FREDHOLM MODULES ASSOCIATED TO BRUHAT-TITS BUILDINGS

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To the memory of our friend Hugh Morris, a young and talented Australian mathematician

1. INTRODUCTION

In his Opus Magnum [Co], Connes defines an even <u>unbounded Fredholm module</u> over a C*-algebra A as a pair $(\mathcal{H}, D)$, where $\mathcal{H}$ is a $\mathbb{Z}/2$-graded Hilbert space carrying a *-representation of A of degree 0, and D is an unbounded self-adjoint operator on $\mathcal{H}$, of degree 1, such that:

i) $(1 + D^2)^{-1}$ is compact

ii) The subalgebra $\mathcal{a} = \{a \in A: [D,a] \text{ is bounded}\}$ is norm-dense in A.

Following [Co2], we say that an unbounded Fredholm module $(\mathcal{H}, D)$ is <u>ϑ-summable</u> if, for any $t > 0$, the operator e^{-tD^2} is trace-class.

This condition is rather natural if one remembers the heat equation proof of the index theorem: Connes simply requires the "heat kernel" to be trace-class. In the case of the Dirac operator D on a compact riemannian spin manifold M, one even has p-summability in the sense of [Co]: $(1 + D^2)^{-p/2}$ is trace-class for $p > \dim M$. In particular $\mathrm{Tr}\, e^{-tD^2} = O(t^{-p/2})$ for $t \to 0$. However, as shown in [Co2], this condition of p-summability is too restrictive, as being too related to finite dimension and commutativity.

If G is a locally compact group, we define an <u>unbounded G-Fredholm module</u> as a pair $(\mathcal{H}, D)$, where $\mathcal{H}$ is now a $\mathbb{Z}/2$-graded Hilbert space carrying a unitary representation of G, and D is as above, with condition ii) replaced by:

ii') For any $g \in G$, the operator $gDg^{-1} - D$ is bounded, and the map $g \to gDg^{-1} - D$ is strongly continuous.

It is easy to see that an unbounded G-Fredholm module gives rise to an unbounded Fredholm module over the C*-algebra of any closed subgroup of G.

In this paper, we associate a ϑ-summable unbounded Fredholm module to any simple algebraic group G over a non-archimedean local field F (the reader is urged to think of F as $\mathbb{Q}_p$, the p-adic field, and of G as $SL_n(\mathbb{Q}_p)$). The construction is geometric, and uses the so-called Bruhat-Tits building of G. When the building is a tree (e.g. for $SL_2(\mathbb{Q}_p)$), we get nothing but the unbounded version of the Fredholm module that we associated to a tree in [JV]. The construction is quite reminiscent of the construction of the dual-Dirac operator on riemannian symmetric spaces of the non-compact type (see [Co], [Co3], [Ka]): it also involves the choice of an origin x_0 and the existence of a unique geodesic between x_0 and a point $x \neq x_0$

Remember that, for any locally compact group G, Kasparov [Ka] organized the G-Fredholm modules into a unital abelian ring $KK_G(\mathbb{C},\mathbb{C})$ which, for compact G, is nothing but the representation ring. If G is a connected Lie group, Kasparov has an idempotent $\gamma \in KK_G(\mathbb{C},\mathbb{C})$ (the celebrated Kasparov obstruction) which embodies both the Dirac and dual-Dirac operators on G/K (K a maximal compact subgroup of G). The above remarks lead us to believe that the Fredholm module described in this paper will be a kind of p-adic analogue of Kasparov's γ; it is easy to see that, for G a simple algebraic group over F, the restriction of our Fredholm module to any compact subgroup is 1. Moreover, if the split rank of G is at least 2 (e.g. $SL_n(F)$, $n \geq 3$), then because of Kazhdan's property (T) (see [DK]), our module is not equal to 1 in $KK_G(\mathbb{C},\mathbb{C})$; in particular it gives a new element in $KK_G(\mathbb{C},\mathbb{C})$ (when G is split (e.g. $G=SL_n(F)$), the only elements of $KK_G(\mathbb{C},\mathbb{C})$ previously known were the multiples of 1). However, we do not know whether or not $\gamma^2 = \gamma$ in $KK_G(\mathbb{C},\mathbb{C})$.

