INTEGRATION OF INFINITESIMALLY UNITARY REPRESENTATIONS

Roe Goodman

1. BANACH LIE ALGEBRAS AND GROUP GERMS

A fundamental part of Lie group theory is the construction of a dictionary that translates properties of Lie algebras into properties of Lie groups, and vice versa. The utility of such a dictionary is clear, since the language of Lie algebras is linear algebra, which is generally more accessible to study than the topological and differential-geometric language of Lie groups.

For finite-dimensional Lie algebras and groups over $\mathbb{R}$ or $\mathbb{C}$, there is a well-known standard such dictionary, started by Lie and completed by the work of E. Cartan, H. Weyl, and many others. For infinite-dimensional Lie algebras and groups, even over $\mathbb{C}$, the situation is considerably more complicated. As the simplest example, suppose we start with a Banach-Lie algebra b. That is, b is both a Banach space, with norm $\|\cdot\|$, and a Lie algebra, with Lie bracket $[\cdot,\cdot]$, and one assumes that

$$\|[x,y]\| \leq C\|x\|\|y\| \,.$$

To set up a Banach-Lie algebra: Banach-Lie group dictionary, we first need to construct a Banach-Lie group B having Lie algebra b. Locally, this was done first by G. Birkoff in the late 1930's and then a decade later by E.B. Dynkin using Dynkin's explication of the Campbell-Hausdorff formula. Recall that this formula is the formal identity

$$e^x e^y = e^{H(x,y)} \,,$$

where

$$H(x,y) = x + y + \tfrac{1}{2}[x,y] + \tfrac{1}{12}[x,[x,y]] + \tfrac{1}{12}[y,[y,x]] + \cdots$$

is an infinite series in iterated commutators of x and y that converges absolutely for x and y near zero. Using this formula we obtain a Banach-Lie group germ B_0 and an exponential map

$$\exp : b \longrightarrow B_0 \ .$$

So far this is similar to the finite-dimensional case, and in fact the early development of Lie group theory dealt mostly with local Lie groups, and not global Lie groups. The finite-dimensional global groups were eventually constructed and studied by exponentiating faithful finite-dimensional linear representations of the Lie algebra (Ado's theorem) [1]. In the infinite-dimensional case, however, van Est [2] showed that there can exist topological obstructions to imbedding an Banach-Lie group germ into a group. Since the use of linear representations was so successful for finite-dimensional Lie algebras and groups, it is natural to try this approach in the infinite-dimensional case.

2. EXPONENTIATION OF LIE ALGEBRAS OF OPERATORS

Suppose we have a faithful representation π of b on a complex inner-product space V (not assumed to be complete). Assume that there is an isometric anti-automorphism $X \mapsto X^*$ on b and that this representation is "infinitesimally unitary":

(1) $$(\pi(X)v, w) = (v, \pi(X^*)w) \ , \quad \text{for } v, w \in V \ .$$

Note that

$$\mathcal{U} = \{X \in b : X^* = -X\}$$

is a real Lie subalgebra of b, and the operators $\pi(X)$ are skew-hermitian, for $X \in \mathcal{U}$. The following questions arise in this situation:

(1) If $x \in \mathcal{U}$, then does the operator $\pi(X)$ generate a one-parameter unitary group on the Hilbert-space completion H of V? If so, then what is the subgroup of the unitary group of H generated by these operators?

(2) For any $X \in b$, can the operator $e^{\pi(X)}$ be defined as a linear automorphism on some (locally convex) completion S of V?

(3) (Assume (2) has an affirmative answer) Does the map $X \mapsto e^{\pi(X)}$, $X \in b$, define a local isomorphism from the group germ B_0 into Aut (S)?

