CLIFFORD MATRICES, CAUCHY KOWALEWSKI EXTENSIONS AND ANALYTIC FUNCTIONALS

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INTRODUCTION

Our aim is to first use classical, and Clifford, analysis to characterize the dual space of the space of analytic functions on the unit sphere in R^{n+1} . Our characterization is given by an infinite set of vector spaces of functions defined on the complement of the sphere. Each space comprises of solutions to a fixed order iterate of the Euclidean Dirac operator. The first of these representation spaces has previously been obtained by Sommen [13].

We then examine the Cauchy-Kowalewski extensions of the analytic functions on the unit sphere in more detail. We use Huygens principle to show that these extensions are determined by the holomorphic extensions of these functions within the complex sphere. This enables us to give a characterization of the space of analytic functionals over the unit sphere, as the completion of a union of subspaces of holomorphic functionals acting over a system of domains of holomorphy lying the complex sphere, and containing the sphere. These domains of holomorphy correspond to ones described by Morimoto in [8].

CLIFFORD ANALYSIS REVISITED

In this section we give, for completeness, a reintroduction to some basic results from Clifford analysis that we require here.

Suppose that $h:U_2 \to \mathbb{C}$ is a harmonic function, where U_2 is a domain in $\mathbb{C}$. Then, trivially, we have

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}h(z)(=\Delta_2 h(z))=0\,,$$

and the operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ couple to give a linear decomposition of the two dimensional Laplacian Δ_2. We could like a similar decomposition for the Laplacian Δ_{n+1} over R^{n+1}, where $n > 1$. In this case we require first order, homogeneous operators ∂_x and $\bar{\partial}_x$ such that $\partial_z \bar{\partial}_z = \Delta_{n+1}$. This requirement is essentially the same as asking for R^{n+1} to be included in some algebra A_n such that for each $x \in R^{n+1}$ we have $x\,\bar{x} = |\,x\,|^2$, where $\bar{x}$ is also in R^{n+1}. Placing $x = x_o e_o + x_1 e_1 + \cdots + x_n e_n$ and $\bar{x} = x_o e_o - x_1 e_1 - \cdots - x_n e_n$ we arrive at the anticommutation relationship

$$e_i e_j + e_j e_i = -2\delta_{ij} e_o$$

for $1 \le i,j \le n$. This anticommutation relationship over the elements $e_1, \cdots, e_n$ gives the necessary information for generating the 2^n dimensional, real, Clifford algebra A_n with basis elements $e_o, e_1, \cdots, e_n, e_1 e_2, \cdots, e_{n-1}e_n, \cdots e_{j_1} \cdots j_r \cdots, e_{1\cdots n}$, where $j_{r_1} < \cdots < j_r$, and $1 \le r \le n$. It should be noted that other algebras, including non-associative algebras like the octonion algebra O *or* $O \oplus O$, also have elements which satisfy (1). However, we shall stick to using the algebra A_n. In this context the generalized Cauchy-Riemann operator is $\frac{\partial}{\partial x_o} + \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}(=\partial_x)$.

DEFINITION: *A function* $f : U_{n+1} \to A_n$, *with* U_{n+1} *a domain in* R^{n+1}, *is called a left regular function (or monogenic function) if* $\partial_x f(x) = 0$ for *all* $x \in U$.

A similar definition may be given for right regular functions.

The study of left (or right) regular functions has been developed by a number of authors (eg [2,4,6,7,9,11,15]) and it is referred to as Clifford analysis.

Examples (A) *Suppose that* $h : U_{n+1} \to R$ *is a harmonic function. Then*

$$\overline{grad h(x)} = \partial_x h(z) = \frac{\partial}{\partial x_o} h(x) - \sum_{j=1}^{n} e_j \frac{\partial h(x)}{\partial x_j}$$

is both a left and right regular function. As a particular example we have $h(x) = H(x) = (n-1)^{-1} |x|^{-n+1}$, then $\overline{grad\, H(x)} = G_1(x) = \overline{x} |x|^{-n-1}$.

(Note that as each element $x \in R^{n+1} \setminus \{0\}$ has an inverse $x^{-1} (= \overline{x} |x|^{-2})$, $\in R^{n+1} \setminus \{0\}$, then $G_1(x) = x^{-1} |x|^{-n+1}$).

