7. SUPERPOSITION OF EVOLUTIONS

The main point of this chapter is to present a vector, or operator, version of the Feynman-Kac formula representing certain perturbations of a given evolution. While for some evolutions, such as the diffusion semigroup, the formula can be stated in terms of classical absolutely convergent integrals, for others, notably the Schrödinger group, the usage of a more general conceptual machinery is inevitable. Needless to say, the notions introduced in earlier chapters will be used here.

A. Let E be a Banach space. The algebra of all bounded linear operators on E is denoted by $\mathrm{BL}(E)$

The basic ingredient of the abstract Feynman-Kac formula, to be stated in the next section, is the $\mathrm{BL}(E)$-valued additive set function determined by an evolution in the space E and a $\mathrm{BL}(E)$-valued spectral measure. In this section, the conventions pertaining to these notions are introduced.

Let Λ be a locally compact Hausdorff space. Although other spaces may be, and indeed are, of considerable interest, in the examples considered in this chapter, Λ will be equal to $\mathbb{R}^d$, for some small or unspecified positive integer d. Let $\mathcal{B} = \mathcal{B}(\Lambda)$ be the σ-algebra of Baire sets in Λ. The $\mathcal{B}$-measurable functions on Λ will be called the Baire functions. (See Section 1D.)

Let $P : \mathcal{B} \to \mathrm{BL}(E)$ be a σ-additive spectral measure. (See Section 6A.) By Corollary 6.6, the spectral measure P is closable. (See Section 6C.) If $\varphi \in E$, by $P\varphi$ is denoted the E-valued set function on $\mathcal{B}$ such that $(P\varphi((B) = P(B)\varphi$, for every $B \in \mathcal{B}$. By the assumption, $P\varphi$ is σ-additive, for every $\varphi \in E$. The integrability with respect to $P\varphi$ is understood in the sense of Proposition 3.13. That is, a function on Λ is called $(P\varphi)$-integrable if it satisfies, *mutatis mutandis*, any of the equivalent conditions (i), (ii) or (iii) of Proposition 3.13.

Given a Baire function, W, on Λ, by

$$P(W) = \int_\Lambda W \mathrm{d}P = \int_\Lambda W(x) P(\mathrm{d}x)$$

will be denoted the operator whose domain consists of the elements, φ, of the space E such that the function W is $(P\varphi)$-integrable and whose value, $P(W)\varphi$, at any such element is given by the formula

$$P(W)\varphi = \int_\Lambda W(x)P(\mathrm{d}x)\varphi .$$

The operator $P(W)$ is bounded if and only if the function W is P-essentially bounded, that is, there exists a Baire set, B_0, such that $P(B_0) = 0$ and W is bounded on the complement of B_0. (See Section 6B.) So, $P(W) \in \mathrm{BL}(E)$ if and only if $W \in \mathcal{L}(P)$. (See Section 6C.)

For any real numbers t' and t'' such that $0 \le t' \le t''$, let $S(t'',t') \in \mathrm{BL}(E)$ be an operator such that

(i) $S(t,t) = I$, the identity opereator, for every $t \ge 0$;

(ii) $S(t''',t') = S(t''',t'')S(t'',t')$, for any t', t'' and t''' such that $0 \le t' \le t'' \le t'''$; and

(iii) the map $S : \{(t'',t') : 0 \le t' \le t''\} \to \mathrm{BL}(E)$ is continuous in the strong operator topology of $\mathrm{BL}(E)$.

Such a map $S : \{(t'',t') : 0 \le t' \le t''\} \to \mathrm{BL}(E)$, with properties (i), (ii) and (iii), is called an evolution, or a propagator, in the space E. If $S(t'',t') = S(t''-t',0)$, for any $0 \le t' \le t''$, then we speak of a continuous semigroup, or a dynamical propagator, and write without ambiguity $S(t) = S(t,0)$, for every $t \ge 0$. Needless to say, the numbers $t, t', t'', \ldots$ entering into arguments of an evolution are intuitively interpreted as instants of time.

Let $t \ge 0$. For every $s \in [0,t]$, let Υ_s be a set of maps $v : [0,s] \to \Lambda$ to be called paths. We assume that $\{v(s) : v \in \Upsilon_t\} = \Lambda$, for every $s \in [0,t]$. To formulate another assumption, for any $s \in [0,t]$, let $\mathrm{pr}_{t,s}$ be the natural projection of Υ_t onto Υ_s. That is, the value, $\mathrm{pr}_{t,s}(v)$, of the map $\mathrm{pr}_{t,s}$ at an element, v, of Υ_t is equal to the restriction, $v|_{[0,s]}$, of v to the interval $[0,s]$. We shall assume that $\{\mathrm{pr}_{t,s}(v) : v \in \Upsilon_t\} = \Upsilon_s$, for every $s \in [0,t]$.

Of main interest are the cases in which $\Upsilon_t = \Lambda^{[0,t]}$, or Υ_t consists of all continuous paths $v : [0,t] \to \Lambda$, or ones which are right-continuous at each point of the interval $[0,t)$ and have a left limit at each point of the interval $(0,t]$, etc.

Let $t \geq 0$. Given an integer $k \geq 1$, sets $B_j \in \mathcal{B}$ and numbers t_j, $j = 1,2,\ldots,k$, such that $0 \leq t_{j-1} < t_j \leq t$ for every $j = 2,3,\ldots,k$, let

$$\text{(A.1)} \qquad Y = \{v \in \Upsilon_t : v(t_j) \in B_j, \ j = 1,2,\ldots,k\} .$$

Whenever it is necessary to indicate the parameters on which the set Y depends we write $Y = Y(t_1,\ldots,t_k; B_1,\ldots,B_k)$.

