PIECEWISE LINEAR FUNCTIONS AND SERIES EXPANSIONS IN TERMS OF DIRICHLET AND FEJÉR KERNELS

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If (x_n) is a sequence of vectors in a Banach space, there are various results which describe, in particular cases, when a subsequence (x_{n_k}) of (x_n) has certain properties, such as being basic, which are not possessed by the original sequence. For example, take the sequence $(e^{int})_{n=-\infty}^{\infty}$ in $C([0,2\pi])$. This sequence is not basic, but any lacunary subsequence of it is basic. Results along similar lines, for different sequences, may be found in [1], [2] and [3].

In this note are announced some results for certain sequences of vectors in $L^p(\mathbb{R})$ and $l^p(\mathbb{Z})$, for $1 \le p < \infty$, these vectors being linear on certain subintervals of $\mathbb{R}$ or $\mathbb{Z}$. This enables a certain characterization to be given of those functions which can be expanded in terms of a lacunary sequence of Dirichlet and Fejér kernels in $L^2(-\pi,\pi)$.

Let $1 \le p \le \infty$. Let $\alpha(0) = 0$ and let $(\alpha(n))$ be a given strictly increasing sequence of positive real numbers. The sequence $(\alpha(n))$ is said to be *lacunary* if there is a $\delta > 1$ such that $\alpha(n+1)\alpha(n)^{-1} \ge \delta > 1$, for all $n \in \mathbb{N}$. A Banach subspace $PL(p,\alpha)$ of $L^p(\mathbb{R})$ is defined as follows: $f \in PL(p,\alpha)$ if and only if $f \in L^p(\mathbb{R})$, f is even, f is zero on $\cap\{t : |t| \ge \alpha(n)\}$ and f is the restriction of a polynomial function of degree at most one upon each interval of the form $[\alpha(n-1),\alpha(n))$, for $n \in \mathbb{N}$. Let $PLC(p,\alpha)$ denote those functions in $PL(p,\alpha)$ which are continuous and let $PC(p,\alpha)$ denote those functions in $PL(p,\alpha)$ which are constant upon each interval of the form $[\alpha(n-1),\alpha(n))$, for $n \in \mathbb{N}$. If $\lim_{n\to\infty}\alpha(n) = \infty$, it is clear that $PLC(p,\alpha) \cap PC(p,\alpha) = \{0\}$. Let (u_n) be the sequence in $PL(p,\alpha)$ given by $u_{2n-1}(t) = 1$ for $|t| \le \alpha(n)$, $u_{2n-1}(t) = 0$ for $|t| > \alpha(n)$, and $u_{2n}(t) =$ maximum $(0, \alpha(n) - |t|)$.

THEOREM 1. *Let $1 \le p < \infty$. Then the following conditions are equivalent.*

(i) $(\alpha(n))$ *is lacunary,*

(ii) $PL(p,\alpha)$ *is the direct sum of* $PLC(p,\alpha)$ *and* $PC(p,\alpha)$,

(iii) (u_n) *is a basis for* $PL(p,\alpha)$, *and*

(iv) *if* $\sum_{n=1}^{\infty} d_n u_n$ *converges in* $PL(p,\alpha)$, *then* $d \in l^p$.

When these conditions hold, $(\|u_n\|_p^{-1} u_n)$ *is equivalent to the standard basis for* l^p.

If $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$, it can be proved that there is a bounded linear projection π from $L^p(\mathbb{R})$ onto $PL(p,\alpha)$ such that $\pi^*(PL(p,\alpha)^*) = PL(q,\alpha)$. This result can be used to obtain the following dual form of Theorem 1.

THEOREM 2. *Let* $1 < q \le \infty$. *For* $f \in L^q(\mathbb{R})$ *and* $n \in \mathbb{N}$, *let*

$$(A(f))_{2n-1} = \alpha(n)^{\frac{1}{q}-1} \int_{-\alpha(n)}^{\alpha(n)} f(t)dt, \text{ and}$$

$$(A(f))_{2n} = \alpha(n)^{\frac{1}{q}-2} \int_{-\alpha(n)}^{\alpha(n)} (\alpha(n) - |t|) f(t)dt.$$

Then $(\alpha(n))$ *is lacunary if and only if* A *is a bounded linear operator from* $L^q(\mathbb{R})$ *onto* l^q. *In this case, the restriction of* A *to the subspace* $PL(q,\alpha)$ *of* $L^q(\mathbb{R})$ *is a bounded invertible operator from* $PL(q,\alpha)$ *onto* l^q.