(B) Suppose that U_n is a domain lying in R^n, the subspace of R^{n+1} spanned by the vectors $e_1, \cdots, e_n$. Suppose also that $g : U_n \to A_n$ is a real analytic function (i.e. about each point in U the function g has an infinite Taylor expansion which converges uniformly to g). Then [2,6,7] there exists a domain $U_{n,g} \subseteq R^{n+1}$ which contain U_n, and the function

$$f_{1,g} : U_{n,g} \to A_n : f_{1,g}(x) = (e^{-x_o \vec{\partial}_{\vec{x}}}) g(\vec{x})$$

is a well defined left regular function, where $\vec{x} = x_1 e_1 + \cdots + x_n e_n$ and $\vec{\partial}_{\vec{x}} = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}$.

The function $f_g(x)$ is called the Cauchy-Kowalewski extension of g with respect to the Euclidean Dirac operator ∂_x. It has previously been shown [2] that this extension is unique.

It is important to note that if g_1 and g_2 are real analytic functions defined on U_n then it is not generally the case that $U_{n,g_1} = U_{n,g_1}$. For example consider

$g_1(x) = 1+|x|^2$ and $g_2(x) = g_1(x)^{-1}$.

We also have the following generalizations of Cauchy's theorem and Cauchy's integral formula:

THEOREM *Suppose that* $f : U_{n+1} \to A_n$ *is a left regular function, and* $M \subseteq U$ *is a compact , (n+1)- dimensional manifold with* $x_o \in \overset{o}{M}$. *Then*

$$\int_{\partial M} Dx\, f(x) = 0$$

and

$$f(x_o) = \frac{1}{\omega_n} \int_{\partial M} G_1(x-x_o)Dx\, f(x) \ ,$$

where $Dx = \sum_{i=0}^{n}(-1)^i\, e_i d\hat{x}_i$, and ω_n is the surface area of the unit sphere in R^{n+1}. ■

Using formula (2) we may deduce the conformal invariance of the solution spaces to the generalized Cauchy Riemann operator introduced here. First we note that for each vector $x_1 \in S^n$ we have from (1) that $x_1 R^{n+1} x_1 = R^{n+1}$ and $|x_1 y x_1| = |y|$ for each $y \in R^{n+1}$. By induction we have that for each $x_1, \cdots, x_p \in S^n$ then $a\ R^{n+1}\tilde{a} = R^{n+1}$ and $|a\ y\ \tilde{a}| = |y|$ for each $y \in R^{n+1}$, where $a = x_p \cdots x_1$ and $\tilde{a} = x_1 \cdots x^p$. The group $Spin(n) = \{x \in A_n : x = x_1 \cdots x_p$ with $x_1, \cdots, x_p \in S^n$ and p { are non fixed positive integer} is a double covering group of the special orthogonal group $SO(n+1)$, and it is isomorphic to the group $Spin(n+1)$ described in [10] and elsewhere.

Suppose now that the function f is left regular with respect to the variable $a\ x\ \tilde{a}$ then from (2) we have that the differential form $Da\ x\ \tilde{a}\ f(a\ x\ \tilde{a})$ is closed. This gives us that the form $a\ Dx\ \tilde{a}\ f(a\ x\ \tilde{a})$ is closed. Consequently the function $\tilde{a}\ f(a\ x\ \tilde{a})$ is left regular with respect to the variable x. Similarly it may be deduced that if f is left regular with respect to the variable x^{-1} then $G(x)f(x^{-1})$ is left regular with respect to the variable x. As ∂_x is a constant coefficient differential operator then its solution spaces remain invariant under translation and dilation. Consequently, the Euclidean Dirac operator's solution spaces remain invariant under actions of the Mobius group in $R^{n+1} \cup \{\infty\}$. In [1] Ahlfors shows that any Mobius transform in R^{n+1} may be written in the form $(ax+b)(cx+d)^{-1})$ where $a = a_1 \cdots a_{p_1}, b = b_1 \cdots b_{p_2}, c = c_1 \cdots c_{p_2}, d = d_1 \cdots d_{p_4}$, with $a_{j_1}, b_{j_1}, c_{j_3}$ and $d_{j_4} \in R^{n+1}$ and $1 \le j_k \le p_k, k \in \{1,2,3,4\}$, and $a\tilde{c}, \tilde{c}d, d\tilde{b}, \tilde{b}a \in R^{n+1}$ and $a\tilde{d} - b\hat{c} \in R \setminus \{0\}$. The set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying the previous conditions form a group $AV(R^{n+1})$ and the quotient group $AV(R^{n+1})/R^+$ is a four fold covering group of the group of Mobius transforms in $R^{n+1} \cup \{\infty\}$ when n is odd, and it is a two fold covering otherwise. In [11] we show that:

THEOREM *Suppose that* $f((ax+b)(cx+d)^{-1})$ *is a left regular function with respect to the variable* $((ax+b)(cx+d)^{-1})$, *then the function* $J_1(cx+d)f((ax+b)(cx+d)^{-1})$ *is left regular with respect to the variable* x, *where* $J_1(cx+d) = (c\tilde{x}+d)|cx+d|^{n+1}$. ■

This result is in complete analogy with the quaternionic result deduced by Sudbery in [15], and for the Minkowskian Dirac operator (see for example [5]).

As $(cn+d)\overline{(cn+d)} \in R^+$, where $\overline{e_{j_1} \cdots e_{j_+}} = (-1)^+ e_{j_+} \cdots e_{j_1}$, then, when defined, $J_1(cx+d)$ is invertible in A_n. Using the Cayley map $(x-e_o)(x+e_o)^{-1}$, and $(x+e_o)(x-e_o)^{-1}$, it now follows [11] that each analytic function $g : S^n \to A_n$ has a Cauchy-Kowalewski extension to a left regular function $f_g : U_g \to A_n$, where U_g is some neighborhood of S^n. This extension has previously been deduced by Sommen [13] using different techniques.

We denote the space of A_n valued analytic functions on the sphere by

$A(S^n, A_n)$ and we denote the space of functionals $T, Q : A(S^n, A_n) \to A_n$ such that (i) $T(g(x_o)+h(x_o)a) = T(g(x_o)) + T(h(x_o))a$ where $g(x_o)h(x_o) \in A(S^n, A_n)$ and $a \in A_n$.

$$(ii) \quad |T(g(x_o))| \leq C_T \sup_{x \in S^n} |g(x_o)| \quad \text{for some} \quad C_T \in R^+$$

$$(iii) \quad (bT+Q)(g(x_o)) = b(T)(g(x_o)) + Q(g(x_o)) \ , \quad \text{where} \quad b \in A_n \ ,$$

by $A'(S^n, A_n)$. This space is the dual space of $A(S^n, A_n)$.

THEOREM *Suppose that* $T \in A'(S^n, A_n)$ and $g \in A(S^n, A_n)$. *Then*

$$T(g(x_o)) = T\left[\frac{1}{\omega_n} \int_{S_{R_1(g)}} - \int_{S_{R_2(g)}} G_1(x-x_o)Dx\, f_g(x)\right] = \frac{1}{\omega_o} \int_{S_{R_1(g)}} T(G_1(x-x_o))Dx\, f_g(x)$$

$$- \frac{1}{\omega_n} \int_{S_{R_2(g)}} T(G_1(x-x_o))Dx f_g(x) \ ,$$

where $f_g(x)$ *is the Cauchy Kowalewski extension of* g, $S_{R_1(g)}$ *is the sphere of radius* $R_1(g) > 1$ and $S_{R_2(g)}$ *is the sphere of radius* $R_2 < 1$. *Moreover*, $S_{R_1(g)}, S_{R_2(g)} \subseteq U_g$.

COROLLARY $TG_1(x-x_o)$ *is a right regular function defined on* $R^{n+1} \setminus S^n$. *Moreover*, $\lim_{x \to \infty} TG_1(x-x_o) = 0$ and $\lim_{x \to \infty} |x|^k TG_1(x-x_o) = 0$

for $k \in (0,n)$. ■

We denote the left A_n module of right regular functions f defined on $U_{outer} = \{x \in R^{n+1} : |x| > 1\}$, and such that $\lim_{x \to \infty} |x|^k f(x) = 0$ for $k \in [0,n)$, by $M_{1outer}(A_n)$. We denote the left A_n module for right regular functions f defined on the interior of the unit disc by $M_{1inner}(A_n)$.

THEOREM *The left module* $A'(S_n, A_n)$ *is isomorphic to the left module* $M_{1outer}(A_n) \oplus M_{1inner}(A_n)$. ■

Using the Clifford matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in AV(R^{n+1})$ we have, on endowing $M_{1outer}(A_n)$ and $M_{1inner}(A_n)$ with the Frechet topology obtained by taking supremums over nested sequences of compact subsets of the open disc, and the complement of its closure on R^{n+1}.