The family of sets (A.1) formed for all choices of $k = 1,2,\ldots$, sets $B_j \in \mathcal{B}$ and numbers t_j, $j = 1,2,\ldots,k$, such that $0 \leq t_{j-1} < t_j \leq t$ for every $j = 2,3,\ldots,k$, is denoted by $\mathcal{R}_t$. It is classical and comparatively easy to show, that $\mathcal{R}_t$ is a semialgebra of sets in the space Υ_t. (See Section 1D.)

Now we define the set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$ determined by the evolution S and the spectral measure P. Namely, if $k \geq 1$ is an integer, $B_j \in \mathcal{B}$ sets and t_j numbers, $j = 1,2,\ldots,k$, such that $0 \leq t_{j-1} < t_j \leq t$ for $j = 2,3,\ldots,k$, and the set Y is given by (A.1), let

$$\text{(A.2)} \qquad M_t(Y) = S(t,t_k)P(B_k)S(t_k,t_{k-1})P(B_{k-1}) \ldots P(B_2)S(t_2,t_1)P(B_1)S(t_1,0) .$$

PROPOSITION 7.1. *For every set $Y \in \mathcal{R}_t$, the operator $M_t(Y)$ is defined by* (A.2) *unambiguously. The resulting set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$ is additive.*

Proof. Let $Y \in \mathcal{R}_t$. If Y is given by (A.1), then $Y = \emptyset$ if and only if $B_j = \emptyset$ for some $j = 1,2,\ldots,k$. So, let $Y \neq \emptyset$. If $Y = Y(t_1,\ldots,t_k; B_1,\ldots,B_k)$, for some integer $k \geq 1$, sets $B_j \in \mathcal{B}$ and pair-wise different numbers t_j, $j = 1,2,\ldots,k$, and also $Y = Y(s_1,\ldots,s_\ell; C_1,\ldots,C_\ell)$, for some integer $\ell \geq 1$, sets $C_m \in \mathcal{B}$ and pair-wise different numbers s_m, $m = 1,2,\ldots,\ell$, then $C_m = B_j$ whenever $s_m = t_j$, $B_j = \Lambda$ for every $j = 1,2,\ldots,k$ such that $t_j \neq s_m$ for every $m = 1,2,\ldots,\ell$, and $C_m = \Lambda$ for every $m = 1,2,\ldots,\ell$ such that $s_m \neq t_j$ for every $j = 1,2,\ldots,k$. Therefore, property (ii) of an evolution and the equality $P(\Lambda) = I$ imply that the operator $M_t(Y)$ is defined unambiguously by (A.2).

To prove the additivity of the set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$, by Proposition 1.8, it suffices to prove that this set function is 2-additive. However, the 2-additivity follows immediately from the following general set theoretical fact: If $X \in \mathcal{R}_t$, $Y \in \mathcal{R}_t$ and $Z \in \mathcal{R}_t$ are sets such that $Y \cap Z = \emptyset$ and $X = Y \cup Z$, then there exist an integer $k \geq 1$, sets A_j, B_j and C_j, belonging to $\mathcal{B}$, pair-wise different numbers t_j, $j = 1,2,\ldots,k$, and an integer $m \in [1,k]$ such that

$$X = \{v \in \Upsilon_t : v(t_j) \in A_j,\ j = 1,\ldots,k\}\ ,\quad Y = \{v \in \Upsilon_t : v(t_j) \in B_j\ ,\ j = 1,\ldots,k\}\ ,$$

$$Z = \{v \in \Upsilon_t : v(t_j) \in C_j\ ,\ j = 1,\ldots,k\}\ ,$$

$A_j = B_j = C_j$ for every $j \neq m$, $j = 1,2,\ldots,k$, $B_m \cap C_m = \emptyset$ and $A_m = B_m \cup C_m$.

It should be noted that the set function $M_t\varphi$, for some given $\varphi \in E$, is usually of more direct interest than M_t itself. To be sure, $M_t\varphi$ is the E-valued function on $\mathcal{R}_t$ whose value at any set $Y \in \mathcal{R}_t$ is equal to $M_t(Y)\varphi$.

Let ρ be a gauge on some quasialgebra $\mathcal{Q} \subset \mathcal{R}_t$ integrating for the restriction of $M_t\varphi$ to $\mathcal{Q}$. Let $f \in \mathcal{L}(\rho,\mathcal{Q})$. By

$$\int_{\Upsilon_t} f(v) M_t(\mathrm{d}_\rho v)\varphi = \int_{\Upsilon_t} f \mathrm{d}_\rho(M_t\varphi) = (M_t)_\rho(f)\varphi$$

will be denoted the 'integral of the function f with respect to $M_t\varphi$,' that is, the value, $\ell(f)$, of the continuous linear functional, ℓ, on $\mathcal{L}(\rho,\mathcal{Q})$ such that $\ell(X) = M_t(X)\varphi$, for every $X \in \mathcal{Q}$. (See Section 3A.) We should note though that, usually, ρ does not integrate for (the corresponding restriction of) M_t, so that the symbol '$(M_t)_\rho(f)$' is meaningless as are other symbols for the 'integral of f with respect to M_t.'