It is conceivable that our Fredholm module could be useful to prove particular cases of the <u>Kaplansky-Kadison conjecture</u> (see [BC]): let Γ be a countable torsion-free group; then any idempotent in the reduced C*-algebra $C^*_r(\Gamma)$ is trivial (either

0 or 1). Our friend Paul Baum likes to say that this conjecture is probably false, because of a principle of Gromov("any non-trivial statement about the class of countable groups has to be false"). But it should be true in interesting cases (e.g. linear groups). Anyway, it is known from work of Pimsner-Voiculescu [PV], Cuntz [Cu], Connes [Co] on the free groups, that this conjecture would follow from the fact that the canonical trace τ is integer-valued on projections of $C^*_\Gamma(\Gamma)$. Assume Γ is a discrete subgroup of G, a p-adic group as above. Let $\mathcal{A} = \{a \in C^*_\Gamma(\Gamma) : [D,a] \text{ is bounded}\}$; this is a dense sub-algebra, stable under holomorphic functional calculus (see [Co]). So any projection in $C^*_\Gamma(\Gamma)$ is equivalent to a projection in $\mathcal{A}$. For such a p, the Fredholm index of

$$pD^+p : p\mathcal{H}^+ \to p\mathcal{H}^-$$

is a well-defined integer which, by the heat equation method, is equal to the super-trace $Tr_s(p.\exp(-tD_p^2))$ for any $t > 0$ (here $D_p = pDp + (1-p)D(1-p)$). Then the conjecture would follow if we could prove something like

$$\lim_{t \to 0} Tr_s(p.\exp(-tD_p^2)) = \tau(p)$$

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2. THE BUILDING OF G

Let G be (the group of rational points of) a simple algebraic group defined over a non-archimedean local field F. Let Δ^0 be the set of maximal compact subgroups of G (a countable set), acted upon by conjugation. Note the following results of Bruhat and Tits [BT], giving information on this action of G:

i) Every maximal compact subgroup coincides with its normalizer.

ii) The number of conjugacy classes of maximal compact subgroups is finite, equal to $\ell+1$, where ℓ is the split rank of G.

EXAMPLE: $G = SL_n(\mathbb{Q}_p)$; here the split rank is n-1; representatives for conjugacy classes of maximal compact subgroups are

$$K_k = G \cap \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$$

where the top left block of this matrix is $k \times k$ $(0 \leq k \leq n-1)$; here $\mathbb{Z}_p$ denotes as usual the ring of p-adic integers. Note that $K_o = SL_n(\mathbb{Z}_p)$.

We say that two elements K, K' of Δ^o are <u>incident</u>, or determine an <u>edge</u>, if $K \cap K'$ is maximal both in K and K'; next, for $2 \leq k \leq \ell$, we say that $k+1$ elements of Δ^o determine a <u>k-face</u> if they are pairwise incident. By definition, the <u>building</u> is the resulting simplicial complex. More precisely, it is a contractible simplicial complex of dimension ℓ, carrying a proper action of G. A simplex of dimension ℓ is a <u>chamber</u> (see [BT]).

The building is not a purely combinatorial object: an important feature is the presence of a metric, for which it is complete and G acts by isometries (see [BT], [Ti]). Moreover two points are joined by a unique geodesic.

In the Bruhat-Tits philosophy, a building is a p-adic analogue of a riemannian symmetric space of the non-compact type (we already publicized this philosophy in [JV]). Note that in these spaces, there are certain distinguished subspaces, namely maximal flat subspaces, which are euclidean in the induced metric. In the building, this rôle is played by the so-called <u>apartments</u> ("an apartment is a flat"); apartments are euclidean spaces, triangulated according to the action of the affine Weyl group of G, the induced metric being the euclidean metric (see [BT], [Ti2], [Ti3]). Any two chambers belong to at least one apartment, so that the building can be seen as a bunch of apartments glued together (in a complicated way). At this point, we feel that some examples will be welcome.

