SPECTRAL SYNTHESIS OF ORBITS OF COMPACT GROUPS

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This talk is a contribution to the spectral synthesis in L^1 convolution algebras of noncommutative groups. Let us begin by recalling some ideas and results from the much better understood case of commutative groups. One way to consider spectral synthesis is as the attempt to classify the closed ideals in $L^1(G)$ for a locally compact abelian group G (or even in more general commutative Banach algebras).

With each closed ideal I in $L^1(G)$ there is associated a closed subset of the structure space $L^1(G)^\wedge \cong \hat{G}$, namely the hull $h(I) := \{\chi \in \hat{G} | \operatorname{Kern}_{L^1(G)} \chi \supseteq I\}$. This is clearly an invariant of the ideal. On the other hand, to each closed subset A of $\hat{G}$ one may form the kernel $k(A) := \bigcap_{\chi \in A} \operatorname{Kern}_{L^1(G)} \chi = \{f \in L^1(G) ; \hat{f} = 0 \text{ on } A\}$ where $\hat{f}$ denotes the Fourier transform of f. It is easy to see that $k(A)$ is the largest ideal I in $L^1(G)$ with $h(I) = A$. There is also a less obvious way to associate an ideal with A, namely

$$j(A) := \{f \in L^1(G) | \operatorname{supp}(\hat{f}) \text{ is a compact subset of } \hat{G} \setminus A\}^- .$$

It turns out that $j(A)$ is the smallest closed ideal I with $h(I) = A$. The classification problem reduces to: Describe (the ideal structure of) the algebra $k(A)/j(A)$ for closed subsets A of $\hat{G}$. The best possible situation is, of course, that $k(A)/j(A)$ is zero. In this case A is called a <u>set of synthesis</u> or a <u>Wiener set</u>. The next better situation is that

$k(A)/j(A)$ is of finite degree of nilpotency.

The following is a sample of "classical results":

i) The empty set is a set of synthesis.

This is one way to formulate Wiener's approximation theorem, proved by him is the case $G = \mathbb{R}$.

ii) All sets A with "scattered" boundary are sets of synthesis.

iii) The 1-sphere S^1 in $\mathbb{R}^2$ is a set of synthesis (C. Herz).

iv) The 2-sphere S^2 in $\mathbb{R}^3$ is not a set of synthesis (L. Schwartz).

N. Varopoulos showed, more generally, that the n-sphere S^n in R^{n+1} has the property that $k(S^n)/j(S^n)$ is of finite degree of nilpotency, and the degree is $[\frac{n}{2}] + 1$.

v) For each non compact G the dual $\hat{G}$ contains at least one closed subset which is not a set of synthesis (P. Malliavin).

Later, Y. Domar and other Swedes as well as D. Müller from Bielefeld considered various submanifolds of $\mathbb{R}^n$.

The investigations of spectral synthesis in the case of nilpotent Lie groups were started by my colleagues H. Leptin and J. Ludwig. The "hull h" (of a closed two sided ideal) and the "kernel k" can be defined for the convolution algebra $L^1(N)$ of a noncommutative group N as soon as one agrees what the structure space of $L^1(N)$ should be. In the case of a simply connected nilpotent Lie group (and this is the only case in which we will present some results) there is only one reasonable candidate: The space Max $L^1(N)$ of maximal closed two sided ideals coincides with the space Priv $L^1(N)$ of primitive ideals, and Max $L^1(N)$ is in bijective correspondence to the unitary dual $\hat{N}$ via the map $\hat{N} \ni [\pi] \to \ker_{L^1(N)} \pi \in \mathrm{Max}\, L^1(N)$.

