Harmonic Analysis and Exceptional Representations of Semisimple Groups

K.M. DAVIS, J.E. GILBERT, R.A. KUNZE

Introduction.

The purpose of this paper is to extend the results announced in the paper of Gilbert et.al. [3]. The authors showed that the concepts and techniques of Euclidean H^p theory can be applied to give realizations of ladder representations of $SO(4,1)$. (cf. Dixmier [2]). They single out for study a first-order differential operator $\mathfrak{d}$, which has the same principal symbol as the Calderon-Zygmund higher gradients operator on $\mathbb{R}^4$. The operator $\mathfrak{d}$ acts on functions with values in the space of spherical harmonics, which transform on the left according to the spherical harmonic representation $(m,0)$ of $SO(4)$. The authors showed:

1) $\mathfrak{d}$ is an elliptic differential operator.
2) The kernel of $\mathfrak{d}$, decomposed under the right-action of $SO(4)$, has a lowest K-type $(m,0)$, and the remaining K types are of the form $(m+j,0)$, $j>0$.
3) There is an embedding of limits of complementary series into the kernel of $\mathfrak{d}$, showing $\ker\mathfrak{d}$ is non-trivial.
4) Under the right action of $SO(4,1)$, the kernel of $\mathfrak{d}$ is irreducible and unitarizable.

The authors of [3] defined $\ker\mathfrak{d}$ as the intersection of the kernels of two Schmid operators (cf. Schmid [7]), and all the results of that paper followed from known results for discrete series. The ellipticity of $\mathfrak{d}$ followed from known embeddings of Schmid kernels into twisted Dirac operators; K-type information could be obtained from the Blattner multiplicity formulæ of Hotta and Parthasarathy ([4]); embeddings followed from known embeddings of discrete series into non-unitary principal series given by Knapp and Wallach [6]. Finally, unitarizability followed from known unitarizability results for limits of complementary series, established by Knapp-Stein [5].

The authors then claimed that their results extended to $SO(2n,1)$, using the same techniques. Unfortunately this is not the case; K-type analysis shows that the situation for $SO(4,1)$ does not extend to other Lorentz groups. Moreover, ladder representations exist for the non-equirank Lorentz groups $SO(2n+1,1)$, and for these the discrete series do not exist.

A theory of ladder representations for all Lorentz groups $SO(k,1)$, was developed by Davis, Gilbert and Kunze [1]; it was necessary to develop entirely new techniques to treat ellipticity, irreducibility, and unitarizability. We show that $\mathfrak{D}$ is elliptic by specifically identifying it with a twisted d, d^* system; the kernel of $\mathfrak{D}$ is shown to lie in an eigenspace of the Casimir, through a generalized Bochner-Weitzenbock formula; K type information follows from the use of classical invariant theory applied to differential forms with coefficients. The representations are identified with a subrepresentation of a non-unitary principal series, using the Szego maps and further computations with invariant theory. Finally, we show an explicit unitary structure for these representations, and give a explicit unitary equivalence with the subrepresentation. The techniques of paper [1] are thus function-theoretic, typically dealing with Hilbert spaces, while the results of [3] were largely infinitesimal, valid for smooth or K-finite functions.

In this paper we shall develop the K-finite theory of exceptional representations, begun in [3]. We begin by defining a first order elliptic system, $\mathfrak{D}$, and prove ellipticity by an infinitesimal embedding of $\ker \mathfrak{D}$ into the kernel of a twisted d, d^* system. We establish a map from a quotient of non-unitary principal series, into the kernel of $\mathfrak{D}$, using the Langlands embedding parameters given by Vogan [8]. We establish multiplicity formulæ somewhat stronger than Blattner-type results, and as we vary our lowest K-type, we exhaust the exceptional representations of $SO(2n,1)$.

Notation.

Let G be a noncompact connected semisimple Lie group with finite center; for much of the paper we shall be concerned with the case $G = SO(2n,1)$. We choose a Cartan involution θ determining maximal compact subgroup K; let $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ be the corresponding Cartan decomposition of the Lie algebra of G.

We shall assume that our Cartan subalgebra $\mathbf{t}$ maybe chosen with $\mathbf{t} \subseteq \mathbf{k}$; the nonzero roots Δ of $\mathbf{t}_{\mathbb{C}}$ acting on $\mathbf{g}_{\mathbb{C}}$ may be divided into compact roots $\Delta(\mathbf{k})$ and noncompact roots $\Delta(\mathbf{p})$. Finally, we let B denote the Killing form and $X \to \overline{X}$ congugation fixing $\mathbf{g}$ in $\mathbf{g}_{\mathbb{C}}$.

We choose a basis $\{E_\alpha, H_\alpha\}_{\alpha\in\Delta}$ for $G_{\mathbb{C}}$, where the E_α are root vectors normalized so that $\overline{E}_\alpha = E_{-\alpha}$, and $B(E_\alpha, E_{-\alpha}) = 2/\langle\alpha,\alpha\rangle$, and $H_\alpha = [E_\alpha, E_{-\alpha}]$. We chose a system of positive roots for $\mathbf{k}$, $\Delta^+(\mathbf{k})$, and a compatible system $\Delta^+(\mathbf{p})$. For a $\Delta^+(\mathbf{k})$ dominant integral form λ, let $(\pi, V) = (\pi_\lambda, V_\lambda)$ denote the corresponding irreducible representation of K with highest weight λ. Then the tensor product representation $\pi_\lambda \otimes Ad$ acting on $V_\lambda \otimes \mathbf{p}_{\mathbb{C}}$ decomposes into irreducible pieces

$$V_\lambda \otimes \mathbf{p}_{\mathbb{C}} = \sum_{\beta\in\Delta(\mathbf{p})} m_\beta V_{\lambda+\beta}\,.$$

Let

$$\Delta_* = \{\alpha \in \Delta^+(\mathbf{p}) : \langle\lambda,\alpha\rangle > 0\}$$

$$\Delta_\# = \Delta(\mathbf{p}) \setminus \Delta_*$$

and

$$V_* = \sum_{\beta\in\Delta_*} m_\beta V_{\lambda+\beta}$$

$$V_\# = \sum_{\beta\in\Delta_\#} m_\beta V_{\lambda+\beta}$$

and let $P_\# : V_\lambda \otimes P_{\mathbb{C}} \to V_\#$ denote the K-equivariant projection.

The Differential Operator.

Fix λ a dominant integral form with corresponding K-module (π, V); smooth sections of the homogeneous bundle $G \times_K V$ can be identified with covariants

$$C^\infty(G,V) = \{f : G \to V : f(kg) = \pi(k)f(g)\} .$$

The gradient operator $\nabla f = \sum_{\alpha\in\Delta(p)} \frac{1}{2}|\alpha|^2 E_\alpha f \otimes \overline{E}_\alpha$ maps $C^\infty(G,V)$ into $C^\infty(G, V \otimes \mathfrak{p}_{\mathbb{C}})$; we define the operator $\mathfrak{D}$ on $C^\infty(G,V)$ as

$$\mathfrak{D}f = (P_\# \circ \nabla)f$$

and a subspace $H^* = \{f \in C^\infty(G,V) : \mathfrak{D}f = 0\}$. As defined, $\mathfrak{D}$ is clearly a homogeneous operator, and H^* is a G-module under the right regular representation of G on $C^\infty(G,V)$. We call this the Hardy module associated to λ.

Remarks.

1.) If λ is $\Delta^+(\mathfrak{p})$ dominant, $\Delta_* = \Delta^+(\mathfrak{p})$ and the operator $\mathfrak{D}$ is the same as the operator introduced by Schmid in [7]. If $G = SO(4,1)$ and $\Delta^+(\mathfrak{p}) = \{e_1, e_2\}$, let $\lambda = me_1$. Then $\mathfrak{D}$ is the higher gradients operator introduced in [3].