EXAMPLE 7.2. Let $\varphi \in E$. Let $\mathcal{Q} \subset \mathcal{R}_t$ be a quasialgebra. Let ρ be a gauge on $\mathcal{Q}$ integrating for the set function $M_t\varphi$ restricted to $\mathcal{Q}$. Let $0 \leq t_1 < t_2 < \ldots < t_{n-1} < t_n \leq t$ and $W_1, W_2, \ldots, W_n$ be Baire functions on Λ such that the function f, defined by

$$f(v) = \prod_{j=1}^{n} W_j(v(t_j))$$

for every $v \in \Upsilon_t$, is ρ-integrable. Then

$$\int_{\Upsilon_t} f \mathrm{d}_\rho (M_t\varphi) = S(t,t_n)P(W_n)S(t_n,t_{n-1})P(W_{n-1}) \dots P(W_2)S(t_2,t_1)P(W_1)S(t_1,0)\varphi .$$

EXAMPLE 7.3. Let $\varphi \in E$. Let $\mathcal{Q} \subset \mathcal{R}_t$ be a quasialgebra. Let ρ be a gauge on $\mathcal{Q}$ integrating for the set function $M_t\varphi : \mathcal{Q} \to E$. Let $0 \le s \le t$. Let $\mathcal{Q}_s = \{Z \subset \Upsilon_s : \mathrm{pr}_{t,s}^{-1}(Z) \in \mathcal{Q}\}$ and $\rho_s(Z) = \rho(\mathrm{pr}_{t,s}^{-1}(Z))$, for every $Z \in \mathcal{Q}_s$. Assume that the gauge ρ_s integrates for the set function on $M_s\varphi : \mathcal{Q}_s \to E$. Let g be a ρ_s-integrable function on Υ_s, W a Baire function on Λ and

$$f(v) = W(v(s))g(\mathrm{pr}_{t,s}(v))$$

for every $v \in \Upsilon_t$. If the function f is ρ-integrable, then

$$\int_{\Upsilon_t} f \mathrm{d}_\rho (M_t\varphi) = S(t_1 s)P(W) \int_{\Upsilon_s} g \mathrm{d}_{\rho_s} (M_s\varphi) .$$

B. We maintain the notation of Section A.

Assume that an evolution, S, in the space E and a spectral measure, P, on $\mathcal{B} = \mathcal{B}(\Lambda)$ are given. Let φ be an element of the space E.

Let $t \ge 0$. Let ι be the Lebesgue measure on the interval $[0,t]$ and $\mathcal{L}(\iota)$ the family of all (individual) Lebesgue integrable functions on $[0,t]$. We write, of course,

$$\iota(f) = \int_0^t f(r)\mathrm{d}r ,$$

for every $f \in \mathcal{L}(\iota)$.

Let $\mathcal{Q}$ be a semialgebra of sets in the space Υ_t such that $\mathcal{Q} \subset \mathcal{R}_t$ and let ρ be a gauge integrating for the restriction of the set function $M_t\varphi$ to $\mathcal{Q}$. For every $s \in [0,t]$, let $\mathcal{Q}_s = \{Z \subset \Upsilon_s : \mathrm{pr}_{t,s}^{-1}(Z) \in \mathcal{Q}\}$ and let $\rho_s(Z) = \rho(\mathrm{pr}_{t,s}^{-1}(Z))$, for every $Z \in \mathcal{Q}_s$.

Let W be a function on $[0,t] \times \Lambda$ such that the function $r \mapsto W(r,v(r))$, $r \in [0,t]$, is ι-integrable for ρ-almost every $v \in \Upsilon_t$. For every $s \in [0,t]$, let e_s be a function on Υ_s , to be called the Feynman-Kac functional, such that

(B.1) $$e_s(v) = \exp\left[\int_0^s W(r,v(r))\mathrm{d}r\right] ,$$

for every $v \in \Upsilon_s$ for which the integral at the right exists.

If e_s happens to be ρ_s-integrable, let

(B.2) $$u(s) = \int_{\Upsilon_s} e_s(v)M_s(\mathrm{d}_{\rho_s} v)\varphi .$$

In particular,

$$u(t) = \int_{\Upsilon_t} e_t(v)M_t(\mathrm{d}_\rho v)\varphi .$$

In order to present concisely an intuitive interpretation of $u(t)$, let us extend the definition of W onto the whole of $[0,\infty) \times \Lambda$ by letting $W(s,x) = W(t,x)$, for every $s \geq t$ and every $x \in \Lambda$. Assume that, for every t' and t'' such that $0 \leq t' \leq t''$,

(B.3) $$T(t'',t') = P\left[\exp \int_{t'}^{t''} W(s,\cdot)ds\right]$$

is a well-defined operator belonging to $\mathrm{BL}(E)$ and the resulting map $T : \{(t'',t') : 0 \leq t' \leq t''\} \to \mathrm{BL}(E)$ is an evolution in the space E .

Then $u(t)$ can be thought of as the element of the space E into which φ evolves under simultaneous action of S and T during the time-interval $[0,t]$. In fact, if the numbers $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$ represent a partition, $\not{p}$, of the interval $[0,t]$, let us denote

$$u_{\not{p}}(t) = T(t_n,t_{n-1})S(t_n,t_{n-1})T(t_{n-1},t_{n-2})S(t_{n-1},t_{n-2}) \ldots T(t_2,t_1)S(t_2,t_1)T(t_1,t_0)S(t_1,t_0)\varphi .$$

Furthermore, let

$$W_j = \exp\left[\int_{t_{j-1}}^{t_j} W(r,\cdot)\mathrm{d}r\right],$$

so that $T(t_{j-1},t_j) = P(W_j)$, for every $j = 1,2,\ldots,n$, and

$$h_{\not{p}}(v) = \prod_{j=1}^{n} W_j(v(t_j)),$$

for every $v \in \Gamma_t$. Then, by Example 7.2,

$$u_{\not{p}}(t) = \int_{\Gamma_t} h_{\not{p}} \mathrm{d}_\rho(\mathrm{M}_t\varphi).$$

Now, if the partition $\not{p}$ is sufficiently fine, then we may expect that $u_{\not{p}}(t)$ will be approximately the outcome of the simultaneous action of the evolutions S and T on φ during the time-interval $[0,t]$. On the other hand, we may also expect that the integral of $h_{\not{p}}$ 'with respect to $M_t\varphi$' will approximate the integral of e_t.