For a closed two sided ideal I in $L^1(N)$ one defines

$$h(I) := \{[\pi] \in \hat{N} \mid I \subset \ker_{L^1(N)} \pi\},$$

and for a closed subset A of $\hat{N}$ one defines

$$k(A) := \bigcap_{[\pi] \in A} \ker_{L^1(N)} \pi \quad .$$

The first result is due to Leptin, [4], who showed that $L^1(N)$ is the only closed two sided ideal I such that $h(I)$ is empty. In other words, the empty set is a set of synthesis. While it is trivial that $k(A)$ is the largest ideal I with $h(I) = A$ it is much less obvious that for each closed subset A of $\hat{N}$ the set of closed two sided ideals I with $h(I) = A$ contains a smallest element. The existence was established by Ludwig, [5], using Dixmier's symbolic calculus, see below. This smallest element is again denoted by $j(A)$. Ludwig also proved, [7], that $k(A)/j(A)$ is of finite degree of nilpotency for each one point subset A of $\hat{N}$. Examples showed that the degree can be larger than one. In other words, it is not even true that <u>points</u> are always sets of synthesis; finite degree of nilpotency seems to be the best result one may hope for.

My interest in questions of spectral synthesis comes from attempts to classify the algebraically irreducible representations of $L^1(G)$ for a solvable connected Lie group G or, more general, to classify the so-called topologically completely irreducible representations, TCI for short. One can attack this problem by doing a "Mackey type analysis" using the restriction of the representation to the nilradical N of G. To be more specific, one tries to describe the kernels of such representations in $L^1(N)$. Originally, I planned to apply the following result of Dixmier, [2], which is true, by the way, in much larger generality:

If π *is a* TCI *representation of* G *in* E *then the annihilator* $\mathfrak{p}$ *of the* C^∞ *vectors of* E *in the universal enveloping algebra* $\mathfrak{Un}$ *of* N *is a prime ideal and the hull* $h(\mathfrak{p})$ *of* $\mathfrak{p}$ *in the primitive ideal space* *of* $\mathfrak{U}\mathfrak{n}$ *is the closure of a* Γ*-orbit where* Γ *denotes the (complex) Zariski closure of the adjoint group of* G .

If in addition π is uniformly bounded and hence a representation of $L^1(G)$ then clearly $\mathfrak{p} * \mathcal{D}(N)$ is contained in $\ker_{L^1(N)} \pi$ where $\mathcal{D}(N)$ denotes as usual the compactly supported C^∞ functions on N . I wanted to show that $\mathfrak{p} * \mathcal{D}(N)$ is a "substantial part" of $\ker_{L^1(N)} \pi$ – whatever that means. The weakest useful information in this direction, as far as I can see, would be that $h(\mathfrak{p})$ is the closure of a Γ-orbit through a point in $\hat{N}$. Observe that $\hat{N}$ can be canonically embedded into the primitive ideal space Priv $\mathfrak{U}\mathfrak{n}$ of $\mathfrak{U}\mathfrak{n}$, the image consists of the self-adjoint primitive ideals. I was unable to prove even this weak assertion directly. In full generality, it is still an <u>open problem</u>, see also below. Therefore, I was looking for other methods. I discovered that by some other considerations I could prove what I first wanted namely that for each algebraically irreducible representation π of $L^1(G)$ the annihilator $\ker_{L^1(N)} \pi$ in $L^1(N)$ is the kernel of the closure of a G-orbit in $\hat{N}$ provided I would know the following result:

THEOREM *Let* N *be a simply connected nilpotent Lie group, let* T *be a compact abelian group of automorphisms of* N *, and let* U *be a connected group of unipotent automorphisms (i.e. unipotent as transformations on* $\mathfrak{n}$*) of* N *. Suppose that* T *normalizes* U *. There exists a natural number* m *depending only on* N *with* $\{k(A)/j(A)\}^m = 0$ *for each* $T \ltimes U$*-orbit* A *in* $\hat{N}$.

A detailed proof of the Theorem can be found in [8].

Concerning the above stated "open problem" it is finally <u>true</u> that $h(\mathfrak{p})$ is the closure of a Γ-orbit through a point in $\hat{N}$ in the case of an algebraically irreducible representation. But the proof is anything else but "direct". Moreover, it doesn't work in the case of general uniformly bounded

TCI representations.

In the following I will try to explain the strategy of the proof of the Theorem. This strategy leads to certain properties, later called (I) - (IV), of closed subsets of $\hat{N}$ which can be studied separately. I will formulate these properties and I will give some comments to them. A always denotes a closed subset of $\hat{N}$.

PROPERTY (I) $k(A) \cap \mathcal{D}(N)$ is L^1-dense in $k(A)$.

