}$, with outputs to $L^2(\mathbf{R}^+, \mathcal{G})$ leaving $\mathcal{H}$ by means of the operator Q, and inputs from $L^2(\mathbf{R}^-, \mathcal{G})$ entering $\mathcal{H}$ by means of the operator $-JQ$ (where J and Q are the operators defined in section 1). More precisely, for $s \geq 0$,

$$U(s)\langle h,f\rangle = \langle h',f'\rangle \tag{3.3}$$

where

$$h' = T(s)h - \int_0^s T(s-t)JQf(-t)\,dt \tag{3.4}$$

and

$$f' = f'(\tau) = f(\tau-s) + \chi_{[0,s]}(\tau)\ \left[QT(s-\tau)h - \int_0^{s-\tau} QT(s-t-\tau)JQf(-t)\,dt\right]. \tag{3.5}$$

Here, and in the sequel, we have adopted the convention (also used in [2]) of using a special symbol τ to denote the independent variable. Thus, for example, $f(\tau)$ represents an element of $L^2(\mathbf{R}, \mathcal{G})$, whereas $f(t)$ represents a vector in $\mathcal{G}$. $f(\tau-s)$ in (3.5) is the function obtained by shifting f to the right by s units. We will also be using λ in the same role when discussing functions in $H^2(0, \mathcal{G})$.

In [2] it is shown that the $U(s)$ defined above is a semigroup on $\mathcal{K}$, and that it is a dilation of $T(s)$ which is unitary in the sense of the indefinite inner product, i.e., $U(s)$ is invertible and $[U(s)k,U(s)k'] = [k,k']$ for every

$k, k' \in \mathcal{K}$ and $s \geq 0$. By defining $U(-s) = U(s)^*$ (where the adjoint is taken in the indefinite inner product), $U(s)$ becomes a unitary group on $\mathcal{K}$, again in the sense of the indefinite inner product.

The characteristic function defined by (1.3) can now be derived in a manner that is similar to that used in [7], [3], and [6] for the characteristic function of a single operator. We consider Fourier transforms of the spaces of input and output functions, and express the relationship between inputs and outputs by means of the characteristic function. The technique can be compared to the systems theory approach of representing inputs and outputs by their Fourier representations and relating them by means of a frequency response function.

We will be considering the subspace $\mathcal{K}_+$ of $\mathcal{K}$ given by

$$(3.6) \qquad \mathcal{K}_+ = \{\langle h,f\rangle \in \mathcal{K} : f \in L^2(\mathbb{R}^+, \mathcal{G})\},$$

and the semigroup

$$(3.7) \qquad U^+(s) = U(s)\,|\,\mathcal{K}_+ , \qquad \text{for } s \geq 0.$$

(Note that $\mathcal{K}_+$ is invariant for $U(s)$, for $s \geq 0$.) Then Theorem 1.1 will be proved by establishing:

THEOREM 3.1 $U^+(s)$ *is similar to a semigroup of operators which are isometries with respect to both an indefinite and a Hilbert space inner product.* ■

The proof of this theorem, and Theorem 1.1, will be completed in section 4 below, after investigating some of the structure of $\mathcal{K}_+$.

We consider two subspaces $\mathcal{M}$ and $\mathcal{M}_*$ of $\mathcal{K}_+$, and their Fourier representations Φ and Φ_*, corresponding to outputs and inputs, respectively. The simplest to describe is the output space $\mathcal{M}$ and its Fourier representation:

$$(3.8) \qquad \mathcal{M} = \{<0,f> \in \mathcal{K} : f \in L^2(\mathbf{R}^+, \mathcal{G})\} \quad \text{and} \quad \Phi<0,f> = \hat{f},$$

where $\hat{f}$ is the holomorphic Fourier transform (2.7). By Theorem 2.5 and (2.9), Φ is a unitary operator from $\mathcal{M}$ onto $H^2(0, \mathcal{G})$, preserving both the Hilbert space and indefinite inner products. Also, $\Phi(U(s)<0,f>)$ is the holomorphic Fourier transform of $f(\tau-s)$, so that

$$(3.9) \qquad \Phi(U(s)m) = e^{-\lambda s}\Phi m \quad \text{for all} \quad m \in \mathcal{M} \quad \text{and} \quad s \geq 0.$$