Turning these heuristics into a solid argument would of course require an appeal to a Trotter-Kato type theorem. However, we shall proceed differently. Namely, assuming that the function e_s is ρ_s-integrable, for every $s \in [0,t]$, we are going to present a sufficient condition for the function $s \mapsto u(s)$, $s \in [0,t]$, to satisfy a Duhamel type integral equation which expresses formally the idea of the superposition of the two evolutions. The condition is stated in terms of $(\iota \otimes \rho)$-integrability. (See Section 5C.)

So, let

$$f(s,v) = W(s,v)\exp\left[\int_0^s W(r,v(r))\mathrm{d}r)\right],$$

for every $s \in [0,t]$ and $v \in \Upsilon_t$ for which the integral at the right exists.

THEOREM 7.4. *If, for every $s \in [0,t]$, the function e_s is ρ_s-integrable and the function f is $(\iota \otimes \rho)$-integrable, then*

(B.4) $$u(t) = S(t,0)\varphi + \int_0^t S(t,s)P(W(s,\cdot))u(s)\mathrm{d}s.$$

Proof. First note that

$$\int_0^t f(s,v)\mathrm{d}s = \exp\left[\int_0^s W(r,v(r))\mathrm{d}r\right] - 1 = e_t(v) - 1 ,$$

for every $v \in \Upsilon_t$ such that $f(\cdot,v) \in \mathcal{L}(\iota)$. Furthermore, by Example 7.3,

$$\int_{\Upsilon_t} f(s,v)M_t(\mathrm{d}_\rho v)\varphi = S(t,s)P(W(s,\cdot))u(s) ,$$

for every $s \in [0,t]$ such that $f(s,\cdot) \in \mathcal{L}(\rho,\mathcal{Q})$. Therefore, by theorem 5.11,

$$u(t) - S(t,0)\varphi = \int_{\Upsilon_t} (e_t(v)-1)M_t(d_\rho v)\varphi =$$

$$= \int_{\Upsilon_t}\left[\int_0^t f(s,v)\mathrm{d}s\right]M_t(\mathrm{d}_\rho v)\varphi = \int_0^t\left[\int_{\Upsilon_t} f(s,v)M_t(\mathrm{d}_\rho v)\varphi\right]\mathrm{d}s =$$

$$= \int_0^t S(t,s)P(W(s,\cdot))u(s)\mathrm{d}s .$$

It should be noted that it may be possible to define $u(s)$ by (B.2), for every $s \in [0,t]$, and to write equation (B.4) independently of whether (B.3) defines an evolution. Indeed, the initial-value problem

$$\dot{u}(t) = P(W(t,\cdot))u(t) ,\ t > 0 ;\ u(0+) = \varphi ,$$

may have a solution for some $\varphi \in E$ but not for others.

Now, assuming that (B.4) holds for every $t \in (0,t_0)$, where $0 < t_0 \le \infty$, formal differentiation gives that

(B.5) $$\dot{u}(t) = A(t)u(t) + P(W(t,\cdot))u(t) ,\ t \in (0,t_0) ,$$

where

$$A(t)\psi = \lim_{r\to 0} r^{-1}(S(t+r,t)\psi-\psi) ,$$

for every $\psi \in E$ such that this limit exists in the sense of convergence in the space E . Furthermore, (B.4) also implies that $u(0+) = \varphi$.

So, another, perhaps more conventional, interpretation of $u(t)$ is that it is the value at t of a generalized solution of the initial-value problem consisting of equation (B.5) and the condition that $u(0+) = \varphi$.

C. Let Λ be a locally compact Hausdorff space. Let $t \mapsto \Sigma_t$, $t \in \mathbb{R}$, be a continuous group of homeomorphisms of the space Λ. That is, for every $t \in \mathbb{R}$, a homeomorphism $\Sigma_t : \Lambda \to \Lambda$ is given such that

(i) $\Sigma_{s+t} = \Sigma_t \circ \Sigma_s$, for every $s \in \mathbb{R}$ and $t \in \mathbb{R}$; and

(ii) for every $x \in \Lambda$, the orbit $t \mapsto \Sigma_t x$, $t \in \mathbb{R}$, of the element x is a continuous map of $\mathbb{R}$ into Λ.

Let $\mathcal{B}$ be the σ-algebra of Baire sets in Λ. Let κ be a Baire measure on Λ. That is to say, there is a vector lattice, $\mathcal{L}(\kappa)$, of functions on Ω and a positive linear functional, κ, on $\mathcal{L}(\kappa)$ such that its restriction to $\mathcal{B}_\kappa = \mathcal{B} \cap \mathcal{L}(\kappa)$ is σ-additive and $\mathcal{L}(\kappa) = \mathcal{L}(\kappa,\mathcal{B}_\kappa)$ and $B\varphi \in \mathcal{L}(\kappa)$ for every set $B \in \mathcal{B}$ and function $\varphi \in \mathcal{L}(\kappa)$. (See Section 3B.) For the sake of simplicity, we assume also that κ is σ-finite, that is, Λ is equal to the union of a sequence of sets belonging to $\mathcal{B}_\kappa$.

Let $1 \le p < \infty$ and $E = L^p(\kappa)$ with the usual norm. (See Section 3C.) To simplify the exposition, we shall use the standard licence and not distinguish between elements of the space E and the individual functions on Λ determining them.

