BETTER GOOD λ INEQUALITIES

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Introduction

In the early 1970s, D. Burkholder and R. Gundy introduced a technique for studying operators on L^p spaces. Their idea was to relate a pair of operators by a distribution function estimate which is now known as a "good-λ" inequality:

$$m(\{x \in \mathbb{R}^n: |Tf(x)| > 2\lambda, |Mf(x)| \le \delta\lambda\}) \le \epsilon\, m(\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}) .$$

Such an inequality implies that the L^p norm of Tf is bounded by the L^p norm of Mf. Thus, integrability results about M can be used to derive corresponding ones about T. Often, the method of proof allows one to replace Lebesgue measure by a weighted measure.

In many instances, this kind of result can be improved. Consider the situation when Tf is a maximal Calderón-Zygmund singular integral operator and Mf is the Hardy-Littlewood maximal function of f. R.R. Coifman and C. Fefferman proved [6]

$$w(\{x \in \mathbb{R}^n: Tf(x) > 2\lambda, Mf(x) \le \delta\lambda\}) \le \epsilon\, w(\{x \in \mathbb{R}^n: Tf(x) > \lambda\}) \tag{0.1}$$

for any weight w in Muckenhoupt's A_∞ class. Our main result is an improved version of (0.1).

Theorem 1: Let $w \in A_\infty$ and $0 < \epsilon < 1$. There is a constant $C > 0$ such that

$$w(x \in \mathbb{R}^n\colon Tf(x) > CMf(x) + \lambda\})$$
$$\leq \epsilon\, w(\{x \in \mathbb{R}^+\colon Tf(x) > \lambda\}), \qquad \lambda > 0\,. \square$$

This result is equivalent to an estimate proved by R.J. Bagby and the author [1].

The conclusion of Theorem 1 implies (0.1). One can see it is an improvement of (0.1) by considering non-increasing rearrangements. From the theorem, we get

$$(Tf)_w^*(t) \leq C(Mf)_w^*(t/2) + (Tf)_w^*(2t), \qquad t > 0\,.$$

This inequality implies sharp estimates on the operator norm of T acting on L^p, for large p. Such estimates cannot be obtained from (0.1).

We use m(E) for the Lebesgue measure of the set E. Given a non-negative, measurable function, w, and $p \geq 1$, set

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p\, w(x)dx\right)^{1/p}.$$

In Section 1, we discuss the good-λ inequality (0.1). A sketch of the proof of Theorem 1 can be found in Section 2. The last two sections contain results about rearrangement functions and applications.

The results of this paper contain joint work done with R.J. Bagby [1]. In particular, the content of Sections 3 and 4 can be found in that paper, where complete proofs are given.

I. The Good-λ Inequality

Let w(x) be a non-negative, measurable weight function and set $w(E) = \int_E w(x)dx$ for any Lebesgue measurable set E

Definition 1.1: $w \in A_\infty$ if given ϵ, $0 < \epsilon < 1$, there is a $\delta > 0$ so that for any cube $Q \subset \mathbb{R}^n$ and measurable set $E \subset Q$, $m(E) < \delta m(Q)$ implies $w(E) < \epsilon\, w(Q)$. □

Let f be a Lebesgue measurable function on $\mathbb{R}^n$ and define the distribution function of f with respect to w by

$$D_{f,w}(\lambda) = w(\{x \in \mathbb{R}^n\colon |f(x)| > \lambda\}) \,,$$

for $\lambda > 0$. For $1 \leq p < \infty$, we have

$$\int_{\mathbb{R}^n} |f(x)|^p\, w(x)dx = p \int_0^\infty \lambda^{p-1}\, D_{f,w}(\lambda)d\lambda \,. \tag{1.2}$$

Let K(x) be homogeneous of degree -n and satisfy the conditions:

(i) $|K(x)| \leq C/|x|^n$

$$\text{(ii)} \int_{\{a<|x|<b\}} K(x)dx = 0 \,, \qquad 0 < a < b \tag{1.3}$$

(iii) $|K(x\text{-}y) - K(x)| \leq C|y|/|x|^{n+1}$, $\qquad |x| \geq 2|y|$.