The main reason to introduce (I) is the following

OBSERVATION For every connected nilpotent Lie group N there exists a number m such that $\{k(A)/j(A)\}^m = 0$ for all closed subsets A of $\hat{N}$ satisfying (I).

Let us recall the basic facts of Dixmier's symbolic calculus, [1], which is crucial for the proof of the observation. There exists a number d depending only on N such that for every compact neighborhood V of the identity in N the Haar measure of V^n is $O(n^d)$ as $n \to \infty$ (the Haar measure is of polynomial growth). Each r-times, $r := d + 4$, continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{C}$ with compact support and with $\varphi(0) = 0$ operates on selfadjoint functions $f = f^*$ in $\mathcal{D}(N)$ in the following sense: There exists a unique element $\varphi\{f\} \in L^1(N)$ with $\tau(\varphi\{f\}) = \varphi(\tau(f))$ for all $\tau \in \hat{N}$ where $\varphi(\cdot)$ is the usual functional calculus in C^* algebras. Moreover, it was shown in [1] that for each $f = f^* \in \mathcal{D}(N)$ there exists a family $\mathcal{F}$ of compactly supported r-times continuously differentiable functions $\varphi : \mathbb{R} \to \mathbb{C}$ vanishing in a neighborhood of 0 such that the r^{th} convolution power f^r can be approximated by $\varphi\{f\}$, $\varphi \in \mathcal{F}$.

Now, suppose in addition that $f = f^* \in \mathcal{D}(N)$ is contained in $k(A)$. The construction of $j(A)$, see [5], shows that $\varphi\{f\}$ is in $j(A)$ for $\varphi \in \mathcal{F}$. It follows that $f^r \in j(A)$ for all $f = f^* \in \mathcal{D}(N) \cap k(A)$. Then a little algebra gives that the same is true for all $f \in \mathcal{D}(N) \cap k(A)$. Since

$\mathcal{D}(N) \cap k(A)$ is dense in $k(A)$ by assumption one gets $f^r \in j(A)$ for all $f \in k(A)$. Finally, a purely algebraic theorem of Nagata-Higman implies that $\{k(A)/j(A)\}^m = 0$ with $m = 2^r - 1 = 2^{d+4} - 1$.

It is very easy to establish property (I) in the case that N is commutative, i.e. isomorphic to $\mathbb{R}^n$, and that A is the orbit of any compact group, say K. Since this fact is not contained in the literature as far as I know I include the short trivial proof although at present I see no way to generalize it to the noncommutative case. The method doesn't work already in the case of $K \ltimes U$-orbits where U is, in the spirit of the Theorem, a connected group consisting of unipotent automorphisms normalized by K. Evidently, the assumption in the Theorem that T is abelian is very unnatural.

To prove the above claim it is sufficient to establish that $k(A)^{(\sigma)} \cap \mathcal{D}(\mathbb{R}^n)$ is dense in $k(A)^{(\sigma)}$ for all $\sigma \in \hat{K}$ where $E^{(\sigma)}$ denotes the σ-isotypic component for each K-space E. We fix σ and choose a concrete realization of σ by matrices, $\sigma(k) = (a_{ij}(k))$, $i,j = 1,\dots,D :=$ $\dim \sigma$. Let $\chi = \chi_\sigma : K \to \mathbb{C}$ be the character of σ, i.e. $\chi(k) = \sum_{i=1}^{D} a_{ii}(k)$, and let $p := D\chi$. For $f \in L^1(\mathbb{R}^n)^{(\sigma)}$ one has

$$\hat{f}(x) = \int_K p(k)\hat{f}(k^{-1}x)\,dk \quad \text{for}$$

all $x \in \hat{N} = (\mathbb{R}^n)^\wedge$. If $A = Kx_o$ this gives in particular

$$\hat{f}(mx_o) = \int_K p(k)\hat{f}(k^{-1}mx_o) \quad \text{for } m \in K .$$

Since $\chi(mk) = \sum_{i,j=1}^{D} a_{ij}(m)a_{ji}(k)$ it follows that

$$\hat{f}(m x_o) = D \sum_{i,j=1}^{D} a_{ij}(m) \int_K a_{ij}(k)\hat{f}(k^{-1} x_o)\, dk \quad .$$

