NON–MONOMIAL REPRESENTATIONS OF ABELIAN GROUPS WITH MULTIPLIERS (ANNOUNCEMENT)

L.W. Baggett, A.L. Carey, William Moran[1]*, and Arlan Ramsay*[2]

INTRODUCTION

If G is a nilpotent Lie group, a well–known result of A. Kirillov says that every irreducible unitary representation of G is monomial, i.e. it can be obtained by inducing a one–dimensional representation from some closed subgroup of G. By using the connection between multiplier representations of G and ordinary representations of a central extension of G [Ma], with some theory of multipliers, Kirillov's result can be extended to multiplier representations. Furthermore, K.C. Hannabuss showed that if G is nilpotent and locally compact (as well as second countable), and the pair (G,σ) is of type I, then every irreducible σ–representation is monomial.

A partial converse to the theorem of Hannabuss was proved by A. Carey and W. Moran [C–M], for the case in which G is discrete and abelian. They have also proved the converse for all discrete nilpotent groups, but the proof is unpublished. Their proof uses the Subgroup Theorem of Mackey in ways that make it seem unlikely that their approach can be extended to general locally compact abelian groups G. The proof we annotate here uses the extended version of the Mackey analysis of group extensions [R], but is not an extension of the method of [C–M]. Before discussing the proof, we need to introduce some notation and recall some results of [C–M].

Let $\sigma : G \times G \to T$ be a Borel function that is a 2–cocycle (multiplier). A unitary–valued Borel function π on G is a σ–representation if

[1]Work by this author was partially supported by a study leave from the University of Adelaide, and parts of it were done at the University of Colorado and at the Centre for Mathematical Analysis of the Australian National University.

[2]Work by this author was partially supported by the National Science Foundation, and part of the work was done at the Centre for Mathematical Analysis of the Australian National University. He thanks the Centre for the opportunity to visit and the staff for their generous hospitality.

$\pi(x)\pi(y) = \sigma(x,y)\pi(xy)$ for all x,y in G. The pair (G,σ) is said to be of type I iff every factor σ–representation ($\pi(G)'' \cap \pi(G)'$ is the scalars) is a multiple of an irreducible σ–representation. We claim that if this fails then G has a non–monomial irreducible σ–representation.

In studying σ, it is useful to consider $\tilde{\sigma}$, defined by $\tilde{\sigma}(x,y) = \sigma(x,y)\overline{\sigma}(y,x)$; this is a bicharacter, and so is separately and hence jointly continuous. If we write $\tilde{\sigma}(x)(y) = \tilde{\sigma}(x,y)$, we can think of $\tilde{\sigma}$ as a homomorphism of G into $\hat{G}$. If this homomorphism is one–one, we say σ is non–degenerate, and it is proved in [C–M] that it suffices to consider this case. If H is a subgroup of G, we also get a homomorphism of G into $\hat{H}$: $\tilde{\sigma}_H(x) = \tilde{\sigma}(x)|_H$.

A subgroup H is called *isotropic* iff $\tilde{\sigma}_H$ maps H into $\{1\}$. This is the same as saying $\sigma(x,y) = \sigma(y,x)$ for x,y in H, or that the central extension of H given by $\sigma|_{H\times H}$ is abelian. It is also the same as saying that $\sigma|_{H\times H}$ is a coboundary, i.e. H has a one–dimensional σ–representation. Thus isotropic subgroups are needed for making monomial representations. It is a fact that the representation $\sigma\text{–}\chi\uparrow_H^G$ induced by a one–dimensional σ–representation of H (see [C–M]), is irreducible iff H is maximal isotropic. We also have a characterization of type I (G,σ) in terms of an arbitrary maximal isotropic subgroup H [C–M], namely (G,σ) is type I iff $\tilde{\sigma}_H(G) = \hat{H}$. If $\tilde{\sigma}_H(G) = \hat{H}$, the method of Mackey can be used to show that $(G,\sigma)\hat{}$ is smooth, and that implies (G,σ) is type I. We therefore study the case in which $\tilde{\sigma}_H(G)$ is properly dense ($\tilde{\sigma}_H(G)$ is dense because σ is non–degenerate.) An example of such a (G,σ) can be obtained by taking $G = \mathbb{R} \times \Delta$ where Δ is a dense subgroup of $\mathbb{R}$. Define $\sigma((x,\delta),(x',\delta')) = e^{i\delta x'}$. Then $H = \mathbb{R} \times \{0\}$ is maximal isotropic and $\tilde{\sigma}_H(G) \simeq \Delta$ is dense.