THEOREM *The left* A_n *– Fréchet modules* $M_{1outer}(A_n)$ and $M_{1inner}(A_n)$ *are topologically isomorphic.* ■

THE MODULE $A'(S^n, A_n)$: We started by considering harmonic functions. We shall now briefly return to this theme. All of the results described in the previous section generalize to this setting. First suppose that $g : U_n \rightarrow A_n$ is an analytic function. Then the function

$$f_{2,g} : U^*_{n,g} \rightarrow A_n : f_{2,g}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x_o^{2n} \Delta^n_n g(\vec{x})$$

is well defined on some domain $U^*_{n,g}$, of R^{n+1}, containing U_n, and it is a harmonic function. The function $f_{2,g}$ is a Cauchy Kowalewski extension of g with respect to the Laplacian in R^{n+1}. However, it is *not* the only Cauchy Kowalewski extension of g with respect to the Laplacian in R^{n+1}. The function $\frac{1}{2} f_{1,g} + \frac{1}{2} f_{2,g}$ defined on $U_{n,g} \cap U^*_{n,g}$ is also a Cauchy-Kowalewski extension of g with respect to the Laplacian on R^{n+1}.

Also, if the function h is harmonic with respect to the variable $(ax+b)(cx+d)^{-1}$ then the function $J_2((cx+d)h((ax+b)(cx+d)^{-1})$ is harmonic with respect to the variable n, where $J_2(cx+d) = |cx+d|^{-n+1}$. Using the Clifford matrices $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and

$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ we now have that each $g(x) \in A(S^n, A_n)$ has infinitely many Cauchy Kowalewski extensions to harmonic functions $h^* : U_g^* \to A_n$ where U_g^* is a domain in R^{n+1}.

Following [12] we now have:

THEOREM *(i) Suppose that* $f : U_{n+1} \to A_n$ *is a left regular function. Then* $|x|^{2p} f(x)$ *is annihilated by the operator* $\Delta_{n+1}^p \partial_x$ *where* $p \in N$.

(ii) Suppose that $h : U_{n+1} \to A_n$ *is a harmonic function. Then* $|x|^{2p} h(x)$ *is annihilated by the operator* Δ_{n+1}^p. ■

As, by definition, $|x| = 1$ on the unit sphere we now have:

COROLLARY *Suppose that* $g(x) \in A(S^n, A_n)$ *then there is a domain* $\hat{U}_g$ in R^{n+1} *which* g *has a Cauchy Kowalewski extension* $f_{q,g}$, where $q \in \{1, \cdots, p, \cdots\}$, and $f_{q,g}$ *is annihilated by the operator* $\Delta_{n+1}^{q/2}$ if q *is even*, and *by* $\Delta_{n+1}^{[q/2]} \partial_n$ *otherwise*. ■

We also have [12] the Cauchy integral formulae:

THEOREM *Suppose that* $f : U_{n+1} \to A_n$ *satisfies the equation* $\Delta_{n+1}^q \partial_n f(x) = 0$, *then for each compact, real,* $(n+1)-$ *dimensional manifold* M *lying in* U, and *each* $x_o \in \overset{o}{M}$ *we have*

$$f(x_o) = \frac{1}{\omega_n} \int_{\partial M} \sum_{k=0}^{\infty} A_{2k+1} G_{2k+1}(x-x_o) D_n \Delta_{n+1}^q f(x) \lambda_1(k,q) + A_{2k+2} G_{2K+2}(x-x_o) \overline{D}x \Delta_{n+1}^k \partial_x f(x) \lambda_2(k,q),$$

where $A_1 = 1$, A_{2k+1}, $A_{2k+2} \in R$, $\partial_x A_{2k+1} G_{2k+1}(x) = A_{2k} G_{2k}(x)$,

$\overline{\partial}_x A_{2k+2} G_{2K+2}(x) = A_{2k+1} G_{2k+1}(x)$, $\lambda_1(k,q) = 1$ if $k \le q$ and *is equal to zero* *otherwise, and* $\lambda_2(k,q) = 1$ if $k \le q-1$ *and is equal to zero otherwise.*