Set

$$T_\epsilon f(x) = \int_{\{y:|x-y|>\epsilon\}} K(x\text{-}y)f(y)dy \,.$$

To study the Calderón-Zygmund singular integral operator $Kf(x) = \lim_{\epsilon \searrow 0} T_\epsilon f(x)$, consider the maximal singular integral operator $Tf(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|$. The operator we use to control T is the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy ,$$

where the supremum is taken over all cubes, Q, which contain x.

In [6], Coifman and Fefferman proved

Theorem 1.4: Let $w \in A_\infty$. Given ϵ, $0 < \epsilon < 1$, there is a $\delta > 0$ so that

$$w(\{x \in \mathbb{R}^n: Tf(x) > 2\lambda, Mf(x) \le \delta\lambda\})$$
$$\le \epsilon \, w(\{x \in \mathbb{R}^n: Tf(x) > \lambda\}) , \quad \lambda > 0 . \square$$

From this theorem, we have

$$D_{Tf,w}(2\lambda) \le w(\{x \in \mathbb{R}^n: Tf(x) > 2\lambda, Mf(x) \le \delta\lambda\}) + D_{Mf,w}(\delta\lambda)$$
$$\le \epsilon \, D_{Tf,w}(\lambda + D_{Mf,w}(\delta\lambda) .$$

Using (1.2) and several changes of variables

$$\int_{\mathbb{R}^n} Tf(x)^p w(x)dx = p\int_0^\infty \lambda^{p-1} D_{Tf,w}(\lambda)d\lambda$$

$$= p\int_0^\infty (2\lambda)^{p-1} D_{Tf,w}(2\lambda)d\lambda$$

$$\leq 2^p\epsilon p\int_0^\infty \lambda^{p-1} D_{Tf,w}(\lambda)d\lambda$$

$$+ \left(\frac{2}{\delta}\right)^p p\int_0^\infty (\delta\lambda)^{p-1} D_{Mf,w}(\delta\lambda)d\lambda$$

$$= 2^p\epsilon \int_{\mathbb{R}^n} Tf(x)^p\, w(x)dx$$

$$+ \left(\frac{2}{\delta}\right)^p \int_{\mathbb{R}^n} Mf(x)^p w(x)dx\ .$$

Combining terms and taking p^{th} roots yields

$$\|Tf\|_{p,w} \leq \frac{2}{\delta(1\text{-}2^p\epsilon)^{1/p}} \|Mf\|_{p,w}\ , \tag{1.5}$$

as long as $\epsilon < 2^{-p}$.

Consider now the A_p condition.

Definition 1.6: Let $1 < p < \infty$. $w \in A_p$ if there is a constant $C > 0$ so that for all cubes $Q \subset \mathbb{R}^n$

$$\left(\frac{1}{|Q|} w(x)dx\right)\left(\frac{1}{|Q|}\int w(x)^{1-p'}\,dx\right)^{p-1} \leq C\ .\ \square$$

Muckenhoupt [8] has shown that $w \in A_p$ implies M defines a bounded operator on the weighted L^p space, L^p_w. Since $w \in A_p$ implies $w \in A_\infty$ (see 6,), (1.5) gives the weighted norm inequality $\|Tf\|_{p,w} \leq C\|f\|_{p,w}$ whenever 1

$< p < \infty$ and $w \in A_p$.

There are two main problems with (1.5). One is that no single ϵ works for all values of p since we need $\epsilon < 2^{-p}$. The other is that, due to the relationship between ϵ and δ, the expression $2/\delta(1-2^p\epsilon)^{1/p}$ is on the order of 2^p while operators like T should have operator norms on the order of p (see [10, p.48]). Both of these problems are caused by the constant 2 by which λ is multiplied in Theorem 1.4. This constant can be replaced by any $\beta > 1$, still yielding estimates with exponential growth, but not by 1, since this would imply the norm of T is bounded for large p.