This equation shows that for $f \in L^1(\mathbb{R}^n)^{(\sigma)}$ the finitely many conditions $\int_K a_{ji}(k)\hat{f}(k^{-1} x_o)\, dk = 0$ for $i,j = 1,\dots,D$ are equivalent to $f \in k(A)$. Hence $k(A)^{(\sigma)}$ is of finite codimension in $L^1(\mathbb{R})^{(\sigma)}$. Since $\mathcal{D}(\mathbb{R}^n)^{(\sigma)}$ is dense in $L^1(\mathbb{R}^n)^{(\sigma)}$ one concludes that $k(A)^{(\sigma)} \cap \mathcal{D}(\mathbb{R}^n)$ is dense in $k(A)^{(\sigma)}$.

How can we establish (I) in the case of non-commutative groups? Even in the case of points it is not evident from the definitions that $\ker_{L^1} \pi \cap \mathcal{D}(N)$ contains any non-zero function. Here the original idea to use the universal enveloping algebra enters the scene again. By its help one can construct elements in $k(A) \cap \mathcal{D}(N)$. As I mentioned above $\hat{N}$ can be considered as part of the primitive ideal space Priv $\mathfrak{U}\mathfrak{n}$: If π is an irreducible unitary representation of N in $\mathcal{H}$ then the annihilator $\ker \pi_\infty$ of the associated representation π_∞ of $\mathfrak{U}\mathfrak{n}$ in the space of C^∞-vectors in $\mathcal{H}$ is a primitive ideal. For A in $\hat{N}$ the ideal $k_\infty(A)$ in $\mathfrak{U}\mathfrak{n}$ is defined by $k_\infty(A) = \bigcap_{[\pi]\in A} \ker \pi_\infty$. We consider

PROPERTY (II) $k_\infty(A) * \mathcal{D}(N)$ is L^1-dense in $k(A)$.

Clearly, $k_\infty(A) * \mathcal{D}(N)$ is contained in $k(A) \cap \mathcal{D}(N)$, hence A satisfies (I) if it satisfies (II). One should take note of the fact that (II) is a very strong property. In general the hull $h(k_\infty(A))$ in Priv $\mathfrak{U}\mathfrak{n}$ will be much larger than A, even the hull of $\{k_\infty(A) * \mathcal{D}(N)\}^{-L^1(N)}$ in $\hat{N}$ which is the intersection of $h(k_\infty(A))$ with $\hat{N}$ will often be larger than A. But the equation $A = h(k_\infty(A)) \cap \hat{N}$ is a <u>necessary condition</u> for (II). This equation means that A is an "algebraic subset" of $\hat{N}$, more precisely, via the Kirillov picture A corresponds to a Zariski closed N-invariant subset of $\mathfrak{n}^*$.

To verify (II) in some cases we introduce the properties (III) and (IV) where $\mathcal{S}(N)$ denotes the space of Schwartz functions on N.

PROPERTY (III) $k_\infty(A) * \mathcal{S}(N)$ is $\mathcal{S}$-dense in $k(A) \cap \mathcal{S}(N)$.

PROPERTY (IV) $k(A) \cap \mathcal{S}(N)$ is L^1-dense in $k(A)$.

It is evident that if A satisfies (III) and (IV) then it satisfies (II) and hence (I). To summarize we have reduced (I) to property (IV) which is weaker because there we are dealing with Schwartz functions instead of test functions. But the prize to pay is that in addition we have to verify the "smooth-harmonic-analysis-property" (III).

COMMENTS TO (III)

The case of an abelian N $(=\mathbb{R}^n)$ was studied by one of my students. He proved that (III) is true for all regular algebraic varieties A. If in addition A is compact one even has that $k_\infty(A) * \mathcal{S}(N)$ is equal to $k(A) \cap \mathcal{S}(N)$.