If $(\alpha(n))$ satisfies the initial assumption and its terms are integers, let $PL_d(p,\alpha)$ be the discrete analogue of $PL(p,\alpha)$. That is, a sequence (d_n) is in $PL_d(p,\alpha)$ if and only if $(d_n) \in l^p(\mathbb{Z})$, $d_n = d_{-n}$ for all $n \in \mathbb{N}$, and (d_n) is piecewise linear in the sense that for each $n \in \mathbb{N}$ there is a scalar θ_n such that $d_{j+1} - d_j = \theta_n$, for all $j \in \{\alpha(n-1), \alpha(n-1)+1, \ldots, \alpha(n)-1\}$. The following is then a discrete analogue of Theorem 1.

THEOREM 3. *Let* $1 \le p < \infty$ *and assume that* $(\alpha(n))$ *is a strictly increasing sequence of positive integers such that* $\alpha(n) - \alpha(n-1) \ge 2$, *for all* $n \in \mathbb{N}$. *Let the sequence* (w_n) *in* $PL_d(p,\alpha)$ *be given by* $w_{2n-1}(j) = 1$ *for* $|j| < \alpha(n)$, $w_{2n-1}(j) = 0$ *for* $|j| \ge \alpha(n)$ *and* $w_{2n}(j) =$ *maximum* $(0, \alpha(n) - |j|)$. *Then the following conditions are equivalent:*

(i) $(\alpha(n))$ *is lacunary,*

(ii) (w_n) *is a basis for* $PL_d(p,\alpha)$, *and*

(iii) *if* $\sum_{n=1}^{\infty} c_n w_n$ *converges in* $PL_d(p,\alpha)$, *then* $c \in l^p$.

When these conditions hold, $(\|w_n\|_p^{-1} w_n)$ *is equivalent to the standard basis in* l^p.

The Fourier transform of w_{2n-1} is the Dirichlet kernel $D_{\alpha(n)}$ given by $D_{\alpha(n)} = \sin(\alpha(n) - \frac{1}{2})t / \sin \frac{t}{2}$. The Fourier transform of $\alpha(n)^{-1} w_{2n}$ is Fejér's kernel $F_{\alpha(n)}$ given by $F_{\alpha(n)} = \alpha(n)^{-1}(\sin^2 \alpha(n)\frac{t}{2})/(\sin^2 \frac{t}{2})$. Plancherel's Theorem thus gives the following corollary of Theorem 3.

COROLLARY. *The condition that* $(\alpha(n))$ *be lacunary is equivalent to the condition that* $\sum_{n=1}^{\infty}(c_n \alpha(n)^{-\frac{1}{2}} D_{\alpha(n)} + d_n \alpha(n)^{-\frac{1}{2}} F_{\alpha(n)})$ *is convergent in* $L^2(-\pi, \pi)$ *if and only if both* $c, d \in l^2$. *In this case, the functions which are the sums of such series are precisely those whose Fourier transforms belong to* $PL_d(2, \alpha)$.

REFERENCES

[1] R.E. Edwards, *Approximate integration formulas and "Sidonicity"*, Australian National University Mathematics Research Report No. 14(1982).

[2] V.I. Gurarii and V.I. Macaev, *Lacunary power sequences in the spaces C and* L^p, Izv. Akad. Nauk. SSR Ser. Math. 30(1966), 3-14(Russian), Amer. Math. Soc. Trans. 72(1968), 9-21(English).

[3] R.V. Nillsen, *Some basic sequences and their moment operators*, Proceedings of the Centre for Mathematical Analysis, Australian National University, 14(1986) (Miniconference on Operator Theory and Partial Differential Equations, Macquarie University, September, 1986), 246-271.

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