2.) The case of greatest interest in this paper is for the so-called exceptional λ, that is, λ which are not the lowest K-types of discrete series or limits of discrete series. In the case of $G = SO(2n,1)$, let

$$\Delta^+(\mathfrak{k}) = \{e_i \pm e_j : i < j\} \quad , \quad \Delta^+(\mathfrak{p}) = \{e_j : 1 \le j \le n\} .$$

An exceptional λ is of the form $\lambda = \sum_{j=1}^k \lambda_j e_j$ where $k < n$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k > 0$. Then $\Delta_* = \{e_1, e_2, \ldots, e_k\}$.

LEMMA. *Let $\rho_* = \frac{1}{2}\sum_{\alpha\in\Delta_*} \alpha$. If $\lambda - 2\rho_*$ is $\Delta(\mathfrak{k})$ dominant, then $\mathfrak{D}$ is elliptic.*

Proof. Ellipticity here means that the principal symbol of $\mathfrak{D}$ is injective; this is the condition necessary to prove regularity of solutions to $\mathfrak{D}f = 0$ and to establish that H^* is a closed set in the Frechet topology on $C^\infty(G,V)$.

Since $\mathfrak{D}$ is a homogeneous differential operator, it is enough to compute the symbol at the identity coset, eK. Moreover the Killing form gives an equivariant isomorphism of the cotangent bundle with $\mathfrak{p}$. Since the projection $P_{\#}$ is just a linear combination of components of ∇f, the symbol of $\eth$ is $P_{\#} \circ$ symbol ∇. But the symbol of ∇ is $\sigma : V \times \mathfrak{p} \to V \otimes \mathfrak{p}_{\mathbb{C}}$, $\sigma(v,\xi) = v \otimes \xi$. We must show that if $\xi \in \mathfrak{p}$, $\xi \neq 0$, the map $V \to P_{\#}(v \otimes \xi)$ is injective.

If $\Delta_* = \{E_{\alpha_1}, \ldots, E_{\alpha_k}\}$ then $X_* = E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_k}$ has weight $2\rho_*$, and $\phi_{\lambda-2\rho_*} \otimes X_*$ has weight λ. This gives an equivariant embedding $\psi : V_\lambda \to V_{\lambda-2\rho_*} \otimes \Lambda^k(\mathfrak{p})$, which is non-zero, so an injection.

Now if $e(X) : \Lambda^s \to \Lambda^{s+1}$; $i(X) : \Lambda^s \to \Lambda^{s-1}$ denote exterior and interior multiplication, then both are equivariant maps, and $i(\xi)e(\xi) + e(\xi)i(\xi) = B(\xi,\bar{\xi})Id$. Define

$$E : V \otimes \mathfrak{p} \longrightarrow V_{\lambda-2\rho_*} \otimes \Lambda^{s+1}(\mathfrak{p})$$
$$I : V \otimes \mathfrak{p} \longrightarrow V_{\lambda-2\rho_*} \otimes \Lambda^{s-1}(\mathfrak{p})$$

by $E(v \otimes \xi) = e(\xi)\psi(v)$; $I(v \otimes \xi) = i(\xi)\psi(v)$; these are equivariant.

Now assume that $P_{\#}(v \otimes \xi) = 0$. It follows that $v \otimes \xi \in V_*$, and by equivariance that $E(v \otimes \xi) \in E\psi(V_*)$; $I(v \otimes \xi) \in I(V_*)$. But the irreducible components of V_* are all of the form $V_{\lambda+\beta}$ for $\beta \in \Delta_*$; we shall show that no weight in $V_{\lambda-2\rho_*} \otimes \Lambda^{s+1}(\mathfrak{p})$ can be of this form. It follows that $E(v \otimes \xi) = 0$, and a similar argument shows that $I(v \otimes \xi) = 0$. Then

$$\begin{aligned} 0 &= i(\xi)E(v \otimes \xi) + e(\xi)I(v \otimes \xi) \\ &= i(\xi)e(\xi)\psi(v) + e(\xi)i(\xi)\psi(v) \\ &= B(\xi,\bar{\xi})\psi(v) = B(\xi,\xi)\psi(v)\,. \end{aligned}$$

Since $\xi \in \mathfrak{p}$, $B(\xi,\xi) \neq 0$, so $\psi(v) = 0$. Since ψ is injective, $v = 0$. So the symbol of $\eth$ is injective.

The weights in $V_{\lambda-2\rho_*} \otimes \Lambda^{s+1}(\mathfrak{p})$ are all of the form $\lambda - 2\rho_* + \sum_{j=1}^{s+1} \gamma_j$ where $\gamma_j \in \Delta(\mathfrak{p})$, and it follows that $\lambda + \beta = \lambda - 2\rho_* + \sum \gamma_j$. We may cancel common terms between $2\rho_*$ and $\sum \gamma_j$, noting however that $2\rho_*$ has s terms, so some γ_j remain. Then $\beta = -\sum \beta_j + \sum \gamma_k$ where $\beta_j \in \Delta_*$ and $\gamma_k \notin \Delta_*$. Thus, $\langle \lambda, \beta \rangle = -\sum \langle \lambda, \beta_j \rangle + \sum \langle \lambda, \gamma_k \rangle$. Since $\beta, \beta_j \in \Delta_*$, $\gamma_k \in \Delta_{\#}$,

$$\langle \lambda, \beta \rangle + \sum \langle \lambda, \beta_j \rangle > 0 \quad , \quad \sum \langle \lambda, \gamma_k \rangle \leq 0$$

and this is a contradiction. Thus $\lambda + \beta$ does not occur, so no weights of V_* occur, so $E(v \otimes \xi) = 0$. Similarly, $I(v \otimes \xi) = 0$.

Remarks.

1.) In the case that λ is an exceptional representation of $SO(2n, 1)$,

$$\lambda = \sum_{j=1}^{k} \lambda_j e_j \quad \text{for} \quad \lambda_j \geq 1 \ ,$$

and

$$2\rho_* = \sum_{j=1}^{k} e_j \ , \quad \text{so } \lambda - 2\rho_* \text{ is always } \Delta^+(\mathbf{k}) \text{ dominant.}$$

2.) When $\eth$ is elliptic, H^* is necessarily closed in the Frechet topology on $C^\infty(G, V)$, and the right regular representation on H^* gives a continuous action.

Szego Map.

The purpose of this section is to prove that the kernel of $\eth$ is non-trivial, which we do by constructing a map from non-unitary principal series into the kernel of $\eth$. We wish to find a map from the principal series representation $U(\sigma, \nu)$ into co-variants $C^\infty(G, V)$. Technically, we must go from lowest K-type information, that is, knowledge of λ, to Langlands data, that is, a parabolic P, an Iwasawa decomposition $G = KAN$, and representations σ of M and ν of $\mathbf{A}$. For the construction of P, M and $\mathbf{A}$, we follow Vogan [8] in Proposition 4.1. The parameter ν we must determine ourselves. Our construction is special to $SO(2n, 1)$.