Let $S(t)\varphi = \varphi \circ \Sigma_t$, for every $t \in \mathbb{R}$ and $\varphi \in E$. Assume that

(i) $S(t)\varphi \in E$, for every $t \in \mathbb{R}$ and $\varphi \in E$;

(ii) for every $t \in \mathbb{R}$, the so defined map $S(t) : E \to E$ is an element of $\mathrm{BL}(E)$; and

(iii) for every $\varphi \in E$ the map $t \mapsto S(t)\varphi$, $t \in \mathbb{R}$, is continuous.

So, $S : \mathbb{R} \to \mathrm{BL}(E)$ is a (continuous) group of operators.

For any set $B \in \mathcal{B}$, let $P(B)$ be the operator of point-wise multiplication by the characteristic function of the set B. Then the map $P : \mathcal{B} \to \mathrm{BL}(E)$ is a σ-additive spectral measure. The integral, $P(W)$, of a Baire function W is the operator of pointwise multiplication by the function W. So, we may write simply $P(W) = W$.

For any $t \geq 0$, let Υ_t be the space of all continuous maps $v: [0,t] \to \Lambda$. For every $x \in \Lambda$ and $r \in [0,t]$, let

$$\gamma_x(r) = \Sigma_{-r} x .$$

Then, by the assumptions, $\gamma_x \in \Upsilon_t$, for every $x \in \Lambda$. For any set $Y \subset \Upsilon_t$, let

$$B_Y = \{x \in \Lambda : \gamma_x \in Y\} .$$

If the set $Y \in \mathcal{Q}_t$ is given by (A.1) with some integer $k \geq 1$, sets $B_j \in \mathcal{B}$ and numbers t_j, $j = 1,2,\dots,k$, such that $0 \leq t_{j-1} < t_j \leq t$ for every $j = 2,3,\dots,k$, let

$$\text{(C.1)} \qquad M_t(Y) = S(t-t_k)P(B_k)S(t_k-t_{k-1})P(B_{k-1}) \dots P(B_2)S(t_2-t_1)P(B_1)S(t_1) .$$

This is of course a version of (A.2) for the case when the evolution S happens to be time-homogeneous, that is, it is a semigroup.

Let $\mathcal{S}_t$ be the σ-algebra of sets generated by $\mathcal{R}_t$.

PROPOSITION 7.5. *If $Y \in \mathcal{S}_t$, then $B_Y \in \mathcal{B}$. If $Y \in \mathcal{R}_t$, then $M_t(Y) = S(t)P(B_Y)$.*

Let $\varphi \in E$. If $\mu(Y) = S(t)P(B_Y)\varphi$, for every $Y \in \mathcal{S}_t$, then μ is an E-valued σ-additive set function on $\mathcal{S}_t$ such that $\mu(Y) = M_t(Y)\varphi$, for every $Y \in \mathcal{R}_t$.

Proof. Because $B_Y \in \mathcal{B}$ for every $Y \in \mathcal{R}_t$ and $\mathcal{B}$ is a σ-algebra, it follows that $B_Y \in \mathcal{B}$ for every $Y \in \mathcal{S}_t$. The equality $M_t(Y) = S(t)P(B_Y)$ can be checked by a direct inspection for any $Y \in \mathcal{R}_t$. Then the last statement is obvious.

Let $\varphi \in E$ and let us keep the notation of Proposition 7.5. Because the set function μ is σ-additive, Proposition 3.13 is applicable. Let $|\varphi' \circ \mu|$ be the variation of the set function $\varphi' \circ \mu$, for any $\varphi' \in E'$. Let

$$\rho(f) = \sup\left\{ \int_{\Upsilon_t} |f| \, \mathrm{d}|\varphi' \circ \mu| : \varphi' \in E' , \|\varphi'\| \leq 1 \right\} ,$$

for every $f \in \mathrm{sim}(\mathcal{R}_t)$. Then, by Proposition 3.13, the seminorm ρ integrates for (the linear extension of) μ.

EXAMPLE 7.6. Let $0 \le t_1 < t_2 < \dots < t_{k-1} < t_k \le t$, let $W_1, W_2, \dots, W_k$ be Baire functions on Λ and let

$$f(v) = \prod_{j=1}^{k} W(v(t_j))$$

for every $v \in \Upsilon$. The function f is ρ-integrable if and only if the function

$$\varphi \prod_{j=1}^{k} W_j \circ \Sigma_{-t_j}$$

(the multiplication is point-wise) determines an element of the space E. Moreover,

$$\int_{\Upsilon_t} f \mathrm{d}_\rho \mu = \left[\varphi \prod_{j=1}^{k} W_j \circ \Sigma_{-t_j} \right] \circ \Sigma_t ,$$

whenever the function f is in fact integrable.

PROPOSITION 7.7. *Let W be a function on $[0,t] \times \Lambda$ such that the function $r \mapsto W(r, \Sigma_r x)$, $r \in [0,t]$, is integrable for κ-almost every $x \in \Lambda$. Let*

$$\text{(C.2)} \qquad V_{t,W}(x) = \exp\left[\int_0^t W(r, \Sigma_{-r} x) \mathrm{d}r \right] ,$$

for every $x \in \Lambda$ such that the integral on the right exists. Then the function e_t is ρ-integrable if and only if the function $V_{t,W}\varphi$ determines an element of the space E. If the function e_t is indeed ρ-integrable, then

$$\int_{\Upsilon_t} e_t(v) \mu(\mathrm{d}_\rho v) = (V_{t,W}\varphi) \circ \Sigma_t .$$

Proof. When the integral in (C.2) exists in the sense of Riemann, then the statement follows easily from Example 7.6. So, the statement is valid for all functions that are κ-almost everywhere limits of functions for which the integral in (C.2) exists in the sense of Riemann.

