TRIGONOMETRIC SUMS AND POLYNOMIAL ZEROS

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1. INTRODUCTION

This is a preliminary report on work in progress on an ARGS project concerned with positive trigonometric sums and their applications.

Consider the cosine series

$$G_m(\theta) = \sum_{j=1}^{\infty} j^{-m} \cos j\theta, \quad m \in \mathbb{N},$$

and its partial sums

$$G_m^n(\theta) = \sum_{j=1}^{n} j^{-m} \cos j\theta.$$

We establish the following

THEOREM *(i)* $G_m(\theta)$ *is decreasing on* $(0,\pi)$,

(ii) *the unique zero of* $G_m(\theta)$ *lying in* $(0,\pi)$ *increases with* m,

(iii) $G_m^n(\theta)$ *is decreasing on* $(0,\pi)$ *for* $m \geq 2$,

(iv) *the unique zero of* $G_m^n(\theta)$ *lying in* $(0,\pi)$ *increases with* $m (\geq 2)$ *for fixed* n.

Apart from the obvious connection with the Riemann zeta function, such series arise in the context of a quadrature-based method for solving boundary integral equations currently being developed by I.H. Sloan and W.L. Wendland [3] : the zeros of $G_m(\theta)$ in $(0,2\pi)$ correspond to the quadrature points, and a consequence of (ii) is the stability of some forms of the method.

The special values $m = 1,2,4,\infty$ give an idea of the general behaviour of $G_m(\theta)$:

$$G_1(\theta) = -\frac{1}{2}\log(2(1 - \cos\theta)),$$

$$G_2(\theta) = \frac{\theta^2}{4} - \frac{\pi\theta}{2} + \frac{\pi^2}{6},$$

$$G_4(\theta) = -\frac{\theta^4}{48} + \frac{\pi\theta^3}{12} - \frac{\pi^2\theta^2}{12} + \frac{\pi^4}{90},$$

$$G_\infty(\theta) = \cos\theta;$$

note that up to a constant $G_{2m}(\theta)$ are the Bernoulli polynomials.

2. PROOF OF THEOREM

(i) For $m = 1$ we see immediately from the explicit formula that $G_1(\theta)$ is decreasing on $(0,\pi)$. For $m > 1$, the series may validly be differentiated termwise [2, 196, 199.4] so that we reduce to proving $H_\beta(\theta) = \sum_{j=1}^{\infty} j^{-\beta}\sin j\theta$ positive on $(0,\pi)$, $\beta > 1$, $\beta \in \mathbb{N}$. In fact that result is valid for all positive real β and Dick Askey showed us how to prove it using the correct kernel: write $j^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{-jt}\,dt$ so that

$$\begin{aligned}
H_\beta(\theta) &= \frac{1}{\Gamma(\beta)}\sum_{j=1}^{\infty}\sin j\theta\int_0^\infty t^{\beta-1}e^{-jt}\,dt \\
&= \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}\sum_{j=1}^{\infty}\sin j\theta\,(e^{-t})^j\,dt \\
&= \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}\frac{e^{-t}\sin\theta}{1 - 2e^{-t}\cos\theta + e^{-2t}}\,dt \\
&> 0 \quad \text{for} \quad \theta \in (0,\pi).
\end{aligned}$$

(ii) Denote by $z(m)$ the unique zero of $G_m(\theta)$ lying in $(0,\pi)$. Notice that $z(1) = \frac{\pi}{3}$ and that for $m > 1$ we have

$G_m(\frac{\pi}{3}) = \frac{1}{2}(1 - 2^{1-m})(1 - 3^{1-m})\zeta(m) > 0$ and $G_m(\frac{\pi}{2}) = -2^{-m}(1 - 2^{1-m})\zeta(m) < 0$.