The Harish-Chandra parameter is defined as follows. With the given orders on compact and non-compact roots for $SO(2n, 1)$, let

$$\rho_c = \tfrac{1}{2} \sum_{\alpha \in \Delta^*(\mathbf{k})} \alpha \quad ; \quad \rho_n = \tfrac{1}{2} \sum_{\alpha \in \Delta^+(\mathbf{p})} \alpha \ .$$

Then

$$\Lambda = \lambda + \rho_c - \rho_n = \sum_{j=1}^{n} (\lambda_j + (n - j) - \tfrac{1}{2}) e_j \ .$$

When λ is exceptional, Λ is not dominant, and the simple non-compact root $\gamma = e_n$ satisfies

$$\frac{\langle \Lambda, \gamma \rangle}{\langle \gamma, \gamma \rangle} = -1 \ .$$

We take $Z_\gamma = \frac{1}{2}(e_n + \bar{e}_n)$, and define $\mathbf{A} = \mathbb{C}Z_\gamma$. M is the centralizer of $\mathbf{A}$ in K; the maximal torus $\mathfrak{t}_1$ of M contains those X in $\mathfrak{t}$ with $\gamma(X) = 0$. The roots of $(\mathfrak{t}_1, \mathfrak{m})$ then are $(e_1, e_2, \dots, e_{n-1})$ and we may form $\lambda\big|_{\mathfrak{t}_1} = \sum_{j=1}^{n-1} \lambda_j e_j$. This is a dominant integral form for M, and determines an irreducible representation (σ, W) of M.

We now introduce the notation for our non-unitary principal series. We let $G = ANK$ be an Iwasawa decomposition with co-ordinate functions given by $g = \exp H(g) n \mathcal{K}(g)$. If ν' is in the dual of $\mathbf{A}$, we look at covariants $f : G \to W$ satisfying $f(manx) = e^{\nu'(\log a)}\sigma(m)f(x)$, and the action of G is the right regular representation. This is the non-compact picture; the compact picture starts with $f \in C^\infty(K, W)$ a covariant under M, and extends f to G by covariance.

In the compact picture the Szego map is the obvious intertwining of $C^\infty(K, W)$ with $C^\infty(G, V)$, *i.e.*,

$$(Sf)(x) = \int_K \pi(k)^{-1} f(kx)\, dk \ .$$

In the non-compact version,

$$Sf(x) = \int_K e^{\nu H(\ell x^{-1})} \pi\big(\mathcal{K}(\ell x^{-1})\big)^{-1} f(\ell)\, d\ell \ ;$$

here $\nu = 2\rho^+ - \nu$ and ρ^+ is half the sum of the restricted positive roots.

PROPOSITION. *If* $\nu(Z_\gamma) = |\Delta_*|$ *then* $S : C^\infty(K, W) \to H^*$.

Proof. We must show that for $f \in C^\infty(K, W)$, $(\mathfrak{d}Sf)(x) = 0$. Since the maps are equivariant, it is enough to show that $(\mathfrak{d}Sf)(e) = 0$. If ϕ is the highest weight vector of (π, v), every covariant f in $C^\infty(G, W)$ can be written as

$$f(k) = \int_M \sigma(m^{-1}) F(m, h) \phi\, dm$$

where F is scalar valued. Then

$$(\mathfrak{d}Sf)(e) = \int_K \mathfrak{d}\Big\{\exp\big(\nu H(kx^{-1})\big)\pi\big(\mathcal{K}(kx^{-1})\big)^{-1}\phi\Big\} F(k)\, dk$$

and it is enough to show $\mathfrak{d}\{\ \} = 0$.

Now $\eth = P_{\#} \circ \nabla$ and in the orthonormal basis $\{\frac{1}{2}|\beta|^2 E_\beta\}_{\beta\in\Delta(\mathbf{p})}$ for $\mathbf{p_C}$, $\overline{E}_\beta = E_{-\beta}$ and so $\nabla f = \frac{1}{2}\sum_{\beta\in\Delta(\mathbf{p})}|\beta|^2 E_\beta f \otimes E_{-\beta}$. Now our choice was $E_\beta = X_\beta + iY_\beta$ with $X_\beta f(e) = \frac{d}{dt} f(\exp tX_\beta)\big|_{t=0}$, and it follows that

$$\eth\{\ \} = \tfrac{1}{2}P_{\#}\sum_\beta |\beta|^2 \frac{d}{dt}\left[e^{\nu H(\exp tX_\beta)}\pi\big(\mathcal{K}(\exp tX_\beta)\big)^{-1}\phi\right]_{t=0} \otimes E_{-\beta} + \frac{i}{2}$$

times a similar term in Y_β. Using the product rule

$$\frac{d}{dt}\left[e^{\nu H(\exp tX_\beta)}\right]_{t=0}\pi\big(\mathcal{K}(e)\big)\phi + e^{\nu H(e)}\frac{d}{dt}\left[\pi\big(\mathcal{K}(\exp tX_\beta)\big)^{-1}\phi\right]_{t=0}$$
$$= \nu(P_{\mathbf{A}}X_\beta)\phi - (P_{\mathbf{k}}X_\beta)\phi$$

where $P_{\mathbf{A}}$, $P_{\mathbf{k}}$ are projections of $\mathbf{g}$ onto the Iwasawa components in the decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{A} \oplus \mathbf{N}$. Thus,

$$\eth\{\ \} = \tfrac{1}{2}P_{\#}\sum_\beta |\beta|^2\{\nu(P_{\mathbf{A}}E_\beta) - \pi(P_{\mathbf{k}}E_\beta)\}\,\phi\otimes E_{-\beta}\ .$$

Although these projections are very easy to compute for $SO(2n,1)$, we shall use the computations in Knapp-Wallach [6] which apply in general:

$$\begin{aligned} P_{\mathbf{A}}(E_\beta) &= Z_\gamma \quad \text{if}\ \ \beta = \pm\gamma \\ &= 0 \qquad \text{otherwise} \\ P_{\mathbf{k}}(E_\beta) &= \tfrac{1}{2}H_{\pm\gamma} \quad \text{if}\ \ \beta = \pm\gamma \\ &= -\frac{2}{p+q}[Z_\gamma, E_\beta] \quad \text{otherwise.} \end{aligned}$$

Here p, q refer to the γ string containing β, $\beta + n\gamma$, $-p \le n \le q$. Using this, and dropping the common factor of $\frac{1}{2}$, $2\eth\{\ \}$ is

$$\begin{aligned} &P_{\#}\left\{|\gamma|^2\nu(Z_\gamma)\phi\otimes E_{-\gamma} + |\gamma|^2\nu(Z_\gamma)\phi\otimes E_\gamma\right\} \\ &- \tfrac{1}{2}|\gamma|^2 P_{\#}\{\pi(H_\gamma)\phi\otimes E_{-\gamma} + \pi(H_{-\gamma})\phi\otimes E_\gamma\} \\ &+ \sum_{\beta\neq\pm\gamma}\frac{|\beta|^2}{p+q}P_{\#}\{\pi[Z_\gamma, E_\beta]\phi\otimes E_{-\beta}\} \end{aligned}$$

First, our choice of γ as orthogonal to λ guarantees that $\pi(H_{\pm\gamma})\phi = 0$; we also claim that if $\beta > 0$, $\pi([Z_\gamma, E_\beta])\phi = 0$. This follows since for $\beta > 0$, $\beta \neq \gamma$, $\beta \pm \gamma$ is a positive root, hence the root vector annihilates the highest weight vector. The remaining terms are

$$|\gamma|^2 P_{\#}\left\{\nu(Z_\gamma)\phi \otimes Z_\gamma\right\} + \sum_{\substack{\beta<0\\ \beta\neq-\gamma}} \frac{|\beta|^2}{p+q} P_{\#}\left\{\pi[Z_\gamma, E_\beta]\phi \otimes E_{-\beta}\right\} .$$