EXAMPLES:

i) $G = SL_2(\mathbb{Q}_p)$; the building is a homogeneous tree of order $p+1$ (see figure 1 for $p=2$); here chambers are edges, and apartments are straight lines. We refer to [Se] for a discussion.

ii) $G = SL_3(\mathbb{Q}_p)$; an apartment is a euclidean plane with the usual tessellation by equilateral triangles (figure 2).

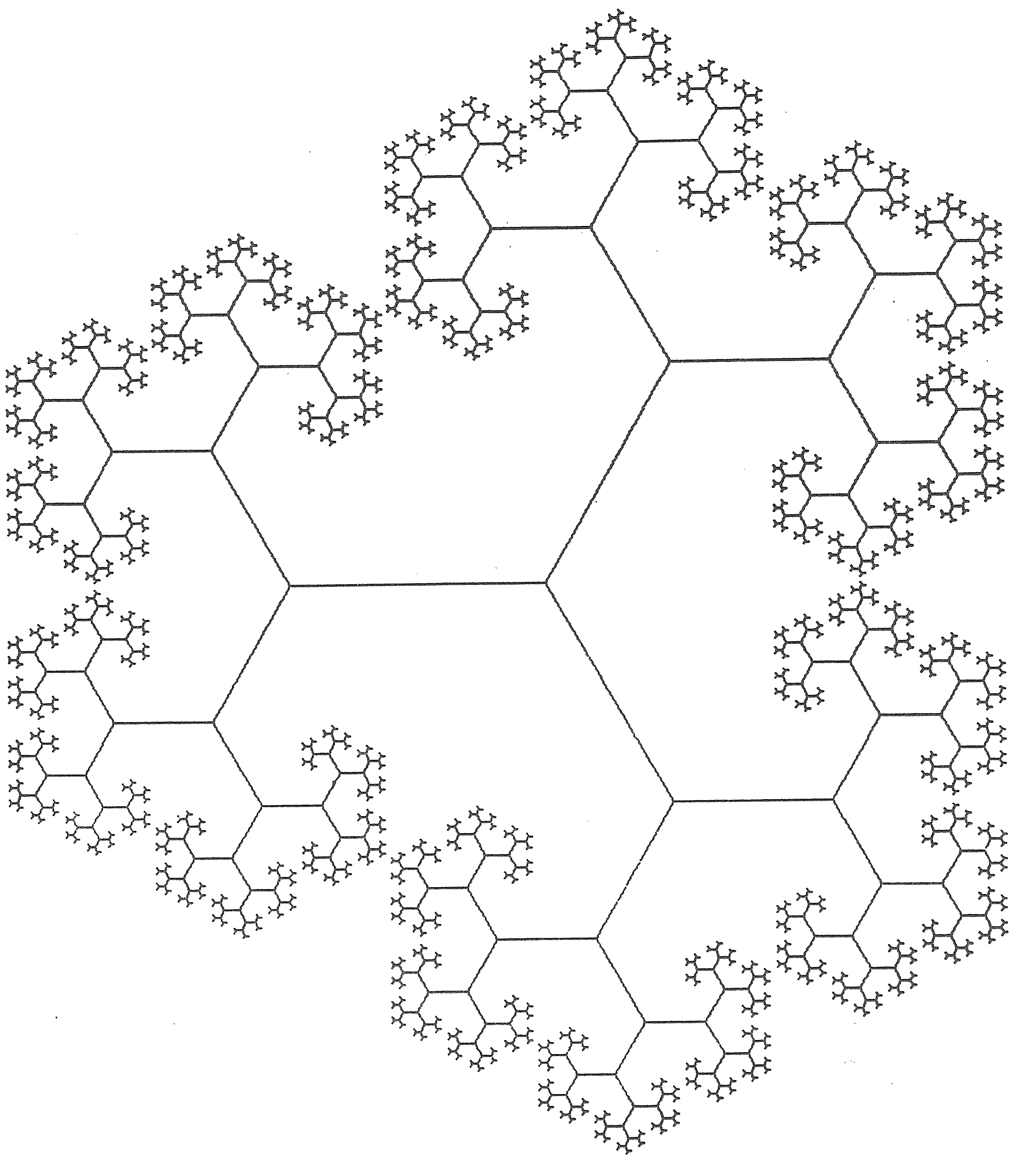

Figure 1

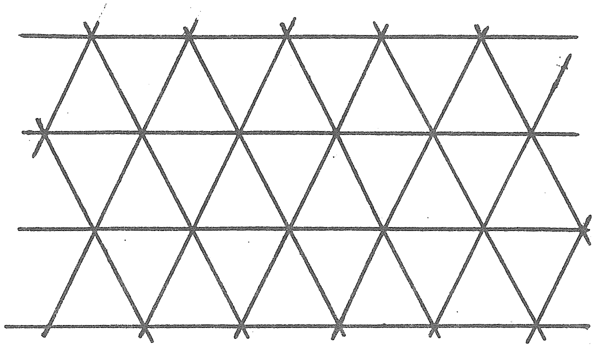

Figure 2

According to [Ti2], one has to figure out the building ramifying along every edge, each one belonging to p+1 apartments. It defies any attempt of drawing.

iii) $G = SL_4(\mathbb{Q}_p)$; here we can only describe a chamber (it will not surprise anyone familiar with the root system A_3). It is a tetrahedron made up of four isosceles triangles looking like figure 3.

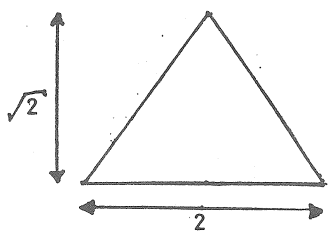

Figure 3

3. THE FREDHOLM MODULE

Fix an origin $x_0 \in \Delta^0$. For any simplex x, denote by $\mu(x)$ its barycentre, and by $\beta(x)$ the unique simplex containing x with interior meeting the geodesic $[x_0\ \mu(x)]$ (see figure 4).

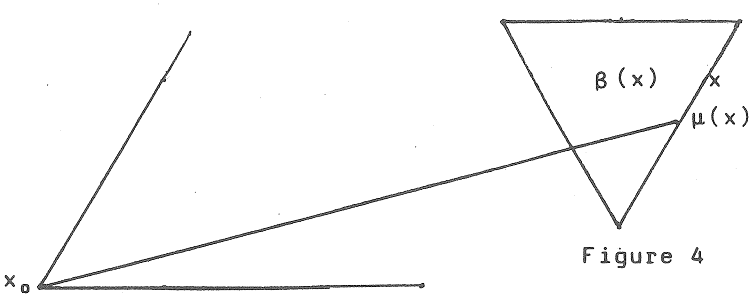

Figure 4

Denote by Δ^k the set of simplices with dimension k; for any simplex σ, define $\Delta_\sigma = \{x \in \coprod_k \Delta^k : \beta(x) = \sigma\}$