The special case when $\Lambda = \mathbb{R}^d$, for some integer $d \ge 1$, and Σ is the fundamental solution of the dynamical system of differential equations $\dot{x} = a(x)$, where $a : \mathbb{R}^d \to \mathbb{R}^d$ is a mapping with components $a_1, a_2, \ldots, a_d$, is of a particular interest. For any $x^0 \in \mathbb{R}^d$, the function $t \mapsto \Sigma_t x^0$, $t \in \mathbb{R}$, is then the solution of this system passing through x^0 at $t = 0$. In this case, the infinitesimal generator of the semigroup S is the differential operator

$$A = \sum_{j=1}^{d} a_j \frac{\partial}{\partial x_j} .$$

If the function φ is smooth enough and

$$u(t,x) = V_{t,W}(\Sigma_t x)\varphi(\Sigma_t x) ,$$

for every $t \geq 0$ and $x \in \mathbb{R}^d$, then u is a solution of the problem

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{d} a_j \frac{\partial}{\partial x_j} + Wu ,\ t > 0 ,\ x \in \mathbb{R}^d ;\ u(0+,x) = \varphi(x) ,\ x \in \mathbb{R} .$$

The case of the Feynman-Kac formula suggested in this section admits many variants. None-the-less the set function M_t it gives rise to can be considered rather 'degenerate'. More complex cases are obtained by introducing another parameter.

For every $y \in [0,1]$, let $t \mapsto \Sigma_t^y$, $t \in \mathbb{R}$, be a group of homeomorphisms of the space Λ. Assume that the map $(x,y,t) \mapsto \Sigma_t^y x$, of the space $\Lambda \times [0,1] \times \mathbb{R}$ into Λ, is continuous.

Let $E = L^p(\kappa \otimes \iota)$, where $\kappa \otimes \iota$ is the tensor product of a σ-finite Baire measure on Λ and the Lebesgue measure on $[0,1]$. For a function φ on $\Lambda \times [0,1]$ and $t \in \mathbb{R}$, let

$$(S(t)\varphi)(x,y) = \varphi(\Sigma_t^y x, y) ,$$

for every $x \in \Lambda$ and $y \in [0,1]$. Assume that, for every $\varphi \in E$ and $t \in \mathbb{R}$, the function $S(t)\varphi$ determines again an element of E, that the resulting map $S(t) : E \to E$ is an operator belonging to $\mathrm{BL}(E)$ and, finally, the so defined map $t \mapsto S(t)$, $t \geq 0$, is a continuous group of operators.

Let $t \geq 0$ and let Υ_t have the same meaning as before. Let

$$\gamma_{x,y}(r) = \Sigma_{-r}^y x ,$$

for every $x \in \Lambda$, $y \in [0,1]$ and $r \in [0,t]$. For a set $Y \subset \Upsilon_t$, let $B_Y^y = \{x : \gamma_{x,y} \in Y\}$.

Let $\varphi \in E$. For every $Y \in \mathcal{S}_t$, let $\mu(Y)$ be the element of the space E such that

$$(S(-t)\mu(Y))(x,y) = B_E^y(x)\varphi(x,y) ,$$

for κ-almost every $x \in \Lambda$ and every $y \in [0,1]$. It is then a matter of direct calculation that $\mu(Y) = M_t(Y)\varphi$, for every $Y \in \mathcal{Q}_t$.

D. Let $d \geq 1$ be an integer. We shall specialize the situation of Sections A and B by taking the d-dimensional arithmetic Euclidean space, $\mathbb{R}^d$, for Λ and the space of all scalar valued σ-additive set functions on the σ-algebra, $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$, of all Baire sets in $\mathbb{R}^d$ for E. Because every element of E has finite and σ-additive variation, we use the standard conventions about integration with respect to elements of E mentioned in Section 3F. Namely, we note that the variation, $|\varphi|$, of an element, φ, of the space E is a gauge on $\mathcal{B}$ which integrates for φ, denote $\mathcal{L}(\varphi) = \mathcal{L}(|\varphi|)$ and do not show the gauge, $|\varphi|$, in symbols for integral with respect to φ. The norm, $\|\varphi\|$, of an element, φ, of the space E is the total variation of φ, that is, the number $|\varphi|(\mathbb{R}^d)$.

The Lebesgue measure on $\mathbb{R}^d$ is denoted by λ. Identifying the elements of $L^1(\lambda)$ with their indefinite integrals, we identify the space $L^1(\lambda)$ with a subspace of E consisting of those elements which are λ-absolutely continuous.

Given a set $B \in \mathcal{B}$, let $P(B)$ be the operator of restriction to the set B. That is, $(P(B)\varphi)(X) = \varphi(B \cap X)$, for every set $X \in \mathcal{B}$ and every $\varphi \in E$. So, on the subspace $L^1(\lambda)$ of E, the operator $P(B)$ acts as point-wise multiplication by the characteristic function of the set B. For every $B \in \mathcal{B}$, the operator $P(B)$ is an element of $\mathrm{BL}(E)$ and the map $P : \mathcal{B} \to \mathrm{BL}(E)$ is a σ-additive spectral measure.

Let D be a strictly positive real number and let

$$p_D(t,x) = (4\pi Dt)^{-\frac{1}{2}d}\exp(-|x|^2/4Dt) ,$$

for every $t > 0$ and $x \in \mathbb{R}^d$. ($|x|$ stands for the usual Euclidean norm of an element x of $\mathbb{R}^d$.)

Let $S(0) = I$ and

$$(S(t)\varphi)(B) = \int_B dx \int_{\mathbb{R}^d} p_D(t,x-y)\varphi(dy) ,$$

for every $t > 0$, $B \in \mathcal{B}$ and $\varphi \in E$. Then the set function $S(t)\varphi$, that is, $B \mapsto (S(t)\varphi)(B)$, $B \in \mathcal{B}$, is an element of the space E. For every $t > 0$, the operator $S(t)$, that is, $\varphi \mapsto S(t)\varphi$, $\varphi \in E$, is an element of $\mathrm{BL}(E)$. Finally, the resulting map $t \mapsto S(t)$, $t \in [0,\infty]$, is a continuous semigroup, $S : [0,\infty] \to \mathrm{BL}(E)$, of operators.