There are three types of $\beta < 0$ with $\beta \neq -\gamma$:

i) $\langle -\beta, \lambda\rangle > 0 \quad i.e. \quad -\beta \in \Delta_*$

ii) $\langle -\beta, \lambda\rangle = 0$

iii) $\langle -\beta, \lambda\rangle < 0$

For exceptional λ in $SO(2n, 1)$, case iii) cannot occur. If $\langle \beta, \lambda\rangle = 0$, $\langle \gamma, \beta\rangle = 0$, so that $\pi[Z_\gamma, E_\beta]\phi = 0$. Thus, only terms involving $-\beta \in \Delta_*$ contribute, and, replacing β by $-\beta$, we have

$$|\gamma|^2 P_{\#}\left\{\nu(Z_\gamma)\phi \otimes Z_\gamma\right\} + \sum_{\beta\in\Delta_*} \frac{|\beta|^2}{p+q} P_{\#}\left\{\pi[Z_\gamma, E_{-\beta}]\phi \otimes E_\beta\right\} .$$

To simplify the second term, we claim that for $\beta \in \Delta_*$, $\phi \otimes E_\beta \in V_*$. Otherwise, $\phi \otimes E_\beta$ occurs in $V_{\#}$, and the weights in $V_{\#}$ are of the form $\lambda + \delta + \sum n_i\alpha_i$, $\delta \in \Delta_{\#}$, $n_i \leq 0$, $\alpha_i \in \Delta^+(\mathbf{k})$. Thus

$$\begin{aligned} \lambda + \beta &= \lambda + \delta + \sum n_i\alpha_i \quad \text{and} \\ \langle \lambda, \beta\rangle &= \langle \lambda, \delta\rangle + \sum n_i\langle \lambda, \alpha_i\rangle \quad \text{or} \\ \langle \lambda, \beta\rangle - \sum n_i\langle \lambda, \alpha_i\rangle &= \langle \lambda, \delta\rangle . \end{aligned}$$

But $\langle \lambda, \beta\rangle > 0$; $-n_i\langle \lambda, \alpha_i\rangle \geq 0$ and $\langle \lambda, \delta\rangle \leq 0$. This is a contradition.

Since $\phi \otimes E_\beta \in V_*$, $P_{\#}(\phi \otimes E_\beta) = 0$, so

$$\begin{aligned} 0 &= \pi_{\#}([Z_\gamma, E_{-\beta}])P_{\#}\{\phi \otimes E_\beta\} \\ &= P_{\#}\Big\{(\pi \otimes Ad)[Z_\gamma, E_{-\beta}](\phi \otimes E_\beta)\Big\} \\ &= P_{\#}\Big\{(\pi[Z_\gamma, E_{-\beta}]\phi) \otimes E_\beta + \phi \otimes \big[[Z_\gamma, E_{-\beta}], E_\beta\big]\Big\} . \end{aligned}$$

Simple computations show that the triple bracket is $2Z_\gamma$, whence

$$P_\#\{\pi[Z_\gamma, E_{-\beta}]\phi \otimes E_\beta\} = -2P_\#\{\phi \otimes Z_\gamma\} ,$$

and the sum collapses to

$$|\gamma|^2 P_\# \left\{ \nu(Z_\gamma)\phi \otimes Z_\gamma - \Big(\sum_{\beta \in \Delta_*} \frac{2|\beta|^2}{p+q} \Big) \phi \otimes Z_\gamma \right\} .$$

For $SO(2n,1)$, $p = q = 1$ and $|\beta| = |\gamma|$, whence we obtain

$$|\gamma|^2\{\nu(Z_\gamma) - |\Delta_*|\} P_\#(\phi \otimes Z_\gamma) = 0 .$$

Remark. We relate these to non-unitary principal series parameters as follows. $\nu' = 2\rho^+ - \nu = \rho^+ + i\mu$ so that $i\mu = \rho^+ - \nu$. But $\rho^+(Z_\gamma) = \frac{2n-1}{2}$, $\nu(Z_\gamma) = |\Delta_*|$, so that $i\mu(Z_\gamma) = \big(1 - \frac{2|\Delta_*|}{2n-1}\big)\rho^+(Z_\gamma)$. For exceptional λ, $2|\Delta_*| < 2n$ so $i\mu$ is real and in the positive chamber. Comparison with Knapp and Stein [5] shows that this parameter is a non-unitary principal series at the limit of complementary series.

LEMMA. *H^* is nontrivial.*

Proof. It is enough to prove that there is an $f \in C^\infty(K, W)$ with $Sf \equiv 0$, in particular, $Sf(e) \neq 0$. Let $P : V \to W$ be the M-equivariant projection, and let $f(\ell) = P\big(\pi(\ell)\phi\big)$. Then f is in $C^\infty(K, W)$, and f is not identically zero since, by our construction, $P(\phi) \neq 0$. But

$$\begin{aligned} (Sf(e), \phi)_V &= \int_K e^{\nu H(\ell)} \Big(\pi\big(K(\ell)\big)^{-1} P\pi(\ell)\phi, \phi\Big)_V \, d\ell \\ &= \int_K \big(P\pi(\ell)\phi, \pi(\ell)\phi\big)_V \, d\ell \\ &= \int_K |P\pi(\ell)\phi|^2 \, d\ell > 0 . \end{aligned}$$

Multiplicity.

In this section we generalize the classical result that if f is holomorphic in the unit disc, f has Fourier coefficients supported on the cone $\mathbb{Z}^+$. The analogous result for Hardy modules concerns those representations which occur when H^* is restricted to K. We obtain estimates on the K-types which occur in this restriction. These follow because the operator $\mathfrak{D}$ is designed to incorporate K-type information. Intuitively, if a K-type π_μ occurs in $f \in C^\infty(G, V)$ then f transforms on the right by π_μ. This means that right invariant vector fields $X \in \mathbf{k}_\mathbb{C}$ map f into a function transforming by π_μ again. But for $X \in \mathbf{p}_\mathbb{C}$, Xf can have values anywhere in $V_\mu \otimes \mathbf{p}_\mathbb{C}$. The gradient operator $\nabla f = \sum_{\beta \in \Delta(\mathbf{p})} E_\beta f \otimes E_{-\beta}$ incorporates all these $\mathbf{p}_\mathbb{C}$ actions. The condition that $\mathfrak{D} f = 0$ is equivalent to $\nabla f \in V_*$, and this restricts the possible K-types which can occur in the right regular representation acting on f. Specifically, the condition is that ∇f can contain only those K-types with highest weights of the form $\lambda + \beta$, $\beta \in \Delta_*$.

For functions which can be recovered from their Taylor series, that is, from repeated applications of ∇, we would expect that $\mathfrak{D} f = 0$ means the only K-types which may occur on H^* are those of the form $\lambda + \sum n_i \beta_i$ where $n_i \geq 0$ and $\beta_i \in \Delta_*$. That is, the K-types lie in a cone based at λ, having generators consisting of the roots in Δ_*.

Multiplicity formulæ make this intuition precise. Our arguments follow the work of Hotta-Parthasarathy [4], and proceed as follows. A linear differential operator on scalar functions has Taylor coefficients which are symmetric tensors, or they may be viewed as symmetric polynomials on $\mathbf{p}$. Functions in $C^\infty(G, V)$ have ℓ^{th} Taylor coefficients with values in $S^\ell(\mathbf{p}) \otimes V$. The operator $\mathfrak{D}$ extends to a map $\mathfrak{D}_\ell : S^\ell(\mathbf{p}) \otimes V \to S^{\ell-1}(\mathbf{p}) \otimes V$, called the polynomialization. Ellipticity implies that we can recover information on H^* from information about the kernels of the $\mathfrak{D}_\ell$.