If x, y are simplices, we say that x is <u>well-contained</u> in y if $x \subseteq y \subseteq \beta(x)$; this is denoted by $x \propto y$. One can see that, if $x \propto y$, then $\beta(x) = \beta(y)$. This implies that $\propto$ is an order relation. Below, we tacitly assume that σ has a nonempty Δ_σ. We omit the proof of the following lemma.

LEMMA: Δ_σ has a unique minimal element σ_{min} with respect to $\propto$. Let m be the dimension of σ_{min}; denote by I_σ the set of simplices of dimension m+1 in Δ_σ; then Δ_σ is in one-to-one correspondence with the power set $\mathcal{P}(I_\sigma)$; in particular, the cardinality of Δ_σ is $2^{\dim \sigma - m}$

Now, remember that G has $\ell+1$ orbits on Δ^0; number these orbits from 0 to ℓ. Then, any k-simplex x of the building can be identified with a subset of cardinality k+1 of $\{0,1,\dots,\ell\}$; this subset is the <u>type</u> of x, denoted by typ x. If $x \in \Delta^k$ is well-contained in $y \in \Delta^{k+1}$, it is then possible to define a sign $\varepsilon(x,y)$ as in simplicial cohomology: consider the unique increasing bijection typ $y \to \{0,1,\dots,k+1\}$; then the image of typ x is $\{0,1,\dots,k+1\} \setminus \{i\}$ for some i; define $\varepsilon(x,y) = (-1)^{i+1}$; ε is clearly a G-invariant function.