In order to parallel the theory for a single operator, we consider inputs to the system in the following way. Take $s > 0$, and let f be a function in $L^2([0,s], \mathcal{G})$. Shift f to the left by s units, so as to get a function in the input space $L^2(\mathbf{R}^-, \mathcal{G})$, and then apply $U(s)$. This gives a vector in $\mathcal{K}_+$; we define $\mathcal{M}_*$ to be the closed linear span of such vectors:

$$(3.10) \qquad \mathcal{M}_* = \bigvee\{U(s)<0,f(\tau+s)> : f \in L^2([0,s], \mathcal{G}), \quad s > 0\}.$$

The Fourier representation Φ_* of $\mathcal{M}_*$ is densely defined on $\mathcal{M}_*$ by

$$(3.11) \qquad \Phi_*[U(s)<0,f(\tau+s)>] = \hat{f}, \qquad s > 0.$$

For all $s > 0$ and for a dense set of vectors $m_* \in \mathcal{M}_*$, namely $m_* = U(u)<0,f(\tau+u)>$, where $f \in L^2([0,u], \mathcal{G})$ and $u > 0$, we have

$$(3.12) \qquad \Phi_*(U(s)m_*) = e^{-\lambda s}\Phi_* m_*.$$

It follows immediately, from the fact that both $U(s)$ and the holomorphic Fourier transform are unitary, that Φ_* preserves the indefinite inner products on $\mathcal{M}_*$ and $H^2(0, \mathcal{G})$. We can, however, draw no conclusion about the boundedness of Φ_* without (as in [3]) first interpreting the characteristic function as a projection on $\mathcal{K}$.

Suppose $m_* \in \mathcal{M}_*$ and $m \in \mathcal{M}$, where $m_* = U(s)<0,f(\tau+s)>$ for some $f \in L^2([0,s], \mathcal{G})$, $s > 0$, and $m = <0,g>$ for some $g \in L^2([0,N], \mathcal{G})$, $N > 0$. By (3.4) and (3.5), $m_* = <h',f'>$, where

$$(3.13) \qquad h' = -\int_0^s T(s-t)JQf(s-t)\,dt = -\int_0^s T(t)JQf(t)\,dt$$

and

$$(3.14)\qquad f' = f(\tau) - \chi_{[0,s]}(\tau)\int_0^{s-\tau} QT(s-t-\tau)JQf(s-t)\,dt$$
$$= f(\tau) - \chi_{[0,s]}(\tau)\int_\tau^{s} QT(t-\tau)JQf(t)\,dt.$$

Therefore

$$(3.15)\qquad [m_*,m] = \int_0^s [f'(u),g(u)]\,du$$
$$= \int_0^s [f(u),g(u)]\,du - \int_0^s\int_u^s [QT(t-u)JQf(t),g(u)]\,dt\,du$$
$$= \int_0^s [f(t),g(t)]\,dt - \int_0^s\int_0^t [QT(t-u)JQf(t),g(u)]\,du\,dt\ .$$

The second integrand in (3.15) can be written as

$$(3.16)\qquad [QT(t-u)JQf(t),g(u)] = (JQT(t-u)JQf(t),g(u))$$
$$= (f(t),JQT(t-u)^*JQg(u))$$
$$= [f(t),QT(t-u)^*JQg(u)]$$

and thus

$$(3.17)\qquad [m_*,m] = \int_0^s [f(t),\ g(t) - h(t)]\,dt\ ,$$

where

$$(3.18)\qquad h(t) = \int_0^t QT(t-u)^*JQg(u)\,du.$$

Note that, for all $t \geq 0$, the integrand of (3.18) is in $L^1(\mathbb{R}^+,\ \mathcal{G})$, since by (1.4)

$$(3.19) \qquad \int_0^t \| QT(t-u)^*JQg(u) \| \, du \le \int_0^N \| QT(t-u)^*JQg(u) \| \, du \le M\|Q\|^2\sqrt{N}\,\|g\|_2,$$

for $g \in L^2([0,N], \mathcal{G})$. Since the right hand side of the inequality (3.19) is independent of t, it also follows from (3.18) and (3.19) that $h \in L^\infty(\mathbf{R}^+, \mathcal{G})$.