The major idea in establishing multiplicity estimates for $\ker \mathfrak{D}_\ell$ is that the desired estimates are true by definition at the level of B-modules. To go from B-modules to K-modules, we use the Borel-Weil-Bott theorem; the various tensor products can be handled through standard arguments from algebraic geometry and cohomology. The only caveat is that $\mathfrak{D}_\ell$ must be embedded into an exact sequence.

We introduce an auxiliary operator which is a combination of a de-Rham and a Dobeault operator, and exactness follows. Our multiplicity results then follow after an analysis of the long exact sequence.

We begin with notation. Let $\mathcal{B}$ denote the Borel subalgebra $\mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha\in\Delta^+(\mathfrak{k})} \mathbb{C}E_\alpha$, with corresponding Borel subgroup B. Let $S = K^{\mathbb{C}}/B$, $s = \dim S$. For any holomorphic B-module $\mathcal{M}$, we form the homogeneous bundle $\nu_{\mathcal{M}} = K^{\mathbb{C}} \times_B \mathcal{M}$, and the sheaf of germs of holomorphic sections $\mathcal{O}(\nu_{\mathcal{M}})$. The sheaf cohomology $H^i(S, \mathcal{O}(\nu_{\mathcal{M}}))$ is written $H^i(\mathcal{M})$. If μ is a dominant integral form, μ extends to a line bundle ℓ_μ.

The Borel-Weil-Bott theorem states:

1) If $\mu - \rho_c$ is singular, $H^i(\ell_\mu) = 0$ for all i,
2) If $\mu - \rho_c$ is non-singular, there is a unique $w \in W(K,T)$ for which $w(\mu - \rho_c)$ is dominant. Let

$$i_\mu = \left|\left\{\alpha \in \Delta^+(\mathfrak{k}) : (\mu - \rho_c, \alpha) > 0\right\}\right| .$$

Then $H^i(\ell_\mu) = 0$ if $i \neq i_\mu$. $H^i(\ell_\mu)$ is an irreducible K-module of highest weight $w(\mu - \rho_c) - \rho_c$ if $i = i_\mu$.

It follows then that $V_\mu = H^s(\ell_{\mu+2\rho_c})$ if μ is dominant. We will also need the result that for a K-module $\mathcal{M}$, $\mathcal{M} \otimes H^i(\mathcal{N}) = H^i(\mathcal{M} \otimes \mathcal{N})$.

We now define the action of $\bar{\mathfrak{d}}$ on Taylor coefficients. Let $\mathfrak{p}_* = \sum_{\beta\in\Delta_*} \mathbb{C}E_\beta$, $\mathfrak{p}_\# = \mathfrak{p}_{\mathbb{C}}/\mathfrak{p}_*$; let $p_\# : \mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}_\#$ denote the projection. Define

$$\bar{\mathfrak{d}}_\ell : \; S^\ell(\mathfrak{p}_{\mathbb{C}}) \otimes V \to S^{\ell-1}(\mathfrak{p}_{\mathbb{C}}) \otimes V_\# \;\text{ by }\; (1 \otimes p_\#) \circ (d \otimes 1) \; ;$$

$$d \otimes 1 : \; S^\ell(\mathfrak{p}_{\mathbb{C}}) \otimes V \to S^{\ell-1}(\mathfrak{p}_{\mathbb{C}}) \otimes \mathfrak{p}_{\mathbb{C}} \otimes V$$

$$1 \otimes p_\# : \; S^{\ell-1}(\mathfrak{p}_{\mathbb{C}}) \otimes \mathfrak{p}_{\mathbb{C}} \otimes V \to S^{\ell-1}(\mathfrak{p}_{\mathbb{C}}) \otimes V_\# \; .$$

To define the Taylor coefficient, set

$$I^\ell = \{ \; f \in C^\infty(G, V) : (D^\alpha f)(\epsilon K) = 0 \;\text{ for }\; |\alpha| \leq \ell - 1 \; \}$$

and let F^ℓ denote the K-finite vectors in $H^* \cap I^\ell$; $F^\ell = H^*(K) \cap I^\ell$.

PROPOSITION. *If $\lambda - 2\rho_n$ is $\Delta^n(\mathfrak{k})$ dominant, there are injections which are K-equivariant, mapping*

$$H^*(K) \longrightarrow \oplus(F^\ell/F^{\ell-1}) \longrightarrow \oplus \ker \eth_\ell .$$

Proof. We can easily map a K-finite vector into $\oplus F^\ell/F^{\ell-1}$ by taking its Taylor series. If this is not an injection on $H^*(K)$, the image of f is in F^ℓ for all ℓ, that is, all derivatives at eK vanish. But f is in the kernel of $\eth$, and the dominance condition on λ guarantees that $\eth$ is elliptic. Thus f is real analytic, hence f is zero identically.

For the next stage, we construct equivariant injections $i : F^\ell/F^{\ell-1} \to \ker \eth_\ell$. We choose local co-ordinate functions $x = (x_1, \ldots, x_n)$ satisfying $x(eK) = 0$. In local co-ordinates,

$$\eth = \sum a_j(x)\frac{\partial}{\partial x_j} + b(x) , \quad \text{where } b(eK) = 0$$

and $a_j(eK) = (1 \otimes p_\#)(dx^j \otimes -)$ is the projection. The map $i_0 : F^\ell \to S^\ell(\mathfrak{p}) \otimes V$ is defined as

$$i_0(s) = \sum_{|\alpha|=\ell} \frac{1}{\alpha!}(dx)^\alpha \otimes \left(\frac{\partial^\ell s}{\partial x^\alpha}\right)(eK) .$$

Notice that $\ker i_0 \cap I^\ell = I^{\ell+1}$, so that i_0 descends to an injection i on $F^\ell/F^{\ell+1}$. To finish, we need to show that if $s \in \ker \eth \cap I^\ell$, $i(s) \in \ker \eth_\ell$. But for $s \in I^\ell$, $(\frac{\partial s}{\partial x^\alpha})(eK) = 0$ for $|\alpha| \leq \ell - 1$, and therefore

$$\begin{aligned} i(\eth(s)) &= i\left(\sum a_j(x)\frac{\partial s}{\partial x_j} + b(x)s\right) \\ &= \sum_{|\beta|=\ell-1} \sum_{j=1}^{n} \frac{1}{\beta!}(dx)^\beta \otimes a_j(eK)\frac{\partial^\ell s}{\partial x^\beta \partial x_j}(eK) \\ &\qquad + \text{terms of lower order derivatives on } s \\ &\qquad + b(eK)\ [\text{derivatives on } s]. \end{aligned}$$

The terms with lower order derivatives on the s are zero since $s \in I^\ell$; $b(eK) = 0$, so in all,

$$i(\eth s) = \sum_{j=1}^{n} \sum_{|\alpha|=\ell} \frac{\alpha_j}{\alpha!}(dx)^{\alpha(j)} \otimes a_j(eK)\left[\frac{\partial^\ell s}{\partial x^\alpha}\right](eK) .$$

Here $\alpha(j)$ indicates that the exponent of α has been decreased by 1 in the j^{th} position.