Thus $z(m) \in \left[\frac{\pi}{3},\frac{\pi}{2}\right)$, and it is enough to show that $(G_{m+1} - G_m)(\theta)$ is positive on $\left[\frac{\pi}{3},\frac{\pi}{2}\right]$, $m \in \mathbb{N}$, for then $G_{m+1}(z(m)) > G_m(z(m)) = 0$ which implies $z(m + 1) > z(m)$ by (i).

Now $G_m(0) = \zeta(m)$ and $G_m(\pi) = -(1 - 2^{1-m})\zeta(m)$ both decrease (to 1 and -1 respectively), whereas $G_m(\frac{\pi}{3})$ and $G_m(\frac{\pi}{2})$ increase with m. In particular, $(G_{m+1} - G_m)(\theta)$ has an even number, at least 2, of zeros in $(0,\pi)$. It is easily verified that $(G_2 - G_1)(\theta)$ and $(G_3 - G_2)(\theta)$ have exactly 2 zeros in $(0,\pi)$; we proceed inductively. Since $(G_{m+3} - G_{m+2})''(\theta) = -(G_{m+1} - G_m)(\theta)$, $(G_{m+3} - G_{m+2})(\theta)$ has precisely 2 points of inflexion in $(0,\pi)$, and since it is negative and concave up at 0 and at π, $(G_{m+3} - G_{m+2})(\theta)$ cannot have more than two zeros in $(0,\pi)$.

Hence $(G_{m+1} - G_m)(\theta)$, $m \in \mathbb{N}$, has exactly two zeros in $(0,\pi)$: one in $(0,\frac{\pi}{3})$ and the other in $(\frac{\pi}{2},\pi)$; in particular $(G_{m+1} - G_m)(\theta)$ is positive on $\left[\frac{\pi}{3},\frac{\pi}{2}\right]$.

(iii) For the partial sums it does not seem possible to mimic the elegant use of the gamma-function kernel. However the classical Jackson-Gronwall result on the positivity of the partial sums of $H_1(\theta)$ gives all the information required (and that result has been given many pretty proofs over the years).

(iv) $z_n(m)$ increases with m, $m \in \mathbb{N}$, $m \geq 2$.

Note first that the assertion is trivial for $n = 1$ since $G^1_m(\theta) = \cos\theta$ and $z_1(m) \equiv \frac{\pi}{2}$, so we suppose $n \geq 2$. Then

$z_n(m) \in \left[\frac{\pi}{4},\frac{\pi}{2}\right]$ since $G^n_m(\frac{\pi}{2}) < 0$ ($G^n_m(\frac{\pi}{2})$ is an alternating sum of terms decreasing in absolute value, the first of which is negative) and since $G^n_m(\frac{\pi}{4}) > 0$ (to see this, pair the jth term with the (j - 4)th, $j \equiv 3,4,5$ mod 8, $j \geq 11$). As before it suffices to prove $(G^n_{m+1} - G^n_m)(\theta) > 0$ on $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$, that is, to prove $\sum_{j=2}^{n} \frac{j-1}{j^{m+1}} \cos j\theta < 0$, $\theta \in \left[\frac{\pi}{4},\frac{\pi}{2}\right]$, $n,m \geq 2$. Summing by parts we see that it is enough to prove

$$C_n(\theta) = \sum_{j=2}^{n} \frac{j-1}{j^2} \cos j\theta < 0,\ \theta \in \left[\frac{\pi}{4},\frac{\pi}{2}\right],\ n \geq 2.$$

Since $\cos 2\theta$, $\cos 3\theta$ and $(\cos 2\theta + \cos 4\theta)$ are negative throughout $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ we have $C_n(\theta) < 0$ on $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ for $n = 2,3,4$. For $n \geq 5$ we sum twice by parts to see that