If $f(x)$ *satisfies the equation* $\Delta_{n+1}^q f(x) = 0$, *then*

$$f(x_o)=\frac{1}{\omega_o}\int_{\partial M}\sum_{k=0}^{\infty} A_{2k+1}G_{2k+1}(x-x_o)Dx\,\Delta_{n+1}^k f(x)\mu_1(k,q) + A_{2k+2}G_{2K+2}(x-x_o)\overline{D}x\,\Delta_{n+1}^k\partial_x f(x)\mu_1(k,q),$$

where $\mu_1(k,q) = 1$ for $k \le q-2$ *and is equal to zero otherwise.* ■

Explicitly we have in the above integral formulae that

(i) when n is even $G_j(x) = G_1(x)(x\overline{x})^{[j-2]}x^{\frac{[1]}{2}-[\frac{j-1}{2}]}$

(ii) when n is odd $G_j(x) = G_1(x)(x-\overline{x})\,[\frac{j-1}{2}]x^{[\frac{j}{2}]}$ for $j \in \{1, \cdots, n\}$

and $G_j(x) = p(n)(\theta_j + \log|x|)$ for $j \ge n+h$ where $\theta_j \in R$ and

$p_j(x) = (x\overline{x})^{[\frac{j-n}{2}]}x^{\left[\frac{j-n+1}{2}-\frac{j-n}{2}\right]}$.

Combining this theorem with the previous corollary and evaluating the Cauchy integral over $S_{R_1(g)} \cup S_{R_2(g)}$ we obtain:

THEOREM *Suppose that* $T \in A'(S^n, A_n)$ *and* $g \in A(S^n, A_n)$. *Then*

$$T(q,(x_o))=\frac{1}{\omega_n}\int_{S_{R_1(g)}}\sum_{k=0}^{\infty} A_{2k+1}T(G_{2k+1}(x-x_o))Dx\,\Delta_{n+1}^k f_{q,g}(x)\lambda^{\lambda_1(k,q)} + A_{2k+2}T(G_{2K+2}(x-x_o))\overline{D}x\,\Delta_{n+1}^k\partial_J f_{q,g}(x)\lambda_2(k,q)$$

$$-\frac{1}{\omega_n}\int_{S_{R_2(g)}}\sum_{k=0}^{\infty} A_{2k+1}T(G_{2k+1}(x-x_o))Dx\,\Delta_{n+1}^k f_{q,g}(x)\lambda_1(k,q) + A_{2k+2}T(G_{2K+2}(x-x_o))\overline{D}x\,\Delta_{n+1}^k\partial_n f_{q,g}(x)\lambda_2(k,q)$$

if q *is odd, and it is equal to*

$$-\frac{1}{\omega_n}\int_{S_{R_1(\varepsilon)}}\sum_{k=0}^{\infty}A_{2k+1}T(G_{2k+1}(x-x_o))Dx\,\Delta_{n+1}^k f_{q,g}(x)\mu_1(k,q)$$
$$+A_{2k+2}T(G_{2K+2}(x-x_o))\bar{D}x\,\Delta_{n+1}^k\partial_n f_{q,g}(x)\mu_1(k,q)$$

$$-\frac{1}{\omega_n}\int_{S_{R_2(\varepsilon)}}\sum_{k=0}^{\infty}A_{2k+1}T(G_{2k+1}(x-x_o))Dx\,\Delta_{n+1}^k f_{q,g}(x)\mu_1(k,q)$$
$$+A_{2k+2}T(G_{2K+2}(x-x_o))\bar{D}x\,\Delta_{n+1}^k\partial_n f_{q,g}(x)\mu_1(k,q)$$

otherwise. ■

COROLLARY *$TG_q(x-x_o)$ is a well defined analytic function on* $R^{n+1}\setminus S^n$ and *when $q=2p$ we have that*

$\Delta^p TG_q(x-x_o)=0$, and *when $q=\alpha p+1$ we have that* $\Delta_p\partial_n TG_q(x-x_o)=0$.

Moreover, when n is even $\lim_{x\to\infty}|x|^{k_q}TG_q(x-x_o)=0$ for $q\in\{1,\cdots,n\}$

and $k_q\in[0,n+1-q)$ and $\lim_{x\to\infty}|x|^{k_q}TG_q(x-x_o)=0$ for $q>n$ and $k_q\in(q-n-1,+\infty)$.

When? n is odd $\lim_{x\to\infty}|x|^{k_q}G_q(x-x_o)=0$ for $q\in\{1,..,n\}$ and $k_q\in[0,n+1-q)$ ■.