Back to the study of Δ_σ, we may - and will - identify I_σ with typ $\sigma \setminus$ typ σ_{min}. For $i \in I_\sigma$, define a map $\alpha_i : \Delta_\sigma \to \Delta_\sigma$ by

$$\alpha_i(x) = \begin{cases} x \cup \{i\} & \text{if } i \notin x \\ x \setminus \{i\} & \text{if } i \in x \end{cases}$$

So, either $x \propto \alpha_i(x)$ or $\alpha_i(x) \propto x$. This leads us to symmetrize the definition of ε in the following way:

$$\varepsilon(x, \alpha_i(x)) = \varepsilon(\alpha_i(x), x)$$

The proof of the following lemma is obvious, once you realize it is true:

LEMMA: For $i,j \in I_\sigma$, $i \neq j$, and $x \in \Delta_\sigma$, one has:

$$\varepsilon(x,\alpha_i(x))\varepsilon(x,\alpha_j(x)) = -\varepsilon(\alpha_{ij}(x),\alpha_i(x))\varepsilon(\alpha_{ij}(x),\alpha_j(x))$$

(where $\alpha_{ij} = \alpha_i \circ \alpha_j = \alpha_j \circ \alpha_i$)

On $\ell^2(\Delta_\sigma)$, define an operator γ_i by $\gamma_i \delta_x = \varepsilon(x,\alpha_i(x))\delta_{\alpha_i(x)}$ (δ_x, $x \in \Delta_\sigma$, is the canonical basis of $\ell^2(\Delta_\sigma)$).

PROPOSITION: The Clifford algebra relations hold, namely:

$$\gamma_i^2 = 1$$

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0 \quad (i \neq j)$$

Moreover $\gamma_i = \gamma_i^*$

Proof: This follows immediately from the preceding lemma and the relations $\alpha_i^2 = 1$, $\alpha_{ij} = \alpha_{ji}$

This is the first part of the construction of our dual-Dirac operator. Now, we would like to define something similar to "Clifford multiplication by a vector pointing from x_0 to σ". To do that, we choose an apartment containing both x_0 and σ; viewing x_0 as the origin of this apartment, we can speak of the vector $X_\sigma = x_0\mu(\sigma_{min})$. Remember that there is a root system based at x_0, such that any wall of the apartment has equation $\langle\beta , X\rangle = k$, for some root β and some integer k. To any $i \in I_\sigma$, we associate a set B_i of roots by requiring that:

- each root in B_i is constant on the unique simplex of type typ $\sigma \setminus \{i\}$ in Δ_σ;
- the scalar product $\lambda_i(X_\sigma) = \langle\beta , X_\sigma\rangle$ is positive and maximal for any $\beta \in B_i$ (see figure 5).

Note that for generic σ, i.e. dim $\sigma = \ell$, the B_i's contain just one root. The scalar $\lambda_i(X_\sigma)$ is independent of the choice of the apartment involved in the construction.