We now compare this with $\mathfrak{d}_\ell(i(s)) = (1 \otimes p_\#)(d \otimes 1)(i(s))$.

$$(d \otimes 1)(i(s)) = d \otimes 1 \Big[\sum_{|\alpha|=\ell} \frac{1}{\alpha!} (dx)^\alpha \otimes \Big[\frac{\partial^\ell s}{\partial x^\alpha}\Big](eK) \Big]$$

$$= \sum_{|\alpha|=\ell} \frac{1}{\alpha!} d(dx)^\alpha \otimes \Big[\frac{\partial^\ell s}{\partial x^\alpha}\Big](eK)$$

$$= \sum_{j=1}^{n} \sum_{|\alpha|=\ell} \frac{\alpha^j}{\alpha!} (dx)^{\alpha(j)} \otimes dx^j \otimes \Big[\frac{\partial^\ell s}{\partial x^\alpha}\Big](eK) .$$

But $1 \otimes p_\#$ maps $dx^j \otimes v$ into $a_j(eK)v$, so that $\mathfrak{d}_\ell(i(s))$ is just

$$\sum_{j=1}^{n} \sum_{|\alpha|=\ell} \frac{\alpha_j}{\alpha!} (dx)^{\alpha(j)} \otimes a_j(eK) \Big[\frac{\partial^\ell s}{\partial x^\alpha}\Big](eK) = i(\mathfrak{d}s) .$$

Our next task is to construct the exact sequence into which $\mathfrak{d}_\ell$ can be embedded. We proceed by constructing the B-module maps first, and then follow through the effects of taking cohomology.

We order the basis $\{E_\beta\}$ of $\mathfrak{p}_\mathbb{C}$ by

$$\Delta_* = \{\alpha_1, \ldots, \alpha_n\}$$

$$-\Delta_* = \{\beta_1, \ldots, \beta_n\}$$

$$\Delta \setminus \{\Delta_* \cup -\Delta_*\} = \{\gamma_1, \ldots, \gamma_p\}$$

Then $\mathfrak{p}_\mathbb{C}$ is co-ordinatized as

$$X = \sum z_i E_{\alpha_i} + \sum \overline{z}_i E_{\beta_i} + \sum t_i E_{\gamma_i} .$$

Let

$$\mathcal{E}^k = \text{ span of } \big\{ a(\overline{z}, t) d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_q} \wedge \cdots \wedge dt_{j_r} : q + r = k \big\} .$$

Then $\Lambda^k(\mathfrak{p}_\#) = \mathcal{E}^k$. The Dobeault-deRham operator $\overline{\partial}+d$ yields the exact sequence:

$$0 \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E}^1 \longrightarrow \cdots \longrightarrow \mathcal{E}^n \longrightarrow 0 ;$$

the sequence with polynomial coefficients is also exact:

$$0 \longrightarrow S^\ell(\mathrm{p}) \longrightarrow S^{\ell-1}(\mathrm{p}) \otimes \Lambda^1(\mathrm{p}_\#) \longrightarrow \cdots \longrightarrow 0\,.$$

Finally, $S^\ell(\mathrm{p}_*)$ embeds into this sequence from the obvious injection of $S^\ell(\mathrm{p}_*)$ into $S^\ell(\mathrm{p})$:

$$0 \longrightarrow S^\ell(\mathrm{p}_*) \longrightarrow S^\ell(\mathrm{p}) \longrightarrow S^{\ell-1}(\mathrm{p}) \otimes \Lambda^1(\mathrm{p}_\#) \longrightarrow \cdots\,.$$

These sequences are defined by maps

$$\partial_k : S^{\ell-k}(\mathrm{p}) \otimes \Lambda^k(\mathrm{p}_\#) \longrightarrow S^{\ell-k-1}(\mathrm{p}) \otimes \Lambda^{k+1}(\mathrm{p}_\#)$$

defined by the composition of maps

$$\begin{aligned} \nabla \otimes 1 :\ & S^{\ell-k}(\mathrm{p}) \otimes \Lambda^k(\mathrm{p}_\#) \longrightarrow S^{\ell-k-1}(\mathrm{p}) \otimes \mathrm{p} \otimes \Lambda^k(\mathrm{p}_\#) \\ 1 \otimes \mathcal{E} \circ p_\# :\ & S^{\ell-k-1}(\mathrm{p}) \otimes \mathrm{p} \otimes \Lambda^k(\mathrm{p}_\#) \longrightarrow S^{\ell-k-1}(\mathrm{p}) \otimes \Lambda^{k+1}(\mathrm{p}_\#)\,. \end{aligned}$$

Since $p_\#(E_{\alpha_i}) = 0$, $p_\#(E_{\beta_i}) = E_{\beta_i}$, $p_\#(E_{\gamma_i}) = E_{\gamma_i}$,

$$\begin{aligned} (1 \otimes \mathcal{E}p_\#)(\nabla \otimes 1)(f \otimes w) &= (1 \otimes \mathcal{E}p_\#)\Big(\sum \frac{\partial f}{\partial x_i} \otimes x_i \otimes w\Big) \\ &= \sum \frac{\partial f}{\partial x_i} \otimes \overline{E}_{\beta_i} \wedge w + \sum \frac{\partial f}{\partial x_i} \otimes E_{\gamma_i} \wedge w \\ &= \overline{\partial}(f \otimes w) + d(f \otimes w)\,. \end{aligned}$$

Therefore the ∂_k complex is exact. Let $\mathcal{L}$ denote the line bundle $\ell_{\lambda+2\rho_c}$; tensoring with $\mathcal{L}$ preserves the exact sequence, so in all,

$$0 \longrightarrow S^\ell(\mathrm{p}_*) \otimes \mathcal{L} \xrightarrow{\ i\ } S^\ell(\mathrm{p}) \otimes \mathcal{L} \xrightarrow{\ \partial_\ell\ } S^{\ell-1}(\mathrm{p}) \otimes \Lambda^1(\mathrm{p}_\#) \otimes \mathcal{L} \longrightarrow \cdots$$

is exact.

PROPOSITION.

$$0 \longrightarrow H^s(S^\ell(\mathrm{p}_*) \otimes \mathcal{L}) \xrightarrow{\ i_*\ } H^s(S^\ell(\mathrm{p}) \otimes \mathcal{L}) \xrightarrow{\ \partial_1^*\ } H^s(S^{\ell-1}(\mathrm{p}) \otimes \Lambda^1(\mathrm{p}_\#) \otimes \mathcal{L})$$

is exact.

Proof. Taking cohomology of a B-module exact sequence leads to a long exact sequence; exactness at the top degree follows from standard arguments if we can show vanishing of low degree cohomology. This basically involves a computation of what representations may occur in decomposing a tensor product. We shall show that $H^i(S^{\ell-q}(\mathbf{p})\otimes\Lambda^q(\mathbf{p}_\#)\otimes\mathcal{L})=0$ and $H^i(S^\ell(\mathbf{p}_*)\otimes\mathcal{L})=0$ if $i<s$.

For the first result, $S^{\ell-q}(\mathbf{p})$ is already a K-module, so that

$$H^i(S^{\ell-q}(\mathbf{p})\otimes\Lambda^q(\mathbf{p}_\#)\otimes\mathcal{L}) = S^{\ell-q}\otimes H^i(\Lambda^q\otimes\mathcal{L})\ .$$

It is enough then to show that $H^i(\Lambda^q\otimes\mathcal{L})=0$. We use a technical device that reduces the computation to B-modules.

Since B is solvable, there is a chain of B-modules V_i satisfying $0=V_0\subset V_1\subset\cdots\subset V_r=\Lambda^q(\mathbf{p}_\#)\otimes\mathcal{L}$, with $0\to V_{j-1}\to V_j\to\ell_{\lambda+2\rho_c+\beta}\to 0$, where β runs through all the weights of $\Lambda^q(\mathbf{p}_\#)$. Such β are of the form $\sum\beta_i$ for $\beta_i\in\Delta_\#$, and since λ is exceptional, $(\lambda+2\rho_c+\beta)-\rho_c$ is already $\Delta^+(\mathbf{k})$ dominant for every such β. It follows that $H^i(\ell_{\lambda+2\rho_c+\beta})=0$ for $i<s$, and the long-exact sequence now gives

$$\longrightarrow H^i(V_{j-1})\longrightarrow H^i(V_j)\longrightarrow H^i(\ell_{\lambda+2\rho_c+\beta})\longrightarrow\cdots\ .$$

Since $H^i(V_0)=0$, an induction gives $0\to H^i(V_j)\to 0$, so that
$0=H^i(V_r)=H^i(\Lambda^q\otimes\mathcal{L})$.