II. The Better Good-λ Inequality

The major problem with Theorem 1.4 is that Tf is considered only for values of x where Mf is relatively small. Notice that in Theorem 1, the two are compared pointwise. We sketch a proof of Theorem 1 for completeness. The proof is taken from [1].

Proof: Fix ϵ and choose δ by the A_∞ condition. Fix $\lambda > 0$. Let $\{Q_k\}$ be Whitney cubes for the set $E = \{x \in \mathbb{R}^n: Tf(x) > \lambda\}$.

Fix k and choose $x_k \notin E$ so that distance $(x_k,Q_k) \leq 4$ diameter Q_k. Let Q be the cube centered at x_k with diameter Q = 20 diameter Q_k. Set $g = f\chi_Q$ and $h = f\chi_{\mathbb{R}^n-Q}$ so that $f = g+h$. If

$$\alpha = C_1 \frac{1}{|Q|} \int_Q |g(y)|\, dy ,$$

then $C_1 Mf(x) \geq \alpha$ for all $x \in Q_k$. By the inequality

$$m(\{x \in \mathbb{R}^n: Tf(x) > \alpha\}) \leq \frac{c}{\alpha} \int |f(x)|\, dx .$$

we get

$$m(\{x \in Q_k\colon Tg(x) > C_1Mf(x)\}) \le m(\{x \in \mathbb{R}^n\colon Tg(x) > \alpha\})$$

$$\le \frac{A}{\alpha} \int |g(x)|\, dx \le \frac{A}{C_1} |Q| \,. \qquad (2.1)$$

Choose C_1 so that $A|Q|/C_1 \le \delta\, |Q_k|$.

Fix $x \in Q_k$ and $\eta > 0$. Let Δ be the symmetric difference of the balls $B(x,\eta)$ and $B(x_k,\eta)$ and let $r = \max\{\eta, \text{distance}(x_k, \mathbb{R}^n\text{-}Q)\}$. Then

$$|T_\eta h(x)| \le |\int_{\{y\colon |x_k-y|>r\}} K(x_k-y)f(y)dy|$$

$$+ \int_{\{y\colon |x_k-y|>r\}} |K(x_k)| - K(x-y)|\; |f(y)|dy$$

$$+ \int_{\Delta} |K(x-y)|\; |f(y)|dy \,.$$

The first term is bounded by $Tf(x_k) \le \lambda$. The second and third terms are bounded by constant multiples of $Mf(x)$, by (1.3, i) and (1.3, iii), respectively. Taking the supremum over $\eta > 0$,

$$|Th(x)| \le C_2\, Mf(x) + \lambda \,. \qquad (2.2)$$

Let $C = C_1 + C_2$. Then (2.1) and (2.2) imply

$$m(\{x \in Q_k\colon Tf(x) > CMf(x) + \lambda\}) \le \delta m(Q_k) \,.$$

Using the A_∞ condition and summing over k completes the proof. □

Suppose $Mf(x) \le \frac{1}{C}\lambda$. Then $Tf(x) > 2\lambda \ge CMf(x) + \lambda$ which implies

$$x \quad \mathbb{R}^n\; Tf(x) > 2\lambda,\, Mf(x) \le \frac{1}{C}\lambda\}$$

$$\subseteq \{x \in \mathbb{R}^n\colon Tf(x) > CMf(x) + \lambda\} \,.$$

Therefore, Theorem 1 implies Theorem 1.4 with $\delta = \frac{1}{C}$. To see that Theorem 1 contains a stronger inequality, we consider rearrangement functions.