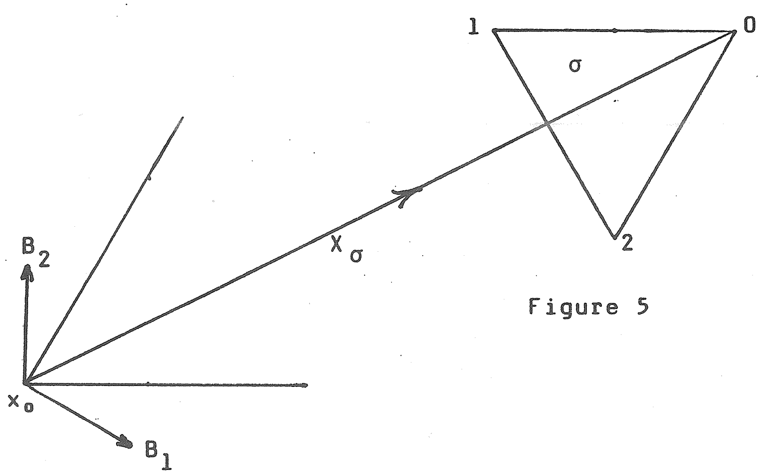

Figure 5

Define an operator $D_\sigma : \ell^2(\Delta_\sigma) \to \ell^2(\Delta_\sigma)$ by

$$D_\sigma = \sum_{i \in I_\sigma} \lambda_i(X_\sigma)\gamma_i$$

Consider then the Hilbert space $\mathcal{H} = \bigoplus_\sigma \ell^2(\Delta_\sigma) = \bigoplus_{k=0}^{\ell} \ell^2(\Delta^k)$, $\mathbb{Z}/2$-graded by the decomposition into even and odd dimensional simplices. Define $D = \bigoplus_\sigma D_\sigma$; this is an unbounded self-adjoint operator.

PROPOSITION: For $t > 0$, the operator e^{-tD^2} is trace-class.

Proof: We have $D_\sigma^2 = \sum_i \lambda_i(X_\sigma)^2$ because of the Clifford algebra relations. So, denoting by ρ_σ the distance from x_0 to $\mu(\sigma_{min})$, we see that there exists a constant $c > 0$ (depending only on the building, not on σ) such that $D_\sigma^2 \geq c\rho_\sigma^2$. Hence

$$e^{-tD^2} \leq \oplus_\sigma e^{-tc\rho_\sigma^2}$$

The trace of the scalar operator $e^{-tc\rho_\sigma^2}$ is less than $2^\ell . e^{-tc\rho_\sigma^2}$ Finally, the function $\sigma \to e^{-tc\rho_\sigma^2}$ is summable, because the number of simplices at distance $\leq n$ from x_0 is in e^{Cn}, where C is a constant only depending on the building.

Note that the inequality $D_\sigma^2 \geq c\rho_\sigma^2$ also implies that $D^+ : \mathcal{H}^+ \to \mathcal{H}^-$ is Fredholm with index 1, its kernel consisting precisely of the multiples of δ_{x_0}.

PROPOSITION: For any $g \in G$, the operator $D - gDg^{-1}$ is bounded.

Sketch of proof: The operator gDg^{-1} is "the same" as D, but with the origin x_0 replaced by gx_0. More generally, let D' be the operator analogue of D, but defined with respect to an origin x_0'. We claim that $D - D'$ is bounded. This is intuitively clear: since D_σ (resp. D_σ') is something like Clifford multiplication by X_σ (resp. X_σ'), the difference should be Clifford multiplication by $X_\sigma - X_\sigma' = x_0' - x_0$ which is constant. However, one has two difficulties to overcome:

- The construction of X_σ involves the choice of an apartment through x_0 and σ. In general, there is no apartment containing simultaneously σ, x_0 and x_0'. This can be arranged by the following trick: since the 1-skeleton of the building is connected, we may assume that x_0 and x_0' are incident, and then find an apartment containing x_0, x_0' and σ.

- The blocks Δ_σ depend on the choice of x_0, so that D, D' are not diagonal in the same decomposition of $\mathcal{H}$. This can be arranged by noticing that, for x far from x_0 and x_0', the simplices $\beta(x)$ and $\beta'(x)$ are close to each other, allowing one to give a bound on D - D' (see figure 6).

From the preceding propositions, we immediately deduce:

THEOREM: The pair $(\mathcal{H}, D)$ is a θ-summable unbounded G-Fredholm module.