If the operators $\pi(X)$, $X \in b$, are bounded, relative to the norm on V associated with the inner product, then of course the answer is "yes" to all these questions, as is well known. Even when b is finite-dimensional, however, the most natural representations on infinite-dimensional spaces occur as unbounded operators, e.g. differential operators. There is a great variety of pathological behaviour exhibited by unbounded operators on a Hilbert space, so we certainly can't expect questions (1) and (2) to have an affirmative answer in general. When $\mathcal{U}$ is a finite-dimensional non-compact semi-simple algebra and V is an irreducible "Harish-Chandra" module, then the answer to (1) is "yes", as a consequence of Harish-Chandra's fundamental work. However, the answer to (2) is "no" in this case [4]. For general finite-dimensional $\mathcal{U}$ Nelson [9] gave a criterion for (1) to hold (essential self-adjointness of the Laplacian for $\mathcal{U}$). Penney [10] determined the precise class of real Lie algebras $\mathcal{U}$ for which (2) has an affirmative answer: these are the algebras for which all eigenvalues of the adjoint representation of $\mathcal{U}$ are purely imaginary.

In Section 4 I will given a specific context in which all the analytical problems can be controlled rather easily. I will then indicate how this can be used to construct Banach-Lie groups corresponding to the following class of infinite-dimensional Lie algebras.

3. COMPLETED DYNKIN DIAGRAMS AND AFFINE ALGEBRAS

Recall that a finite-dimensional simple Lie algebra g is uniquely determined by its *root system* R. This is a finite set of non-zero vectors $\{\alpha_1, \ldots, \alpha_r\}$ that spans an l-dimensional Euclidean space, is stable under the orthogonal reflections sending

$\alpha_i \to -\alpha_j$, for each i, and satisfies the integrality condition

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \in Z \qquad \text{for all } i, j \ . \tag{2}$$

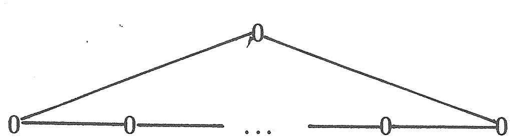
Any such R admits a *base* $B = \{\alpha_1, \ldots, \alpha_l\}$ so that every root is an integral linear combination, with coefficients all of the same sign, of the elements of B. Furthermore, B uniquely determines R (if R is "reduced"), so the whole structure of a simple Lie algebra is encoded in the set of l vectors in B.

The classification of root systems was done first in terms of the $l \times l$ *Cartan matrix* $[C_{ij}]$, which satisfies the conditions

$$C_{ii} = 2 \ , \quad C_{ij} \leq 0 \qquad \text{for } i \neq j \ . \tag{3}$$

Later an equivalent classification was carried out using the *Dynkin diagram* of B. This is the graph with l vertices labelled by the elements of B. Vertex i is joined to vertex j by $|C_{ij}|$ lines (this integer is 0, 1, 2, or 3), and the relative lengths of the vectors in B is indicated. Connectedness of the graph is equivalent to irreducibility of the root system. After classifying connected Dynkin diagrams one arrives at the four infinite series of "classical" root systems A_l, B_l, C_l, D_l, and the five exceptional systems E_6, E_7, E_8, F_4, G_2 as the only possibilities for the roots of a simple finite-dimensional complex Lie algebra g. Around 1950, Chevalley and Harish-Chandra showed that the Lie algebra structure of g could be presented in a uniform way in terms of generators and relations, using the Cartan matrix.

Let B be a base for an irreducible root system R. There is a unique "largest" element $\tilde{\alpha}$ in R (each coefficient in the expansion relative to B being maximal). Set $\alpha_0 = -\tilde{\alpha}$. Then α_0 is a root, and the Dynkin diagram for the set $\tilde{B} = B \cup \{\alpha_0\}$ is called a *completed Dynkin diagram.* For example, the completed Dynkin diagram for the A_l system is



For every root system, one thus has a corresponding *extended* Cartan matrix $[C_{ij}]$, $0 \leq i, j \leq l$. In the late 1960's, R.V. Moody and V. Kac began to study the classes of infinite-dimensional Lie algebras obtained by imitating the Chevalley, Harish-Chandra procedure for constructing simple Lie algebras, but now using a *generalized* Cartan matrix (any integral matrix satisfying conditions (3)). The algebra $\hat{g}$ that they constructed from an extended Cartan matrix of a simple Lie algebra g is called the *affine* algebra corresponding to g. Much of the structure and representation theory of g carries over to $\hat{g}$: There is a class of irreducible (infinite-dimensional) "standard" representations, whose highest weights are labelled by $l+1$ non-negative integers, one for each vertex in the extended Dynkin diagram. The *fundamental representations* $\pi_i, 0 \leq i \leq l$, correspond to the vertices of the extended diagram, as in the case of g. Every standard representation then occurs as a subrepresentation of a tensor product of fundamental representations (cf. [8] for further details).