For arbitrary nilpotent Lie groups N the best result I know is that (III) is true for A being an orbit of an abelian compact group, see [8]. The proof goes as usual by induction and is somewhat lengthy and tedious, hence it is not repeated. The commutativity of the transformation group was used at several points. It seems to me that the most crucial use was made in the proof of the following

PROPOSITION *Let* $\mathfrak{n}$ *be a real nilpotent Lie algebra, let* K *be a compact group acting from the left homomorphically and continuously on* $\mathfrak{n}$ *by Lie algebra automorphisms, and let* $\mathfrak{z}$ *be a non-zero* K*-invariant subspace of the center of* $\mathfrak{n}$. *Suppose there is given a self-adjoint maximal ideal* Ω *in the complex universal enveloping algebra* $\mathfrak{U}\mathfrak{n}$. *Let* H *be the stabilizer in* K *of the maximal ideal* $\Lambda := \Omega \cap \mathfrak{U}\mathfrak{z}$ *in* $\mathfrak{U}\mathfrak{z}$ *(of codimension 1).*

Then

$$\bigcap_{h\in H} {}^{h}\Omega = \bigcap_{x\in K} {}^{x}\Omega + \Lambda\mathfrak{U}\mathfrak{n} \quad .$$

In [8], I proved this proposition only for abelian K because I applied a result of Borho on the structure of certain quotients of universal enveloping algebras of solvable Lie algebras. Meanwhile, I found that the proposition is a more or less immediate consequence of a simple lemma. I include the proofs of the lemma and of the proposition.

LEMMA *Let* K *be a compact group, let* H *be a closed subgroup of* K *, and let* E *be any complex vector space. Suppose that there is given a complex subspace* B *of the space* $M(K,E)$ *of all functions from* K *into* E *satisfying the following conditions:*

(i) B *is invariant under left translations with elements* $x \in K$ *, i.e. if* $f \in B$ *then also* xf *defined by* $(xf)(y) = f(x^{-1}y)$ *belongs to* B .

(ii) B *is the union (or sum) of finite dimensional* K*-invariant subspaces on which* K *acts continuously.*

(iii) B *is invariant under multiplication with elements in the ring* $R(K/H)$ *consisting of all representative functions on* K *which are constant on cosets modulo* H .

Then the subspace B_H *of all functions in* B *vanishing on* H *is equal to* $R_o(K/H)B$ *where* $R_o(K/H)$ *denotes the ideal of all elements in* $R(K/H)$ *vanishing at* eH .

PROOF Clearly, $R_o(K/H)B$ is contained in B_H . So, let $f \in B_H$ be given, and we may assume $f \neq 0$. By (ii), there exists a finite dimensional K-invariant subspace W of B containing f . f lies in the subspace $V := B_H \cap W$ of W . On W , there exists a Hilbert space structure $<,>$ such that K acts by unitary operators, and we may suppose in addition that $<f,f> = 1$. Hence there exists an orthonormal basis $f_1,\dots,f_r, f_{r+1},\dots,f_n$ of W such that $f = f_1$; $f_1,\dots,f_r$ is a basis of V , and $f_{r+1},\dots,f_n$

is a basis of $V^\perp \subset W$. Let's write down the action of K on W in the chosen basis:

There is a homomorphism α from K into the unitary group $U(n)$, $\alpha(x) = (a_{ij}(x))_{i,j}$, such that

$$x f_i = \sum_{j=1}^{n} a_{ji}(x) f_j \quad \text{for all } x \in K ,$$

or

$$f_i(x^{-1}y) = \sum_{j=1}^{n} a_{ji}(x) f_j(y) \quad \text{for all } x,y \in K .$$

Putting $y = e$ and $i = 1$ we get in particular

$$f(x^{-1}) = \sum_{j=1}^{n} a_{j1}(x) f_j(e) \quad .$$

As $f_j(e) = 0$ for $j \leq r$ one even has

$$f(x^{-1}) = \sum_{j>r} a_{j1}(x) f_j(e) \quad .$$

For $\ell = 1,\ldots,n$ let

$$\psi_\ell(x) := \sum_{k=r+1}^{n} a_{k1}(x^{-1}) a_{\ell k}(x) \quad , \quad x \in K \quad .$$

Clearly, ψ_ℓ is contained in $R(K)$. In fact, it is one of the matrix coefficients of $id_{V^\perp} \in Hom_{\mathbb{C}}(V^\perp, V^\perp)$ considered as a subspace of the K-module $Hom_{\mathbb{C}}(W,W)$ which proves that ψ_ℓ sits in $R(K/H)$. This can, of course, also be verified by a direct computation:

$$\psi_\ell(xh) = \sum_{k>r} a_{\ell k}(xh) a_{k1}(h^{-1}x^{-1}) =$$

$$= \sum_{k>r} \sum_i \sum_j a_{\ell i}(x) a_{ik}(h) a_{kj}(h^{-1}) a_{j1}(x^{-1}) \quad .$$