We also need vanishing for $H^i(S^\ell(\mathbf{p}_*)\otimes\mathcal{L})$; this is less simple, since $S^\ell(\mathbf{p}_*)$ has more complicated weights β and $\lambda+\rho_c+\beta$ need not be dominant. Let $\mathcal{F}^q=S^{\ell-q}(\mathbf{p})\otimes\Lambda^q(\mathbf{p}_\#)$ and let $W^q=$ image at the $q-1$ place in the sequence

$$0\longrightarrow S^\ell(\mathbf{p}_*)\longrightarrow S^\ell(\mathbf{p})\longrightarrow S^{\ell-1}(\mathbf{p})\otimes\Lambda^1(\mathbf{p}_\#)\longrightarrow\cdots\ .$$

We want to prove $H^i(W^0\otimes\mathcal{L})=0$, and of course we proceed backwards through the long exact sequence, inductively.

We have B-module exact sequences $0\to W^q\to\mathcal{F}^q\to W^{q+1}\to 0$. yielding $0\to W^q\otimes\mathcal{L}\to\mathcal{F}^q\otimes\mathcal{L}\to W^{q+1}\otimes\mathcal{L}\to 0$, and yielding a long exact sequence.

$$\longrightarrow H^i(W^q\otimes\mathcal{L})\longrightarrow H^i(\mathcal{F}^q\otimes\mathcal{L})\longrightarrow H^i(W^{q+1}\otimes\mathcal{L})\longrightarrow H^{i+1}\cdots\ .$$

At the last position, ($m = \dim \mathfrak{p}_\#$) $W^m = \text{image}\, \partial_{m-1} = \ker \partial_m = S^{\ell-m}(\mathfrak{p}) \otimes \Lambda^m(\mathfrak{p}_\#)$ so that for $i < s$,

$$H^i(W^m \otimes \mathcal{L}) = S^{\ell-m}(\mathfrak{p}) \otimes H^i(\Lambda^m(\mathfrak{p}_\#) \otimes \mathcal{L}) = 0$$

by the above computation. Now, inductively assume that for all $i < s$, $H^i(W^{q+1} \otimes \mathcal{L}) = 0$. We claim that $H^i(W^q \otimes \mathcal{L}) = 0$. This occurs because

$$H^{i-1}(W^{q+1} \otimes \mathcal{L}) \longrightarrow H^i(W^q \otimes \mathcal{L}) \longrightarrow H^i(\mathcal{F}^q \otimes \mathcal{L}) \longrightarrow (W^{q+1} \otimes \mathcal{L})$$

or $0 \to H^i(W^q \otimes \mathcal{L}) \to H^i(\mathcal{F}^q \otimes \mathcal{L}) \to 0$, that is,

$$H^i(W^q \otimes \mathcal{L}) \cong H^i(\mathcal{F}^q \otimes \mathcal{L}) \cong S^{\ell=q}(\mathfrak{p}) \otimes H^i(\Lambda^q(\mathfrak{p}_\#) \otimes \mathcal{L}) = 0\,.$$

Inductively, $H^i(W^0 \otimes \mathcal{L}) = 0$, but $W^0 = \text{image}\, i = S^\ell(\mathfrak{p}_*)$.

COROLLARY. $\ker \partial_\ell^* = H^s(S^\ell(\mathfrak{p}_*) \otimes \mathcal{L})$.

Proof. This is the meaning of the exactness of the sequence in the first place.

To finish the proof, we need to show that $\ker \eth_\ell \subseteq \ker \partial_\ell^*$. We begin by remarking that ∂_ℓ^* has domain $H^s(S^\ell(\mathfrak{p}) \otimes \mathcal{L}) = S^\ell(\mathfrak{p}) \otimes H^s(\mathcal{L}) = S^\ell(\mathfrak{p}) \otimes V =$ domain $\eth_\ell$. Now ∂_ℓ^* is the lift from B-modules of the composition $(1 \otimes p_\# \otimes 1) \circ (d \otimes 1)$, whilst $\eth_\ell$ is the composition $(1 \otimes P_\#) \circ (d \otimes 1)$. It is enough then to see that $\ker P_\# \subseteq \ker(1 \otimes p_\#)^*$. Since the maps are equivariant, and $P_\#$ is a projection, it is enough to prove that the multiplicity of V_* in $H^s(\mathfrak{p}_\# \otimes \mathcal{L})$ is zero.

LEMMA. *Let $\mathcal{S} = \{$weights of $S^\ell(\mathfrak{p}_*)\}$, and choose any $\beta \in \mathcal{S}$. Assume there is a $w \in W(K,T)$ such that $w(\lambda + 2\rho_c + \beta)$ is dominant. Let i_β be the parity of*

$$\{\ \alpha \in \Delta^+(\mathfrak{k}) : (w(\lambda + \rho_c + \beta) - \rho_c, \alpha) > 0\ \}\,.$$

Then

$$H^s(S^\ell(\mathfrak{p}_*) \otimes \mathcal{L}) = (-1)^s \sum_{\beta \in \mathcal{S}} i_\beta V_{w(\lambda+\rho_c+\beta)-\rho_c}\,.$$

Proof. As always, there is a chain of B-modules

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = S^\ell(\mathbf{p}_*) \otimes \mathcal{L}$$

with $0 \longrightarrow V_{j-1} \longrightarrow V_j \longrightarrow \ell_{\lambda+2\rho_c+\beta} \longrightarrow 0$. Here β is a weight of $S^\ell(\mathbf{p}_*)$, and as j varies, β runs through all $\mathcal{S}$. Since the Euler characteristic is additive on short exact sequences, and $\chi(V_0) = 0$,

$$\begin{aligned}
\chi(V_r) &= \sum (-1)^i H^i(S^\ell(\mathbf{p}_*) \otimes \mathcal{L}) \\
&= (-1)^s H^s\big(S^\ell(\mathbf{p}_*) \otimes \mathcal{L}\big) = \chi(V_{r-1}) + \chi(\ell_{\lambda+2\rho_c+\beta}) \\
&= \cdots = \sum_{\beta \in \mathcal{S}} \chi(\ell_{\lambda+2\rho_c+\beta}) \\
&= \sum_{\beta \in \mathcal{S}} \sum (-1)^i H^i(\ell_{\lambda+2\rho_c+\beta}) \\
&= \sum_{\beta \in \mathcal{S}} i_\beta V_{w(\lambda+\rho_c+\beta)-\rho_c} \quad \text{by the Borel-Weil-Bott theorem.}
\end{aligned}$$

LEMMA. *If V_μ is a K-module of non-zero multiplicity in V_*, it has zero multiplicity in $H^s(\mathbf{p}_\# \otimes \mathcal{L})$.*

Proof. Because of the cohomology vanishing results we have proved for $H^i(\mathbf{p}_\# \otimes \mathcal{L})$, the Euler characteristic computations above apply, and

$$H^s(\mathbf{p}_\# \otimes \mathcal{L}) = (-1)^s \sum_{\beta \in \mathcal{S}} i_\beta V_{w(\lambda+\rho_c+\beta)-\rho_c} .$$

But for our exceptional λ, $\lambda + 2\rho_c + \beta$ is already dominant for $\beta \in \mathbf{p}_\#$, hence $w = id$ and $i_\beta = (-1)^s$. Thus, $H^s(\mathbf{p}_\# \otimes \mathcal{L}) = \sum_{\beta \in \Delta_\#} V_{\lambda+\beta}$. Since $V_* = \sum_{\beta \in \Delta_*} m_\beta V_{\lambda+\beta}$ and $\Delta_* \cap \Delta_\# = \phi$, V_* has no components in $H^s(\mathbf{p}_\# \otimes \mathcal{L})$.