When n is even we denote the left A_n module of functions $f(x)$ defined on U_{outer}, satisfying $\Delta_{n+1}^l\partial x f(x)=0$, and with $\lim_{x\to\infty}|x|^{k_2}f(x)=0$ for $q\in\{1,\cdots,n\}$ and $k_q\in[0,n+1-q)$, or with $\lim_{x\to\infty}|x|^{-k_q}f(x)=0$ for $q>n$ and $k_q\in(q-n-1,+\infty)$, by $M_{2l+1,outer}(A_n)$, where $q=2l+1$. From the above corollary a similar definition may be given for $M_{2l,outer}(A_n)$.

We denote the left A_n module of functions $f(x)$ defined on the interior of the unit disc, and satisfying $\Delta_{n+1}^l\partial_x f(x)=0$ by $M_{2l+1,inner}(A_n)$, and the left A_n module of functions defined on the interior of the unit disc, and satisfying $\Delta_{n+1}^l f(x)=0$ by

$M_{2l,inner}(A_n)$.

We now have:

THEOREM *When n is even the left A_n* module $M_{k,outer}(A_n) \oplus M_{k,inner}(A_n)$ *projects onto the left A_n module* $A'(S_n, A_n)$ for $k=1,2,\cdots$ ■

From [12] we have that if $f(x^{-1})$ satisfies the equation $\Delta_{n+1}^{l} f(x^{-1})=0$ with respect to the variable then $x(x\bar{x})^{l-1}G_1(x)f(x)^{-1}$ is annihilated by the operator Δ_{n+1}^{l}, with respect to the variable x, and if $f(x^{-1})$ satisfies the equation $\Delta_{n+1}^{l}\partial_{x^{-1}} f(x^{-1})=0$ then $\Delta_{n+1}^{l}\partial_x (x\bar{x})^{l} G_1(x)f(x^{-1})=0$.

From this and the previous theorem we have:

THEOREM *When n is even the left A_n module $M_{k,outer}(A_n)$ is conformally isomorphic to the left A_n module $M_{k,inner}(A_n)$*, for $k=1,2\cdots$. ■

By taking the supremum of each $f(x)\in M_{k,outer}(A_n)$ over a suitable nested sequence of compact sets, on $U_{outer}\cup\{\infty\}$ we may deduce:

THEOREM *When n is even the left A_n Fréchet module $M_{k,outer}(A_n)$ is topologically isomorphic to the left A_n Fréchet module $M_{k,inner}(A_n)$ via the conformal isomorphism, for $k\in\{1,\cdots,n\}$.* ■

By taking the supremum of $|x|^{-q}f(x)$, for $f(x)\in M_{q,outer}(A_n)$, over the same sequence of compact sets, and taking the supremum of $g(x)\in M_{q,inner}(A_n)$ we have induced Fréchet topologies on these modules for $q\geq n+1$.

THEOREM *When n is even and $q\geq n+1$ the left A_n Fréchet module $M_{k,outer}(A_n)$*

is topologically isomorphic to the left A_n Fréchet module $M_{k,inner}(A_n)$ via the conformal isomorphism.

When n is odd we denote the left A_n module of functions, $f(x)$, defined on U_{outer}, satisfying $\Delta_{n+1}^{l}\partial_n f(x)=0$, with $\lim_{x\to\infty} |x|^{k_q} f(x)=0$ for $q\in\{1,\cdots,n\}$ and $k_q\in[0,n+1-q)$, by $M_{2l+1,outer}(A_n)$, where $2l+1=q$. A similar definition may be given for $M_{2l,outer}(A_n)$, with $2l\in\{1,\cdots,n\}$.

From the previous arguments we now have:

THEOREM *Suppose that n is odd and that the $k\in\{1,\cdots,n\}$ then the left A_n module $M_{k,outer}(A_n)\oplus M_{k,inner}(A_n)$ projects onto the left A_n module $A'(S^n,A_n)$. Moreover, $M_{k,outer}(A_n)$ is conformally isomorphic to $M_{k,inner}(A_n)$ and this isomorphism is a topological isomorphism with respect to the Fréchet topologies.* ∎

When n and k are odd and $k\geq n+1$ we denote the left A_n module of functions, $f(x)$, defined on U_{outer} satisfying $\Delta_{n+1}^{[k/2]}\partial_x f(x)=0$ and with $\Delta_{n+1}^{\frac{k-n}{2}} f(x)\in M_{k,outer}(A_n)$, by $M_{k,outer}(A_n)$. A similar definition may be given for $M_{k,outer}(A_n)$ when n is odd and k is even, with $k\geq n+1$.