III. Rearrangement functions

Define the non-increasing rearrangement function of f with respect to w by

$$f_w^*(t) = \inf\{\lambda > 0: D_{f,w}(\lambda) \leq t\} ,$$

for $t > 0$. Since f and f_w^* are equi-measurable,

$$\int_{\mathbb{R}^n} |f(x)|^p\, w(x)dx = \int_0^\infty f_w^*(t)^p\, dt .$$

Setting $\lambda = (Tf)_w^*$ in Theorem 1, we get the equivalent inequality

$$\begin{aligned} &w(\{x \in \mathbb{R}^n: Tf(x) > CMf(x) + (Tf)_w^*(2t)\}) \\ &\qquad \leq \epsilon\, w(\{x \in \mathbb{R}^n: Tf(x) > (Tf)_w^*(2t)\}) . \end{aligned} \tag{3.1}$$

Fix γ, $0 < \gamma < 1$ and set $\epsilon = \frac{1-\gamma}{2}$. Since the definition of f_w^* implies $D_{f,w}(f_w^*(t)) \leq t$, by (3.1),

$$\begin{aligned} &w(\{x \in \mathbb{R}^n: Tf(x) > C(Mf)_w^*(\gamma t) + (Tf)_w^*(2t)\}) \\ &\quad \leq w(\{x \in \mathbb{R}^n: Tf(x) > CMf(x) + (Tf)_w^*(2t)\}) \\ &\qquad + w(\{x \in \mathbb{R}^n: Mf(x) > (Mf)_w^*(\gamma t)\}) \leq \epsilon(2t) + \gamma t = t . \end{aligned}$$

Therefore, we get

Lemma 3.2: Let $w \in A_\infty$. For γ, $0 < \gamma < 1$, there is a $C > 0$ so that

$$(Tf)_w^*(t) \leq C(Mf)_w^*(\gamma t)\, (Tf)_w^*(2t) , \qquad t > 0 . \square$$

Set $\gamma = \frac{1}{2}$ and iterate the conclusion of the lemma to get

$$(Tf)^*_w \leq \sum_{k=0}^{\infty} (Mf)^*_w(2^{k-1}t) + \lim_{s \to \infty} (Tf)^*_w(s) .$$

If the sum is finite, one can show that the limit is 0. Since f^*_w is non-increasing, we have

Theorem 3.3: Let $w \in A_\infty$. There is a $C > 0$ such that

$$(Tf)^*_w(t) < C(Mf)^*_w\left(\frac{t}{2}\right) + C\int_t^\infty (Mf)^*_w(s)\,\frac{ds}{s}$$

$$\leq C\int_{t/4}^\infty (Mf)^*_w(s)\,\frac{ds}{s}, \qquad t > 0 . \square$$

Results of this nature also appear in [3,4].

IV. Applications

Suppose $w \in A_\infty$. By Hardy's inequality,

$$\int_0^\infty (\int_t^\infty g(u)du)^p dt \leq p^p \int_0^\infty (ug(u))^p\, du,$$

and Theorem 3.3, we get

$$\int_{\mathbb{R}^n} Tf(x)^p w(x)dx = \int_0^\infty (Tf)_w^*(t)^p dt$$

$$\leq C^p \int_0^\infty \left(\int_{t/4}^\infty (Mf)_w^*(s) \frac{ds}{s} \right)^p dt$$

$$\leq 4C^p p^p \int_0^\infty (Mf)_w^*(t)^p dt$$

$$= 4(Cp)^p \int_{\mathbb{R}^n} Mf(x)^p \, w(x)dx \, .$$

Thus, $||Tf||_{p,w} \leq 4Cp||Mf||_{p,w}$ for $1 \leq p < \infty$. By the boundedness of M on weighted L^p spaces, $||Tf||_{p,w} \leq C||f||_{p,w}$ for $1 < p < \infty$ and $w \in A_p$. Using some results about A_∞ weights (see [6]), we have

Corollary 4.1: Let $w \in A_\infty$. There is a $p(w) > 1$ and a $C > 0$ so that for $p(w) \leq p < \infty$,

$$||Tf||_{p,w} \leq Cp||f||_{p,w} \, . \; \square$$

The linear rate of growth of the norm of T is the best possible for general Calderón-Zygmund singular integral operators. Note also that we get all of the L^p results, $1 < p < \infty$, using only $\gamma = \frac{1}{2}$, instead of having to vary ϵ as in (1.5).