COROLLARY. $\ker \eth_\ell \subseteq H^s\big(S^\ell(\mathbf{p}_*) \otimes \mathcal{L}\big)$.

COROLLARY. $H^*(K) \subset \sum_{\ell=0}^{\infty} \sum_{\beta \in \mathcal{S}} (-1)^s i_\beta V_{w(\lambda+\rho_c+\beta)-\rho_c}$.

Proof. $H^*(K)$ injects equivariantly into $\bigoplus_{\ell=0}^{\infty} \ker \eth_\ell$, whose K-module components are as written.

COROLLARY. *The K-types in $H^*(K)$ are all of the form*

$$\lambda + \sum n_i\beta_i ,$$

where $n_i \geq 0$ and $\beta_i \in \Delta_*$. Moreover, λ occurs with multiplicity 1 in $H^*(K)$.

Proof. Every K-type in $H^*(K)$ is of the form $w(\lambda + \rho_c + \beta) - \rho_c$, where $w(\lambda + 2\rho_c + \beta)$ is dominant, and $\beta \in \mathcal{S}$. Since β is necessarily a sum of roots in Δ_*, the $w \in W(K,T)$ which occur necessarily only permute the Δ_* roots. It follows that $w(\lambda + \rho_c + \beta) - (\lambda + \rho_c + \beta) = \sum n_i\beta_i$ where $n_i \geq 0$ and $\beta_i \in \Delta_*$. Then $w(\lambda + \rho_c + \beta) - \rho_c = \lambda + \beta + \sum n_i\beta_i$, as claimed.

To prove that V_λ occurs with multiplicity one, we remark that we may only obtain λ if $w(\lambda+\rho_c+\beta)-\rho_c = \lambda$. But $w(\lambda+\rho_c+\beta)-\rho_c$ is $\lambda+\sum n_i\beta_i$, so $n_i = 0$. Thus $w(\lambda + \rho_c + \beta) = \lambda + \rho_c$. Now $w(\lambda + \rho_c + \beta) - (\lambda + \rho_c + \beta)$ is of the form $\sum m_i\gamma_i$ where $m_i \geq 0$ and $\gamma_i \in \Delta^*(\mathbf{k})$, but is is also equal to $-\beta$. The only way this can occur is $\beta = 0$. But $\lambda + \rho_c$ is already dominant, so $w = id$. But the only time $\beta = 0$ occurs in $S^\ell(\mathrm{p}_*)$ is for $\ell = 0$. Thus, λ occurs with multiplicity one.

Irreducibility.

The purpose of this section is to prove that H^* contains a unique irreducible subrepresentation H_λ, which may be characterized either as the space generated by the K-type π_λ, or as the image of the Szego map. This latter identifies it as a limit of complimentary series.

LEMMA. *The multiplicity of π in H^* is one.*

Proof. Since $\mathfrak{d}$ is elliptic, H^* is admissible, and, the multiplicity of π is given by the computations in the previous section; it is at most 1. To complete the proof, we must find a vector which transforms on the right by π. Recalling the construction of the Szego map, let $P : V \to W$ denote the M-equivariant projection, and ϕ the highest weight vector of π.

Now the projection onto the subspace of H^* which transforms on the right by π is given by convolution on the right with $d_\pi\chi_\pi$; here χ_π = trace π and $d_\pi = \chi(e)$.

Then a change of variables shows that

$$d_\pi S(P(\pi\phi)) * \chi_\pi = d_\pi S(P(\pi * \chi_\pi)\phi)$$
$$= d_\pi S\Big(P\Big(\frac{1}{d_\pi}\pi\phi\Big)\Big) = S(P(\pi\phi)) \ ,$$

so that $S(P(\pi\phi))$ is invariant under the projection.

We shall next show that H^* contains a unique irreducible closed invariant subspace, that generated by the lowest K-type.

LEMMA. *Let $\mathcal{F}$ be a closed invariant subspace of H^* which is non-trivial. Then the multiplicity of π in $\mathcal{F}$ is one.*

Proof. The multiplicity can be at most one. Since $\mathcal{F}$ is non-trivial, there is an $f \in \mathcal{F}$ and a $g \in G$ with $f(g) \neq 0$. Since $\mathcal{F}$ is invariant, there is an $h \in \mathcal{F}$ with $h(e) \neq 0$. Since $\mathcal{F}$ is closed, $d_\pi h * \chi_\pi$ in in $\mathcal{F}$ again, and this is the K-isotypic component of h. We wish to show it is non-zero. But

$$d_\pi h * \chi_\pi(e) = d_\pi \int h(k^{-1})\chi_\pi(k)\, dk$$
$$= d_\pi \int \pi(k^{-1})\chi_\pi(k)h(e)\, dk$$
$$= h(e) \neq 0 \ .$$

COROLLARY. *There is a unique irreducible subrepresentation of H^*, generated by the lowest K-type.*

Proof. The intersection of all non-trivial closed invariant subspaces of H^* is irreducible. But it contains the lowest K-type, so contains the closure of the span.

We denote the unique irreducible subrepresentation of H^* as H_λ. Our next task is to identify H_λ as a known representation of G. We have shown that $S : U(\sigma, \nu') \to \ker \bar{\mathfrak{d}}$, and that the image of S contains the K-type π. It follows then that H_λ is a quotient representation of the non-unitary principal series $U(\sigma, \nu')$. But this has a unique irreducible quotient, by Langlands, so H_λ is isomorphic to the limits of complementary series.

References

[1] K.M. DAVIS, J.E. GILBERT and R.A. KUNZE, *Invariant Differential Operators in Analysis, I. H^p-theory and Polynomial Invariants*, preprint.

[2] J. DIXMIER, *Représentations intégrables du groupe de De Sitter*, Bull. Soc. Math. France, **89** (1961), 9–41.

[3] J.E. GILBERT, R.A. KUNZE, R.J. STANTON and P.A. TOMAS, *Higher Gradients and Representations of Lie Groups* in Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol.II (W. Beckner, A.P. Calderon, R. Fefferman, P. Jones, eds.) Wadsworth International (1981), 416–426.

[4] R. HOTTA and R. PARTHASARATHY, *Multiplicity formulæ for discrete series*, Inventiones Math., **26** (1974), 133–178.

[5] A.W. KNAPP and E.M. STEIN, *Intertwining operators for semi-simple groups*, Ann. of Math., **93** (1971), 489–578.

[6] A.W. KNAPP and N.R. WALLACH, *Szegö kernels associated with Discrete series*, Inventiones, **34** (1976), 163–200.

[7] W. SCHMID, *On the realization of the discrete series of a semi-simple Lie group*, Rice University Studies, **56** (1970), 99–108.

[8] D. VOGAN, *Algebraic structure of irreducible representations of semi-simple Lie groups*, Ann. of Math., **109** (1979), 1–60.

K.M. Davis and J.E. Gilbert

Department of Mathematics
The University of Texas
Austin, Texas 78712

R.A. Kunze

Department of Mathematics
University of Georgia
Athens, Georgia 30602