On the algebra $\hat{g}$ there is a natural choice of an anti-linear involution $*$, so that the subalgebra of skew-symmetric elements is the analogue of the compact real form of g. Garland [3] proved that there is a positive-definite inner product on any standard $\hat{g}$-module (π, V) which satisfies the infinitesimal unitary condition (1). Nolan Wallach and I then "exponentiated" these representations [5] using the following general technique.

4. GENERAL EXPONENTIATION THEOREM

We return to the context of a Lie algebra b with involution $*$ and a representation (π, V) satisfying (1). Suppose that there is a monotone *scale* of norms $\|\cdot\|_t$ on V, such that for $s > t > 0$:

$$\|v\| \leq \|v\|_t \leq \|v\|_s \qquad v \in V\,.$$

We assume that these norms are compatible, so that there are inclusions

$$V \subset V_s \subset V_t \subset H$$

for $s > t > 0$, where V_t is the completion of V in the norm $\|\cdot\|_t$. We then form the inductive limit space

$$S = \lim_{t \to 0} V_t \, .$$

We want to define $e^{\pi(x)}$, for $x \in b$, as an operator on S by constructing it as a bounded operator from V_s to V_t for all $s > t$. For this, it suffices to have control over the order of singularity of the operator norm of $\pi(x)$, acting from V_s to V_t, as $s \to t^+$. A sufficient condition is the following:

THEOREM 1. *Assume that for some number p, $1 < p < \infty$, and all $s > t$, $x \in b$, and $v \in V$, one has an estimate*

$$\|\pi(x)v\|_t \leq M(s-t)^{-1/q}\|x\|\|v\|_s \, , \tag{4}$$

where q is the conjugate exponent to p and M is a fixed constant. Then:

(i) The power series for $e^{\pi(x)}v$ converges absolutely in the norm $\|\cdot\|_t$ for all $t > 0$, and one has

$$\|e^{\pi(x)}v\|_t \leq A \exp[B\|x\|^p]\|v\|_s \, ,$$

for $s > t$, with suitable constants A and B independent of x and v.

(ii) The group generated by the operators $e^{\pi(x)}$, $x \in b$, on S is a complex Banach-Lie group B with Lie algebra b. The matrix-entry functions

$$b \to (b \cdot v, w) \, , \quad \textit{for } v, w \in S$$

are holomorphic on B.

(iii) The operators $e^{\pi(x)}$, $x \in \mathcal{U}$, extend to unitary operators on H. The group generated by these operators is a real Banach-Lie group with Lie algebra $\mathcal{U}$.

The proof of (i) is obtained by iterating the *a priori* estimate (4) in steps of $\epsilon = (s-t)/n$ to obtain a bound for $\|\pi(x)^n v\|_t$ in terms of $\|v\|_{t+n\epsilon}$. The other statements are standard consequences of these estimates and the Campbell-Hausdorff formula [5, §4].

5. LOOP ALGEBRAS AND AFFINE ALGEBRAS

To apply the abstract exponentiation result just described to the affine algebras $\hat{g}$ and their standard representations, we exploit the fact that they can be presented "concretely" as central extensions of loop algebras, in the following way. We start with a finite-dimensional simple Lie algebra g over $\mathbf{C}$. From finite-dimensional representation theory the corresponding simply-connected complex group G can be faithfully represented as an algebraic subgroup of $SL_n(\mathbf{C})$ for some n. We set

$$\tilde{g} = g \otimes \mathbf{C}[t, t^{-1}] ,$$

which we can view as the functions from the circle S^1 to g having finite Fourier series (set $t = e^{i\theta}$). The invariant form $\langle \cdot, \cdot \rangle$ on g extends to a non-degenerate invariant form on $\tilde{g}$ by

$$\langle x, y \rangle := \mathrm{Res}_{t=0} t^{-1} \langle x(t), y(t) \rangle .$$