Calderón [5] has shown that if an operator S is weak-type (1,1) and bounded on L^∞ then

$$(Sf)^*(t) \leq C \frac{1}{t} \int_0^t f^*(s)ds \, . \tag{4.2}$$

Suppose w satisfies the A_1 condition, $Mw(x) \leq Cw(x)$ for almost every x. Then, for the Hardy Littlewood maximal function we have

$$(Mf)_w^*(t) \le C\,\frac{1}{t}\int_0^t f_w^*(s)\,ds\,.$$

Plugging this estimate in the first inequality of Theorem 3.3 and performing the integration yields

Corollary 4.3: Suppose $w \in A_1$. Then there is a $C > 0$ so that

$$(Tf)_w^*(t) \le C\,\frac{1}{t}\int_0^t f_w^*(s)ds + C\int_t^\infty f_w^*(s)\,\frac{ds}{s}\,, \qquad t > 0\,. \square$$

Thus, we get a weighted version of (4.2) even though T is not bounded on L^∞ (see [1,3]). Averaging the conclusion of the corollary over the interval (0,t) yields an analog of a result proved by O'Neil and Weiss [9] for the Hilbert transform.

For finite p, the Marcinkiewicz space weak-L^p properly contains L^p, while the two spaces coincide when $p = \infty$. In order to extend the Marcinkiewicz Interpolation Theorem to include operators that are unbounded on L^∞, Bennett, De Vore, and Sharpley [2] introduced a space called weak-L^∞.

Define the averaged rearrangement function of f by

$$f_w^{**}(t) = \frac{1}{t}\int_0^t f_w^*(s)ds\,.$$

We say $f \in$ weak-L^∞ if $f_w^*(t)$ is finite for all $t > 0$ and

$$\|f\|_{\text{weak-}L^\infty} := \sup_{t>0}\,\{f_w^{**}(t) - f_w^*(t)\} < +\infty\,.$$

Another iteration of the conclusion of Lemma 3.2 yields

$$(Tf)_w^{**}(t) \le C(Mf)_w^{**}(t) + (Tf)_w^*(t)\,.$$

If $f \in L^\infty$ and $(Tf)^*_w(t)$ is finite for a single t, then $(Tf)^*_w(t)$ is finite for all t, by Lemma 3.2. Therefore, we have

Corollary 4.4: Suppose $w \in A_\infty$ and $f \in L^\infty$. If $(Tf)^*_w(t)$ is finite for some t then $Tf \in$ weak-L^∞ and

$$\|Tf\|_{\text{weak-}L^\infty} \leq C\|f\|_\infty . \square$$

As a consequence of Corollary 4.4, one can prove results about local exponential integrability for T.

Versions of Theorem 1 are true for kernels satisfying conditions weaker than (1.3). Let $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$. Suppose K is positively homogeneous of degree -n and $\int_\Sigma K(x)\, d\sigma(x) = 0$. Set

$$\omega_r(t) = \sup_{|\rho| \leq t} \|K\circ\rho - K\|_{L^r(\Sigma)} ,$$

where ρ is a rotation of Σ and $|\rho| = \sup_{x \in \Sigma} |\rho x - x|$. We say $K \in L^r$-Dini, $1 < r \leq \infty$, if $K \in L^r(\Sigma)$ and

$$\int_0^1 \omega_r(t) \frac{dt}{t} < \infty .$$

Analogs of Theorem 1 for Dini kernels can be found in [1] We also note that similar results for Littlewood-Paley operators are known [7].

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