The semigroup S can be interpreted as a mathematical description of isotropic and homogeneous diffusion in $\mathbb{R}^d$ with the diffusion coefficient D. It is called the Poisson semigroup. Its infinitesimal generator is the (closure of the) operator $D\Delta$, where Δ is the Laplacian in $\mathbb{R}^d$.

Given a $t \geq 0$, let Υ_t be the set of all continuous paths $v : [0,t] \to \Lambda$. Because S is a semigroup, the formula (A.2), defining the set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$, takes the form (C.1), for every set $Y \in \mathcal{R}_t$ given by (A.1) with some integer $k \geq 1$, sets $B_j \in \mathcal{B}$ and numbers t_j, $j = 1,2,\ldots,k$, such that $0 \leq t_{j-1} < t_j \leq t$, for each $j = 2,3,\ldots,k$.

Let $\varphi \in E$ be a non-negative measure. Let

$$\rho_\varphi(Y) = \|M_t(Y)\varphi\| = (M_t(Y)\varphi)(\mathbb{R}^d) ,$$

for every $Y \in \mathcal{R}_t$. Then ρ_φ is a non-negative σ-additive set function on $\mathcal{R}_t$ and so, it generates a measure in the space Υ_t. This fact, dating back to N. Wiener, is classical; see, for example, [11], Theorem VIII.2.2. If φ is a probability measure on $\mathbb{R}^d$, then ρ_φ is called the d-dimensional Wiener measure of variance $2D$ per unit of time with initial distribution φ. (See Example 4.33.)

Now, if φ is an arbitrary element of the space E, then $\rho = \rho_{|\varphi|}$ is a gauge on $\mathcal{R}_t$ which integrates for φ.

Let W be a Baire function on $[0,t]\times\mathbb{R}^d$. Mark Kac noted, see [26],Chapter IV, that, if the function $r\mapsto W(r,v(r))$, $r\in[0,t]$, is Riemann integrable in $[0,t]$, for ρ-almost every $v\in\Upsilon_t$, then the Feynman-Kac functional,

$$e_t(v)=\exp\left[\int_0^t W(r,v)\mathrm{d}r\right],\quad v\in\Upsilon_t,$$

is ρ-measurable on Υ_t. Consequently, if e_t is also bounded then it is ρ-integrable. This happens, for example, when $W(r,x)=W(0,x)$, for every $r\in[0,t]$ and $x\in\mathbb{R}^d$, and the function $W(0,\cdot)$ is bounded above and continuous on the complement of a set of capacity zero in $\mathbb{R}^d$, because the set of paths $v\in\Upsilon_t$ that avoid a given set of capacity zero has the Wiener measure equal to zero.

Now, if e_t is indeed ρ-integrable, for every $t>0$, the element

$$u(t)=\int_{\Upsilon_t} e_t(v)M_t(\mathrm{d}_\rho v)\varphi \tag{D.1}$$

of the space E belongs to $L^1(\lambda)$, for every $\varphi\in E$. Let us abuse the notation and denote by $x\mapsto u(t,x)$, $x\in\mathbb{R}^d$, the density of $u(t)$. In terms of densities, the integral equation (B.4) can be re-written in the form

$$u(t,x)=\int_{\mathbb{R}^d} p_D(t,x-y)\varphi(\mathrm{d}y)+\int_0^t\int_{\mathbb{R}^d} p_D(t-s,x-y)\,W(s,y)\,u(s,y)\mathrm{d}y\mathrm{d}s, \tag{D.2}$$

for $x\in\mathbb{R}^d$ and $t>0$. This equation represents the initial-value problem

$$\begin{aligned}&\dot u(t,x)=D\Delta u(t,x)+W(t,x)u(t,x),\ t>0,\ x\in\mathbb{R}^d\\ &\lim_{t\to 0+}\int_B u(t,x)\mathrm{d}x=\varphi(B),\quad B\in\mathcal{B}.\end{aligned} \tag{D.3}$$

If $d\geq 2$, it is easy to produce functions W such that $u(t)$ is well-defined by (D.1) for every $t\geq 0$, but, for many $\varphi\in E$, the integral equation (D.2) does not have a solution. Then the problem (D.3) does not have a solution either. For example, $W(t,x)=-|x|^{-d}$, $t\geq 0$, $x\in\Lambda$, $x\neq 0$, is such a function. Still, $u(t)$ has a perfectly good physical interpretation. (See Section 0C.)

E. Let $d \geq 1$ be an integer. We take, again, $\Lambda = \mathbb{R}^d$. Let $E = L^2(\lambda)$, where λ is the Lebesgue measure in $\mathbb{R}^d$. Elements of the space E and functions on $\mathbb{R}^d$ representing them will not be distinguished. The norm of an element, φ, of E will be denoted by $\|\varphi\|$.

For any $B \in \mathcal{B} = \mathcal{B}(\mathbb{R}^d)$, let $P(B)$ be the operator of point-wise multiplication by the characteristic function of the set B. That is, $P(B)\varphi = B\varphi$, for every $\varphi \in E$. Then $P : \mathcal{B} \to \mathrm{BL}(E)$ is a σ-additive spectral measure.