But $\sum_{k>r} a_{ik}(h) a_{kj}(h^{-1})$ is 1 for $i = j > r$ and zero otherwise. Since $a_{k1}(e) = 0$ for all $k > r$, ψ_ℓ is even contained in $R_o(K/H)$.

The proof is concluded by showing that

$$f = f_1 = \sum_{\ell=1}^{n} \psi_\ell f_\ell \quad .$$

But

$$\sum_{\ell=1}^{n} (\psi_\ell f_\ell)(x) = \sum_\ell \sum_{k>r} \sum_j a_{k1}(x^{-1}) a_{\ell k}(x) a_{j\ell}(x^{-1}) f_j(e) \quad .$$

Since $\sum_\ell a_{\ell k}(x) a_{j\ell}(x^{-1}) = \delta_{jk}$, one obtains

$$\sum_{\ell=1}^{n} (\psi_\ell f_\ell)(x) = \sum_{k>r} a_{k1}(x^{-1}) f_k(e) = f(x) \quad .$$

PROOF OF THE PROPOSITION To apply the lemma let $E := \mathfrak{M}/\Omega$ be the quotient algebra and denote by ν the quotient homomorphism $\mathfrak{M} \to E$. Define $\mu : \mathfrak{M} \to M(K,E)$ by $\mu(u)(x) := \nu({}^{x^{-1}}u)$. The image B of μ clearly satisfies (i) and (ii). Concerning (iii) one observes that $\mu(\mathfrak{Z})$ multiplies B. Since Ω and hence Λ is self-adjoint it turns out that $\mu(\mathfrak{Z})$ is equal to $R(G/H)$. Pedantically, one has $\mu(\mathfrak{Z}) = R(G/H) \otimes e$, where e denotes the unit elements in the algebra E.

The lemma gives

$$B_H = R_o(G/H)B = \mu(\Lambda)\mu(\mathfrak{M}) = \mu(\Lambda\,\mathfrak{M})$$

because $R_o(G/H)$ equals to $\mu(\Lambda)$. Taking preimages one gets $\Lambda\,\mathfrak{M} + \ker\mu = \mu^{-1}(B_H) = \{u \in \mathfrak{M} \mid {}^{x}u \in \ker\nu \text{ for all } x \in H\} = \bigcap_{x\in H} {}^{x}\Omega$. As $\ker\mu = \bigcap_{x\in K} {}^{x}\Omega$

the desired equality follows.

COMMENTS TO (IV)

As I pointed out in the discussion of (I), in the case of an abelian N property (IV) is satisfied by orbits of compact groups. For arbitrary nilpotent Lie groups, again the best result I know is that (IV) holds true for orbits of compact abelian groups, [8]. To give some flavour let me briefly discuss the case of a one element set $A = \{[\pi]\}$. The following considerations are due to Ludwig, [6]. A basic tool is Howe's description of the quotient space $S(N)/\ker\pi \cap S(N)$, [3]. There exists a realization of π in $L^2(\mathbb{R}^n)$ such that for $f \in S(N)$ the operator $\pi(f)$ is given by a kernel $K_f \in S(\mathbb{R}^{2n})$, $(\pi(f)\xi)(x) = \int_{\mathbb{R}^n} K_f(x,y)\xi(y)dy$ for $\xi \in L^2(\mathbb{R}^n)$. The map $f \to K_f$ is a surjection form $S(N)$ onto $S(\mathbb{R}^{2n})$, it is continuous w.r.t. the Fréchet space structure and it allows a continuous inverse. Clearly, the map is multiplicative if $S(\mathbb{R}^{2n})$ is endowed with the multiplication $(P * Q)(x,y) = \int_{\mathbb{R}^n} P(x,z)Q(z,y)dz$.