Here is the basic idea of the proof: Let $\mathcal{M}$ denote the set of measure classes on $\hat{H}$ of representation $\pi|H$, π a monomial σ–representations of G. We give a property of all the measures in $\mathcal{M}$ and produce an irreducible σ–representation π such that $\pi|H$ has a measure class without that property.

1. MONOMIAL MEASURE CLASSES ON $\hat{H}$.

If π is an irreducible monomial σ–representation of G , it is induced from some maximal isotropic subgroup K . The dual group $(G/K)^{\wedge}$ can be identified with the annihilator of K in $\hat{G}$, $K^{\perp}$. Define a homomorphism $\psi : G \times K^{\perp} \to \hat{H}$ by $\psi(g,\gamma) = \tilde{\sigma}_H(g)(\gamma|_H)$. Then ψ gives an action of $G \times K^{\perp}$ on $\hat{H}$.

THEOREM 1.1 *If* π *is a monomial* σ*–representation of* G , *then the measure class of* $\pi|_H$ *is the image of Haar measure on* $G \times K^{\perp}$ *on one of the orbits of the action of* $G \times K^{\perp}$ *on* $\hat{H}$.

2. USES OF STRUCTURE THEORY

One of the main tools in our proof is the fact that a locally compact abelian group always has an open subgroup that is isomorphic to the product of a vector group and a compact group. We apply this structure theorem first to choose a particular kind of maximal isotropic subgroup H , and then several times in analysing the measure classes. There often will be more than one of these special open subgroups, but we denote one choice by $S(\cdot)$.

Let $G_1 = S(G)$ and suppose $G_1 = V \times C$, where V is a vector group and C is compact. If necessary we can change σ to put it in standard form on V . (This is the same as one does for skew–symmetric bilinear forms.) We get $V = V_0 \times V_1 \times V_2$, where σ is identically 1 on $V_0 \times V_0$ and $\sigma((x_1,x_2),(y_1,y_2)) = e^{i(x_2 \cdot y_1 - x_1 \cdot y_2)}$ for $x_1,y_1 \in V_1$, $x_2,y_2 \in V_2$. Let $W = V_0 \times V_1$. Then W is an isotropic subgroup, maximal in V . We let H be any maximal isotropic subgroup of G containing W . Notice that $\tilde{\sigma}_H(W) = \{1\}H = \ker \tilde{\sigma}_H$, and that W is the vector part of $S(H)$. Thus the vector part of $S(\hat{H})$ is isomorphic to $\hat{W}$. We choose one $S(\hat{H})$ and call it L , $L \simeq \hat{W} \times C_1$, C_1 compact. Because L is open in $\hat{H}$, if ν is a measure on L and $\nu|_L \perp \mathcal{M}|_L$ then $\nu \perp M$, so it suffices to study $\tilde{\sigma}_H(G) \cap L$ and $\psi(G \times K^{\perp}) \cap L$.

Now $\tilde{\sigma}_H(V_2) \subsetneq L$, and projects onto the copy of $\hat{V}_1 \subsetneq \hat{W}$. It follows that $\tilde{\sigma}_H(V_2)$ is closed, and $\tilde{\sigma}_H(C)$ is compact, so $\tilde{\sigma}_H(V_2 \times C)$ is closed. Also, $\tilde{\sigma}_H(V_2 \times C)$

has only countably many cosets in $\tilde{\sigma}_H(G)$, so if we form $M = L/(L \cap \tilde{\sigma}_H(V_2 \times C))$, the image of $L \cap \tilde{\sigma}_H(G)$ in M is countable. To understand the image of $\psi(G \times K^{\perp})$ in M, it suffices to take $K_1 = S(K^{\perp})$ and understand $\psi(K_1)$ because the image of G is countable. Notice that M is isomorphic to $\hat{V}_0 \times C_2$, where C_2 is compact.