When n is odd and $k\geq n+1$ it is no longer the case that $M_{k,inner}(A_n)$ is conformally isomorphic to $M_{k,outer}(A_n)$. For example, the function $\log|x-1|\in M_{n+1,outer}(A_n)$. Moreover, as M_{inner} is contractable to a point the holomorphic continuations of each member of $M_{k,inner}(A_n)$ to the L i.e. ball lying in C^{n+1} are unique. Consequently, the holomorphic continuations of the conformal image of each element of $M_{k,inner}(A_n)$ under inversions, is uniquely defined in a domain in C^{n+1}. However, log $|x-1|$ does not have a unique holomorphic continuation to this domain.

However, it is straightforward to see: that

THEOREM *Suppose that* n *is odd, then the left* A_n *module* $M_{k,outer}(A_n) \oplus M_{k,inner}(A_n)$ *projects onto the left* A_n module $A'(S^n, A_n)$. ■

Putting together the previous results we obtain the following infinite sequence of A_n left modules which project onto $A'(S^n, A_n)$:

$$\begin{array}{ccc} \cdots \xrightarrow{\partial_z} M_{2l+2,outer}(A_n) \oplus M_{2l+1,inner}(A_n) & \xrightarrow{\overline{\partial}_z} M_{2l,outer}(A_n) \oplus M_{2l,inner}(A_n) \xrightarrow[\partial_n]{} & \cdots \\ \downarrow & \downarrow & \\ A'(S^n, A_n) & A'(S^n, A_n)\,. & \end{array}$$

In this last part we replace the A_n modules by $A_n(C)$ modules, where $A_n(C)$ is the complex Clifford algebra over $\mathbb{C}^n$.

By replacing the Cayley transform $(1-x)(x+1)^{-1}$ by $(1-z)(z+1)^{-1}$ where $z \in \mathbb{C}^{n+1} \setminus \{z \in \mathbb{C}^{n+1} : (z+1)(\overline{z}+1) = 0\}$, and by replacing the transform $(x+1)(x-1)^{-1}$ by $(z+1)(z-1)^{-1}$ we may now easily extend our previous arguments to obtain the following generalization of a result appearing in [11]:

THEOREM *For* U' *a domain lying in the complex* $n-$*dimensional manifold* $S^n_{\mathbb{C}} = \{z : z\overline{z} = 1\}$, and $f : U' \rightarrow A_n(\mathbb{C})$ *a holomorphic function there exists a domain* $U_f' \subseteq \mathbb{C}^{n+1}$ *with* $U' \subseteq U_f'$ *and Cauchy-Kowalewski extensions* $f_k : U_f' \rightarrow A_n(\mathbb{C})$ *such that* $\Delta^l_{\mathbb{C},n+1}\partial_z f_{2l+1}(z) = 0$ and

$$\Delta^l_{\mathbb{C}} f_{2l}(z) = 0 \;, \quad \textit{where} \quad \Delta_{\mathbb{C},n+1} = \sum_{j=0}^{n} \frac{\partial^2}{\partial z_j{}^2} \quad \text{and} \quad \partial_{\overline{z}} = \sum_{j=0}^{n} c_j \frac{\partial}{\overline{\partial z_j}} \,. \qquad ■$$

If we now take an element $a \in A(S^n, A_n)$ then this element may be extended to some holomorphic function $\hat{a}$ defined over a domain of holomorphy U_a lying in the complex sphere $S^n_{\mathbb{C}}$.

Suppose now that $x \in R^{n+1} \setminus \{0\}$ then we denote the real, n-dimensional space orthogonal to x by $R^n_{x,1}$, and we denote the hyperbolic sphere $\{y \in \lambda x + iR^n_{x,1} : y\bar{y}=1$ and $\lambda \in R\}$ by S^n_x. For $x \in S^n$ and $\lambda \in R \setminus \{-1,0,1\}$ the set $N(\lambda x) \cap S^n_x$ is the $(n-1)$- dimensional sphere $\left\{\frac{(1+\lambda^2)}{2\lambda}x + i\frac{(1-\lambda^2)}{2\lambda}\omega : x\bar{\omega}+\omega\bar{x}=0 \text{ and } \omega \in S^n\right\}$.

where $N(\lambda x) = \{z \in \mathbb{C}^{n+1} : (z-\lambda x)\overline{(z-\lambda x)} = 0\}$.