The function h has the form of a convolution, and so it is to be expected that its holomorphic Fourier transform will be of the form of a product of two functions, one operator-valued and the other vector-valued. In fact

$$(3.20) \qquad \hat{h}(\lambda) = Q\Big(\int_0^\infty e^{-\lambda t}\, T(t)^*\, dt\Big) JQ \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda u} g(u)\, du$$

$$= Q(\lambda - A^*)^{-1} JQ\, \hat{g}(\lambda) \qquad \text{by (1.5)}$$

$$= (I - \Theta(\lambda))\, \hat{g}(\lambda).$$

We are assuming that $\Theta(\lambda)$ is bounded for $\operatorname{Re}\lambda > 0$ (1.6). Since $g \in L^2(\mathbf{R}^+, \mathcal{G})$, then $\hat{g} \in H^2(0, \mathcal{G})$, and it follows immediately from the boundedness of Θ and the relation (3.20) that $\hat{h} \in H^2(0, \mathcal{G})$. Thus, by Theorem 2.4, $\hat{h}$ is the holomorphic Fourier transform of a function in $L^2(\mathbf{R}^+, \mathcal{G})$. The uniqueness theorem (Theorem 2.3) then implies that $h \in L^2(\mathbf{R}^+, \mathcal{G})$.

Now we can rewrite (3.17) as an inner product in $L^2(\mathbf{R}^+, \mathcal{G})$ and, using the fact that the holomorphic Fourier transform preserves the indefinite inner product (2.9), as an inner product in $H^2(0, \mathcal{G})$:

$$(3.21)\qquad [m_*,m] = [f,\ g - h] = [\hat{f},\ \hat{g} - \hat{h}] = [\hat{f},\ \Theta\hat{g}]$$
$$= [\Phi_* m_*,\ \Theta\Phi m]$$

for all m_* of the form $U(s)\langle 0, f(\tau+s)\rangle$ ($f \in L^2([0,s],\ \mathcal{G})$, $s > 0$) and for all m of the form $\langle 0,g\rangle$ ($g \in L^2([0,N],\ \mathcal{G})$, $N > 0$). We are using Θ in (3.21) to represent the bounded operator on $H^2(0,\ \mathcal{G})$ defined by $(\Theta F)(\lambda) = \Theta(\lambda)F(\lambda)$, for $\operatorname{Re}\lambda > 0$. If C is the bound for Θ, as given in (1.6), then for the operator Θ on $H^2(0,\ \mathcal{G})$ we have $\|\Theta\| \le C$.

(3.21) is valid for a dense set of vectors $m \in \mathcal{M}$. Since Φ is a bounded operator from $\mathcal{M}$ to $H^2(0,\ \mathcal{G})$ and Θ is a bounded operator on $H^2(0,\ \mathcal{G})$, (3.21) is in fact valid for all $m \in \mathcal{M}$. In order to extend (3.21) to all $m_* \in \mathcal{M}_*$, it is necessary to first establish the boundedness of Φ_*. This can be done by using an approach that is formally the same as that used in the study of a single operator in [3], and thus the details can be omitted. As in [3], we get the estimates

$$(3.22)\qquad \|\Phi_*\|^2 \le 1 + 2C^2,\quad \|\Phi_*^{-1}\|^2 \le 1 + 2C^2$$

and therefore (3.21) can be extended to give

$$(3.23)\qquad [m_*,m] = [\Phi_* m_*,\ \Theta\Phi m]\quad \text{for all } m \in \mathcal{M},\ m_* \in \mathcal{M}_*.$$

4. PROOFS OF THEOREMS 1.1 AND 3.1

The relation (3.12) can now be extended to all of $\mathcal{M}_*$, using the boundedness of Φ_*, i.e.

(4.1) $\quad \Phi_*(U(s)m_*) = e^{-\lambda s}\Phi_* m_*$ for all $m_* \in \mathcal{M}_*$ and $s > 0$.

Thus the semigroup $\{U(s)|\mathcal{M}_* : s \geq 0\}$ on $\mathcal{M}_*$ is similar to the semigroup $\{W(s) : s \geq 0\}$ of operators on $H^2(0, \mathcal{G})$, where $W(s)$ denotes multiplication by the function $e^{-\lambda s}$. It is readily checked that each $W(s)$ is an isometry on $H^2(0, \mathcal{G})$, with respect to both the indefinite and Hilbert space inner products (it corresponds, via the holomorphic Fourier transform, to the shift to the right by s units on $L^2(\mathbf{R}^+, \mathcal{G})$).

One of our objectives was to prove Theorem 3.1 by showing that $U^+(s)$ (defined as the semigroup $\{U(s)|\mathcal{K}_+ : s \geq 0\}$) is similar to a semigroup of isometries. We have in fact already done this, since it is the case that $\mathcal{M}_* = \mathcal{K}_+$:

THEOREM 4.1 *If* $T(t)$ *is a semigroup such that* $T(t)^*$ *converges strongly to zero as* $t \to \infty$, *then* $\mathcal{M}_* = \mathcal{K}_+$.