Now $G_2 = \tilde{\sigma}_H{}^{-1}(L)$ has a naturally defined homomorphism φ into M, and $\varphi(G_2)$ is countable and dense. The projection of $\varphi(G_2)$ into $\hat{V}_0$ is then dense, so there is a subgroup Γ of $\varphi(G_2)$ that projects one–one onto a $\Gamma' \subset \hat{V}_0$ such that $\hat{V}_0/\Gamma'$ is compact. Then $Q = M/\Gamma$ is compact, and we are able to reduce the problem of finding our desired measure class to a problem involving a compact group. This can be solved by the result of the next section. Notice that K_1 has a quotient group with a one–one continuous homomorphism into Q.

3. CONSTRUCTION OF MEASURE

The measures we need are supplied by a lemma we state here. The proof involves ideas from [B–M1, B–M2].

LEMMA 3.1 *Let* Q *be a compact abelian group with a countable dense subgroup* Δ. *There exists a probability measure* μ *on* Q *that is ergodic and quasi–invariant for* Δ *and has also this singularity property:* (S) *If* D *is a locally compact abelian group,* λ *is a finite measure absolutely continuous with respect to Haar measure on* D *and* φ *is an injective continuous homomorphism of* D *into* C, *then for any* $q \in Q$ *the measure* $q \cdot \varphi_*(\lambda)$ *is mutually singular with respect* to μ.

The method of proof is as follows. Choose a neighbourhood basic at 1 in Q, (W_n), so that $W_n^2 \subset W_{n-1}$ for $n \geq 2$. Construct measures (ρ_n) so that ρ_n has finite support contained in $\Delta \cap W_n$, and at the same time choose a sequence (γ_n) in $\hat{Q}$, so that $\gamma_n | \Delta$ converges to 1 in $\hat{\Delta}$ and if $\mu_n = \rho_1 * \ldots * \rho_n$ then $|\hat{\mu}_n(\gamma_i) - \frac{1}{2}| < \frac{1}{i}$. Then let $\mu = \rho_1 * \rho_2 * \ldots$ (this is the image of a measure on the cartesian product $\prod_{n \geq 1} \mathrm{supp}(\rho_n)$ under $x \mapsto \lim_{n \to \infty} x_1 x_2 \ldots x_n$). It follows that $\gamma_n \to \frac{1}{2}$ in the weak* topology on $L^{\infty}(\mu)$. Ergodicity is not hard to prove and from $\gamma_n \to \frac{1}{2}$ we derive (S).

4. THE COMPLETION OF THE PROOF

With a little effort the results stated so far allow us to find a measure μ on $\hat{H}$ that is ergodic and quasiinvariant for G and singular with respect to every measure in $\mathcal{M}$ (monomial classes). Now we need to show that there is an irreducible σ–representation π of G such that $\pi|_H$ $[\mu]$ as its spectral type. Let λ be the Haar measure on G/H. Then the initial little group associated to μ is $(\hat{H}\times(G/H),\mu\times\lambda)$. Since G/H is abelian, this groupoid is amenable and by [C–F–W] the group $H^2(\hat{H}\times G/H,\pi)$ is trivial. Thus by the results of [R], we only need to find one irreducible one–cocycle for the action of G/H on $\hat{H}$, with no multiplier involved. There are many of these.

REFERENCES

[B–M1] G. Brown and W. Moran, *A dichotomy for infinite convolutions of discrete measures*, Proc. Camb. Phil. Soc. **73**(1973), 307–316.

[B–M2] G. Brown and W. Moran, *Bernoulli measure algebras*, Acta Math. **132**(1974), 77–109.

[C–M] A. Carey and W. Moran, *Non–monomial unitary representations of nilpotent groups. I, The abelian case*, Proc. L.M.S. III Ser., vol.**46**(1983), 53–82.

[C–F–W] A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation,* Erg. Th. and Dyn. Syst. 1(1981),431–450.

[H] K.C. Hannabuss, *Representations of nilpotent locally compact groups,* J. Functional Analysis vol. **34**(1974), 146–165.

[M] G.W. Mackey, *Unitary representations of group extensions.* I, Acta Math. **99**(1958), 265–311.

[R] A. Ramsay, *Nontransitive quasiorbits in Mackey's analysis of group extensions*, Acta Math. **137**(1976), 17–48.