Using the Huygens principle described for the wave equation by Garabedian in [3, Chapter 6] and for the Dirac equation in by Soucek in [14] we now have from the previous theorem:

THEOREM *Suppose that n is odd and $a \in A(S^n, A_n)$ then $\lambda x \in U'_{\hat{a}}$ for $x \in S^n$ and $\lambda \in R \setminus \{-1,0,1\}$ if* $\left\{\frac{(1+\lambda^2)}{2\lambda}x + i\frac{(1-\lambda^2)}{2\lambda}\omega : x\bar{\omega}+\omega\bar{x}=0 \text{ and } \omega \in S^n\right\} \subseteq U_a$. ■

Using the integral formulae described for the wave equation in odd dimensions in [3, Chapter 6] we also have:

THEOREM *Suppose that n is even and $a \in A(S^n, A_n)$. Suppose also that for $n \in S^n$ and $\lambda \in \{-1,0,1\}$ we have that*

$\left\{\frac{(1+\lambda^2)}{2\lambda}x + i\frac{(1-\lambda^2)}{2\lambda}\omega : x\bar{\omega}+\omega\bar{x}=0 \text{ and } w \in S^n\right\} \leq U_a$ *and this set is the boundary of a real n dimensional manifold lying in $S^n_{\mathbb{C}}$. Then $\lambda \in U'_{\hat{a}}$.* ■

Suppose now that U' is a domain of holomorphy lying in the complex sphere and $O(U', A_n(\mathbb{C}))$ denotes the right module of $A_n(\mathbb{C})$ valued holomorphic functions defined over U'. Then $O'(U', A_n(\mathbb{C}))$ denotes its right $A_n(\mathbb{C})$ dual module.

From the previous results we now have:

THEOREM *Suppose that* $T \in A'(S^n, A_n)$ and $S^n \subseteq \Omega$. *Suppose that* $(TG_1(x-x_o)) \in M_{1,outer}(A_n)$ and $TG_1(x-x_o) \in M_{1,inner}(A_n)$ *have holomorphic extensions to* U'. *Then* for *each* $f \in O(U', A_n(\mathbb{C}))$ *we have*

$$T(f) = \frac{1}{\omega_n} \int_{\theta(S_{R_1(\theta)})} TG_1(z-z_o) Dz\, f_1(z) - \frac{1}{\omega_n} \int_{\theta(S_{R_2(\theta)})} TG_1(z-z_o) Dz\, f_1(z),$$

where $\theta \in O(C^{n+1}) = \{(a_{ij}) : 1 \leq 1, j \leq n+1,\ a_{ij} \in C$ and $\{(a_{ij})(\overline{a}_{ij})^T = (a_{ij})\}$, $R_1(\theta) \in (1, +\infty)$, $R_2(\theta) \in (0,1)$, $Dz = \sum_{j=0}^{n} e_j (-1)^j d\hat{z}_j$ and $\theta(S_{R_{1(\theta)}}), \in (S_{R_{2(\theta)}}) \subseteq U_g'$.

The set of all elements in $A'(S^n, A_n)$ which satisfy the conditions given in the previous theorem is a proper submodule of $A'(S^n, A_n)$. We denote this submodule by $A'_{U'}(S^n, A_n)$.

Suppose that $T \in A'(S^n, A_n)$ and either $T(G_1(x-x_o)) \in M_{1,outer}(A_n)$ or $T(G_1(x-x_o)) \in M_{1,inner}(A_n)$ cannot be analytically continued onto S^n. The set of all such functionals is a proper submodule of $A'(S^n, A_n)$, and is denoted by $A'_{singular}(S^n, A_n)$.

Trivially we have the following characterization of $A'(S^n, A_n)$:

PROPOSITION *Suppose that* $\{U'_m\}$ *is a system of domains of holomorphy lying in* S^n_C *with* $\bigcup_Z U'_m = S^n_C$ *and* $\bigcap_Z U'_m = S^n$, *and* $d(U'_{m_1}) \Subset U'_{m_2}$ *for* $m_1 < m_2$. *Then* $A'(S^n, A_n) = \bigcup_{m \in Z} A'_{u_m}(S^n, A_n) \cup A'_{singular}(S^n, A_n)$.

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