To prove that $\ker\pi \cap S(N)$ is dense in $\ker\pi$ we take a bounded linear functional φ on $L^1(N)$ with $\varphi = 0$ on $\ker\pi \cap S(N)$, and we claim that φ is zero on $\ker\pi$. First, we regularize φ using arbitrary functions $p,q \in S(N)$, i.e. we consider the linear functional $f \to \varphi(p*f*q)$. We claim that this functional is even C*-continuous. By Howe's theorem there exists a tempered distribution $\varphi' \in S(\mathbb{R}^{2n})'$ with

$$\varphi(f) = \varphi'(K_f)$$

for all $f \in S(N)$. From the known structure of tempered distributions, see e.g. L. Schwartz, Théorie des distributions, it follows that there exist a continuous function $\varphi_o \in L^2(\mathbb{R}^{2n})$ and a differential operator D on $\mathbb{R}^{2n}$

with polynomial coefficients such that

$$\varphi'(g) = \int_{\mathbb{R}^{2n}} Dg(x,y)\varphi_o(x,y)\,dx\,dy$$

for all $g \in S(\mathbb{R}^{2n})$.

In particular, one gets

$$\varphi(p*f*q) = \varphi'(K_{p*f*q}) = \varphi'(K_p*K_f*K_q) =$$

$$= \int_{\mathbb{R}^{2n}} [D(K_p*K_f*K_q)](x,y)\varphi_o(x,y)\,dx\,dy =$$

$$= \sum_{j=1}^{m} \int_{\mathbb{R}^{2n}} (P_j*K_f*Q_j)(x,y)\varphi_o(x,y)\,dx\,dy$$

with some functions $P_j, Q_j \in S(\mathbb{R}^{2n})$ depending on K_p, K_q and D . Now it is elementary to deduce that

$$|\varphi(p*f*q)| \leq E \,||\pi(f)||$$

where E is a constant depending on p,q,φ_o and D . Clearly, this inequality holds true for all $f \in L^1(N)$, in particular for $f \in \ker \pi$. It follows that $\varphi(p*\ker\pi*q) = 0$ for all $p,q \in S(N)$ and hence $\varphi(\ker\pi) = 0$ as desired.

An application of this method to more general A depends heavily on a description of the quotient $S(N)/k(A) \cap S(N)$. This can be done in a satisfactory way for A being an orbit of a compact abelian group, [8], and possibly for orbits of arbitrary compact groups.

The reader may wonder why I claimed the theorem for $T \ltimes U$-orbits while I only said that (III) and (IV) are true for T-orbits. In fact, I can prove

(III) and (IV) only for T-orbits which gives that (II) is true for T-orbits. But (II) is also true for $T \rtimes U$ orbits in $\hat{N}$. To prove this one applies (II) to T orbits in the dual of a certain semidirect product $U \ltimes (N \times N)$ and does some computations in universal enveloping algebra, details can be found in [8].

Let me summarize and conclude with some remarks. First of all, very little is known on spectral synthesis in L^1 algebras of nilpotent Lie groups, not to speak about more general groups. What we have seen is just the peak of a possible iceberg. We have discussed a certain strategy to attack the spectral synthesis problem leading to four properties (I) - (IV) which can be investigated separately. This strategy is very close to algebraic varieties. By analytic continuation the next step might be the study of orbits of arbitrary compact groups or of more general algebraic groups. Also the complement in $\hat{N}$ of the points in "general position" is a reasonable candidate for further investigations.

Since the original problem has nothing to do with algebraic varieties it is also possible that - as in the case of abelian spectral synthesis - completely different mathematical weapons give better results.

At the end I would like to mention a related

PROBLEM *Let* N *be a connected nilpotent Lie group on which a connected solvable Lie group* G *acts. A closed two sided* G*-invariant ideal* P *in* $L^1(N)$ *is called* G-<u>prime</u> *if for all* G*-invariant two sided ideals* I *and* J *in* $L^1(N)$ *the inclusion* $IJ \subset P$ *implies that* I *or* J *is contained in* P. *It is true that each such* P *is of the form* $P = K(\overline{G\pi})$ *for a certain* $\pi \in \hat{N}$?

The answer is affirmative if G is a unipotent group. A positive solution would have consequences for the classification of uniformly bounded TCI representations of solvable Lie groups.

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