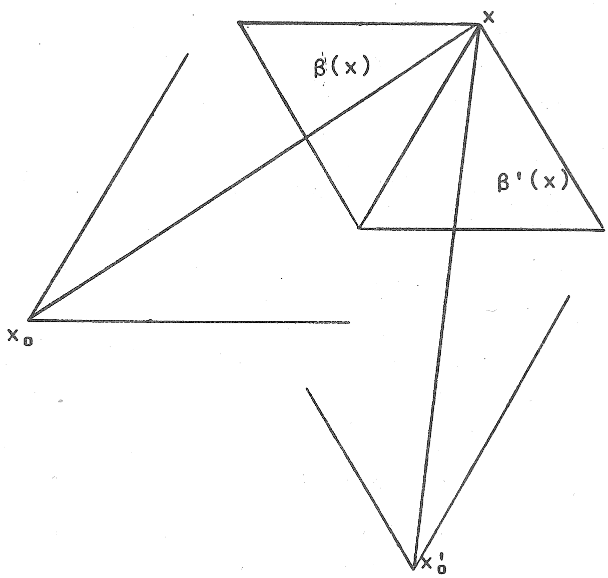

Figure 6

REMARKS:

i) Assume for a moment that the building is a tree. Then the Δ_σ's are of two kinds: $\Delta_\sigma = \{x_0\}$ and, for $x \in \Delta^0 \setminus \{x_0\}$, $\Delta_\sigma = \{x, \beta(x)\}$. So the decomposition of the building in Δ_σ's is a generalization of the bijection $\beta: \Delta^0 \setminus \{x_0\} \to \Delta^1$ exhibited in [JV]. Note that in the case of the tree, the operator D has the following form:

$$D\delta_x = \rho(x)\delta_{\beta(x)} \quad (x \in \Delta^0)$$

$$D\delta_b = \rho(\beta^{-1}(b))\delta_{\beta^{-1}(b)} \quad (b \in \Delta^1)$$

which is nothing but the unbounded version of the Fredholm module in [JV]. As an anecdote, let us mention that Connes noticed that, if the tree is homogeneous (with each vertex of order q+1, say), then $\mathrm{Tr}\, e^{-tD^2}$ is essentially given by

$$\sum_{n=0}^{\infty} q^n e^{-tn^2}$$

and the asymptotic development of this function may be estimated by means of the Poisson formula: for $t \to 0$, this function is equivalent to $t^{-\frac{1}{2}}. \exp((\log q)^2/4t)$.

ii) The reader may wonder why the authors, who stuck to bounded Fredholm modules (in the sense of [Co]) in their previous papers [JV], [JV2], are suddenly dealing with unbounded modules. The reason is that an unbounded Fredholm module contains more information than a bounded one (the Dirac operator has more to say than its phase). Simply think of the fact that an unbounded Fredholm module gives for free a dense subalgebra stable under holomorphic functional calculus (the subalgebra $\mathcal{Q}$), while in the bounded case one has to require p-summability to get this (see [Co]).

It is easy to turn our unbounded module $(\mathcal{H}, D)$ into a bounded one: simply replace X_σ by $X_\sigma/||X_\sigma||$ in the construction. Then, in the case of the tree, one really gets the 1-summable Fredholm module of [JV]. However, as was pointed out to us by A. Connes and G. Skandalis, our bounded module in rank ≥ 2 is not p-summable for any $p \geq 1$. So, working with unbounded modules re-establishes some balance between rank 1 and higher rank.

iii) To conclude, we mention that our unbounded Fredholm module is not p-summable for any $p \geq 1$ (in any rank): this follows from the exponential growth of the building. But our module cannot even be homotopic to a p-summable unbounded G-Fredholm module $(\mathcal{H}', D')$, with the representation of G on $\mathcal{H}'$ weakly contained in the left regular representation of G: indeed, by restricting the module to a lattice Γ in G and using non-amenability for Γ, one would contradict a result of Connes [Co2] asserting that there is no p-summable unbounded Fredholm module over $C^*_\Gamma(\Gamma)$ when Γ is a countable non-amenable group.

For other examples of ϑ-summable unbounded Fredholm modules which are not p-summable for any $p \geq 1$, we refer to [Co2].

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