PROOF. Suppose $k \in \mathcal{K}_+$ is such that $[k, m_*] = 0$ for a dense set of vectors $m_* \in \mathcal{M}_*$, namely

(4.2) $\quad [k, U(s)\langle 0, g(\tau+s)\rangle] = 0$

for all $s > 0$ and all $g \in L^2([0,s], \mathcal{G})$. Since the indefinite inner product on $\mathcal{K}_+$ is nondegenerate, it will follow that $\mathcal{M}_* = \mathcal{K}_+$ if we can show $k = 0$.

Let k be given by $k = \langle h_0, f \rangle$, where $h_0 \in \mathcal{H}$ and $f \in L^2(\mathbf{R}^+, \mathcal{G})$. Define a function h by

$$(4.3) \qquad h(t) = T(t)^* h_0 + \int_0^t T(t-u)^* JQf(u)\, du, \qquad t \geq 0.$$

(This function is analogous to the sequence $\{h_n\}$ used in [6, Theorem 4.2].) h has the properties that $h(0) = h_0$, $f(t) = Qh(t)$ for almost all $t \geq 0$, and $T(t)h(s) = h(s-t)$ for all s and t satisfying $0 \leq t \leq s$. The first of these properties is obvious, but the other two require some proving; details are omitted in this paper for the sake of brevity.

The theorem will be proved by showing the function h must necessarily be zero. We begin by showing that h is bounded.

Since the functions $f_s(\tau) = \chi_{[0,s]}(\tau) f(\tau)$ converge strongly to f in $L^2(\mathbf{R}^+, \mathcal{G})$, as $s \to \infty$, it follows that $[f,f]$ can be obtained as the limit of $[f_s, f_s]$, where

$$(4.4) \qquad [f_s, f_s] = \int_0^s [f(t), f(t)]\, dt = \int_0^s [Qh(t), Qh(t)]\, dt$$

$$= \int_0^s [Qh(s-t), Qh(s-t)]\, dt = \int_0^s [QT(t)h(s), QT(t)h(s)]\, dt.$$

The integrand can be written in the form

$$(4.5)\qquad (JQT(t)h(s), QT(t)h(s)) = -(T(t)^*GT(t)h(s), h(s)),$$

which is the derivative, with respect to t, of $-(T(t)^*T(t)h(s), h(s))$ (by (1.1)). Thus we have

$$(4.6)\qquad [f_s, f_s] = \|h(s)\|^2 - \|T(s)h(s)\|^2 = \|h(s)\|^2 - \|h(0)\|^2.$$

Since the limit of $[f_s, f_s]$ exists, it follows that $\|h(s)\|$ is bounded.

Finally, we observe that for all $h' \in \mathcal{H}$, and for all $t \geq 0$ and $s \geq t$, we have

$$(4.7)\qquad |(h(t), h')| = |(T(s-t)h(s), h')| = |(h(s), T(s-t)^*h')| \leq \|h(s)\|\,\|T(s-t)^*h'\|.$$

Let $s \to \infty$. Since $h(s)$ is bounded, and since, by assumption, $T(s-t)^*h' \to 0$, it follows that $(h(t), h') = 0$ for all $t \geq 0$ and for all $h' \in \mathcal{H}$. Thus $h = 0$, and hence $k = 0$, and the proof that $\mathcal{M}_* = \mathcal{K}_+$ is complete. ■

As was observed at the start of this section, Theorem 4.1 provides the final link in the proof of Theorem 3.1. All that remains is to supply a proof of Theorem 1.1.

Let P denote the projection on $\mathcal{K}$ defined by

$$(4.8)\qquad P\langle h, f\rangle = \langle h, 0\rangle.$$

Thus P is the orthogonal projection onto $\mathcal{H}$, with respect to both the indefinite and Hilbert space inner products on $\mathcal{K}$. The dilation property of $U(s)$ may be described as $T(s) = PU^+(s)|\mathcal{H}$. Note that, by (3.8), $\mathcal{K}_+ = \mathcal{H} \oplus \mathcal{M}$, and that $\mathcal{M}$ is invariant for $U^+(s)$. Thus, $\mathcal{H}$ is invariant for $U^+(s)^*$, and we may write the dilation property as

$$(4.9) \qquad T(s)^* = U^+(s)^*|\mathcal{H}.$$

Since $U^+(s)$ is similar to a semigroup of Hilbert space isometries (Theorem 3.1), it follows from (4.9) that $T(s)^*$ is similar to a contraction semigroup. Thus, $T(t)$ is similar to a contraction semigroup, and Theorem 1.1 is proved.

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