$$\begin{aligned}
2\sin^2\frac{\theta}{2}\, C_n(\theta) &= \frac{1}{4}\sin^2\frac{\theta}{2} - \frac{5}{18}\sin^2\theta - \frac{1}{144}\sin^2\frac{3\theta}{2} \\
&\quad + \sum_{j=3}^{n-2}\left(\frac{j-1}{j^2} - \frac{2j}{(j+1)^2} + \frac{j+1}{(j+2)^2}\right)\sin^2\frac{(j+1)\theta}{2} \\
&\quad + \left(\frac{n-2}{(n-1)^2} - \frac{n-1}{n^2}\right)\sin^2\frac{n\theta}{2} \\
&\quad + \frac{n-1}{n^2}\sin(2n+1)\frac{\theta}{2}\sin\frac{\theta}{2} \\
&\leq \frac{1}{4}\sin^2\frac{\theta}{2} - \frac{5}{18}\sin^2\theta - \frac{1}{144}\sin^2\frac{3\theta}{2} + \frac{5}{144} \\
&\quad + \frac{n-1}{n^2}\sin(2n+1)\frac{\theta}{2}\sin\frac{\theta}{2} \\
&= f\left(\sin^2\frac{\theta}{2}\right) + \frac{n-1}{n^2}\sin(2n+1)\frac{\theta}{2}\sin\frac{\theta}{2}
\end{aligned}$$

where $f(t) = \frac{1}{144}(5 - 133t + 184t^2 - 16t^3)$. Because f is concave up we have $f\left(\sin^2\frac{\theta}{2}\right) \le \max\left\{f\left(\sin^2\frac{\theta_1}{2}\right), f\left(\sin^2\frac{\theta_2}{2}\right)\right\}$ on $[\theta_1,\theta_2]$, and $C_n(\theta) < 0$ on $[\theta_1,\theta_2]$ whenever $F(\theta_1,\theta_2,n) = \max\left\{f\left(\sin^2\frac{\theta_1}{2}\right), f\left(\sin^2\frac{\theta_2}{2}\right)\right\} + \frac{n-1}{n^2}\sin\frac{\theta_2}{2} < 0$. Also, since $f\left(\sin^2\frac{\theta}{2}\right) < 0$ on $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ we have $C_n(\theta) < 0$ on any subinterval where $\sin(2n+1)\frac{\theta}{2} \le 0$.

For $n \ge 9$ we have $F\left(\frac{\pi}{4},\frac{\pi}{2},n\right) \le F\left(\frac{\pi}{4},\frac{\pi}{2},9\right) < 0$; for $5 \le n \le 8$ it is necessary to subdivide the interval:

for $n = 8$ we have $F\left(\frac{\pi}{4},\frac{6\pi}{17},8\right) < 0$, $F\left(\frac{8\pi}{17},\frac{\pi}{2},8\right) < 0$ and $\sin\frac{17\theta}{2} \le 0$ on $\left[\frac{6\pi}{17},\frac{8\pi}{17}\right]$,

for $n = 7$ we have $F\left(\frac{4\pi}{15},\frac{6\pi}{17},7\right) < 0$ and $\sin\frac{15\theta}{2} \le 0$ on $\left[\frac{\pi}{4},\frac{4\pi}{15}\right] \cup \left[\frac{6\pi}{15},\frac{\pi}{2}\right]$,

for $n = 6$ we have $F\left(\frac{\pi}{4},\frac{\pi}{3},6\right) < 0$, $F\left(\frac{\pi}{3},\frac{\pi}{2},6\right) < 0$ and

for $n = 5$ we have $F\left(\frac{4\pi}{11},\frac{\pi}{2},5\right) < 0$ and $\sin\frac{11\theta}{2} \le 0$ on $\left[\frac{\pi}{4},\frac{4\pi}{11}\right]$.

❑

3. REMARKS

Statement (i) of the theorem is valid for arbitrary real numbers $\alpha \ge 1$, as the proof shows. We will discuss the extension of the remainder of the theorem to non-integral m on another occasion, [1]. For $\alpha < 2$ no even partial sum is decreasing; nevertheless it seems that these partial sums still have a unique zero in $(0,\pi)$. If $\alpha \geq \frac{9}{8}$ this can be proved using Vietoris' methods (see [1], [4]).

REFERENCES

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