(This is just the integral of $\langle x(e^{i\theta}), y(e^{i\theta}) \rangle$ over S^1.) The operator

$$d := t \frac{d}{dt}$$

acts as a derivation on $\tilde{g}$. The bilinear form

$$\Omega(x, y) := \langle dx, y \rangle$$

on $\tilde{g}$ is then skew-symmetric and satisfies the cocycle identity. The affine algebra $\hat{g}$, originally presented via generators and relations using the extended Cartan matrix of g, turns out to be isomorphic to the central extension of $\tilde{g}$ obtained from this cocycle:

(5) $$0 \to \mathbf{C} \to \hat{g} \to \tilde{g} \to 0 .$$

Replacing the ring $\mathbf{C}[t, t^{-1}]$ by suitable Banach algebras $\mathcal{A}$ of C^∞ functions on the circle S^1 which contain the functions with finite-Fourier series as a dense subalgebra, we can similarly obtain loop algebras $\tilde{g}_{\mathcal{A}} = g \otimes \mathcal{A}$ and the corresponding central extensions $\hat{g}_{\mathcal{A}}$, which will be Banach-Lie algebras:

$$0 \to \mathbf{C} \to \hat{g}_{\mathcal{A}} \to \tilde{g}_{\mathcal{A}} \to 0 \,. \tag{6}$$

Corresponding to the loop algebras $\tilde{g}_{\mathcal{A}}$ one also has a loop group $\tilde{G}_{\mathcal{A}}$, by taking matrices in G with coefficients in $\mathcal{A}$.

THEOREM 2. *Let (V, π) be a standard module for $\hat{g}$. For any $p > 2$ there is a scale of norms $\|\cdot\|_t$ on V so that*

(i) For suitable Banach-algebras $\mathcal{A}$ of "Gevrey class" functions on S^1, the representation π extends to a representation of $\hat{g}_{\mathcal{A}}$ which satisfies estimate (4) in Theorem 1.

(ii) The Banach-Lie group $\hat{G}_{\mathcal{A}}$ generated by $\pi(\hat{g}_{\mathcal{A}})$ as in Theorem 1 is a central extension of the loop group $\tilde{G}_{\mathcal{A}}$. This extension corresponds to the central extension (6) of Lie algebras.

Remarks. 1. Just as in the case of a finite-dimensional simple algebra, the group $\hat{G}_{\mathcal{A}}$ can depend on the choice of π; however, there is a universal finite covering which is the desired central extension of the loop group $\tilde{G}_{\mathcal{A}}$.

2. The construction of the norms $\|\cdot\|_t$ and the verification of estimate (4) is based on the following properties. The degree operator d has a natural semi-simple action on any standard module, and can be normalized to be positive. The invariant bilinear form on $\hat{g}$ then gives rise to a "Casimir operator" identity of the sort

$$\pi(d) = \sum_i \pi(x_i^* x_i) \,,$$

where the infinite sum is over a suitable basis for an "upper triangular" subalgebra of $\hat{g}$. This sort of formula is familiar in quantum field theory where x_i and x_i^* appear as " annihilation and creation" operators; however, in the present case the commutation relations among x_i and x_i^* are more complicated than the usual Heisenberg relations.

Using the infinitesimal unitarity of Garland's inner product on V, we thus obtain the *a priori* estimate

$$(\pi(x)v,v) = \sum_i \|\pi(x_i)v\|^2 . \tag{7}$$

This estimate implies that the operators $\pi(x), x \in \hat{g}$ are of "order 1/2", roughly speaking, relative to $\pi(d)$, and gives rise to the condition $p > 2$ in Theorem 2. The norms $\|\cdot\|_t$ are then defined in terms of suitable functions of the self-adjoint operator $\pi(d)$, and (7) is used to establish the estimates needed in Theorem 1 (cf. [5]).

3. For application of Theorem 2 to completely integrable Hamiltonian systems, see [6].

4. The Lie algebra of vector fields on the circle has a central extension (the *Vivasoro* algebra) that has many algebraic similarities to the Kac-Moody algebras. The problem of integrating infinitesimally unitary "positive energy" irreducible representations of the Vivasoro algebra (which is not a Banach-Lie algebra) was solved in [5] and [7].

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Department of Mathematics
Rutgers University
New Brunswick
NJ 08903 USA