Let m be a strictly positive number. Let $S(0) = I$ and, for every $t \in \mathbb{R}$, $t \neq 0$, let $S(t) \in \mathrm{BL}(E)$ be the operator such that

$$(S(t)\varphi(x) = \left[\frac{m}{2\pi t \mathrm{i}}\right]^{\frac{1}{2}d} \int_{\mathbb{R}^d} \varphi(y) \exp\left[\frac{m\mathrm{i}}{2t} |x-y|^2\right] \mathrm{d}y ,$$

for every $x \in \mathbb{R}^d$ and every $\varphi \in L^1 \cap L^2(\lambda)$. The root is determined from the branch that assigns positive real values to positive real numbers. It is well-known, and can easily be shown using the Plancherel theorem, say, that such an operator $S(t)$ exists, for every $t \in \mathbb{R}$, is unique and the resulting map $t \mapsto S(t)$, $t \in \mathbb{R}$, is a unitary group of operators. It is called the Schrödinger group. The infinitesimal generator of the Schrödinger group, $S : \mathbb{R} \to \mathrm{BL}(E)$, is (the closure of) the operator

$$A = \frac{\mathrm{i}}{2m} \Delta ,$$

where Δ is the Laplacian on $\mathbb{R}^d$.

Let $t \geq 0$. Let Υ_t be the set of all continuous paths $v : [0,t] \to \mathbb{R}^d$. Let the set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$ be defined by the formula

$$M_t(Y) = S(t-t_k)P(B_k)S(t_k-t_{k-1})P(B_{k-1}) \dots P(B_2)S(t_2-t_1)P(B_1)S(t_1) ,$$

for every set

$$Y = \{v \in \Upsilon_t : v(t_j) \in B_j ,\ j = 1,2,\dots,k\} ,$$

where $k \geq 1$ is an integer, the sets B_j belong to $\mathcal{B}$ and the numbers t_j, $j = 1,2,\dots,k$, satisfy the conditions $0 \leq t_{j-1} < t_j \leq t$ for every $j = 2,3,\dots,k$. Let φ be an element of the space E.

Our aim is to produce a gauge on $\mathcal{R}_t$ which integrates for the set function $M_t\varphi$. Actually, a suggestion for producing such a gauge is presented in Example 4.33, because the construction exhibited there for $d = 1$ can be easily adapted for arbitrary d. However, we present now another construction.

By a special partition of Υ_t we shall understand any $\mathcal{R}_t$-partition, $\mathcal{P}$, obtained in the following manner. (See Section 1D.) Assume that $k \geq 1$ is an integer, $\mathcal{A}_j$ are $\mathcal{B}$-partitions of $\mathbb{R}^d$ and t_j are numbers, $j = 0,1,\ldots,k$, such that $t_0 = 0$, $t_{j-1} < t_j$, for every $j = 1,2,\ldots,k$, and $t_k = t$. The partition $\mathcal{P}$ then consists of all sets of the form

$$Y = Y(t_0,t_1,\ldots,t_k;B_0,B_1,\ldots,B_k) = \{v \in \Upsilon_t : v(t_j) \in B_j, j = 0,1,\ldots,k\},$$

with arbitrary sets B_j belonging to the partition $\mathcal{A}_j$, for every $j = 0,1,\ldots,k$. We say that the partition $\mathcal{P}$ is determined by the numbers t_j and partitions $\mathcal{A}_j$, $j = 0,1,\ldots,k$. The set of all special partitions of Υ_t will be denoted by Γ.

Our construction uses the fact, proved in the following proposition, that the set function $M_t\varphi$ has finite 2-variation with respect to the set of partitions Γ. (See Section 4A.)

PROPOSITION 7.8. *For every special partition,* $\mathcal{P}$, *we have* $v_2(M_t\varphi,\mathcal{P};\Upsilon_t) = \|\varphi\|^2$. *Consequently,* $v_2(M_t\varphi,\Gamma;\Upsilon_t) = \|\varphi\|^2$.

Proof. Let the partition $\mathcal{P}$ be determined by the numbers t_j and the $\mathcal{B}$-partitions $\mathcal{A}_j$, $j = 0,1,\ldots,k$. Because the operators $S(t-t_j)$ and $S(t_j-t_{j-1})$ are unitary and $\mathcal{A}_j$ is a $\mathcal{B}$-partition of $\mathbb{R}^d$, we have

$$\|M_t(Y(t_0,\ldots,t_{j-1};B_0,\ldots,B_{j-1}))\varphi\|^2 = \sum_{B_j \in \mathcal{A}_j} \|M_t(Y(t_0,\ldots,t_{j-1},t_j;B_j,\ldots,B_{j-1},B_j))\varphi\|^2,$$

for any sets $B_\ell \in \mathcal{A}_\ell$, $\ell = 0,1,\ldots,j-1$, and any $j = 1,2,\ldots,k$. Moreover, because $S(t)$ is a unitary operator and $\mathcal{A}_0$ is a $\mathcal{B}$-partition of $\mathbb{R}^d$,

$$\|\varphi\|^2 = \|M_t(\Upsilon_t)\varphi\|^2 = \sum_{B_0 \in \mathcal{A}_0} \|M_t(Y(t_0;B_0))\varphi\|^2.$$

Therefore,

$$\|\varphi\|^2 = \sum_{Y\in\mathcal{P}} \|M_t(Y)\varphi\|^2 = v_2(M_t\varphi,\mathcal{P};\Upsilon_t)\ .$$

Now, let μ be a non-negative real-valued σ-additive set function on $\mathcal{B}$ such that $\mu(B) = 0$ if and only if $\lambda(B) = 0$ and $\mu(\mathbb{R}^d) = 1$. Let ι be the d-dimensional Wiener measure of variance one, say, per unit of time with initial distribution μ. (See Section D.)

Given a partition $\mathcal{P} \in \Gamma$, let

$$\sigma_{\mathcal{P}}(X) = \sum_{Y\in\mathcal{P}} \frac{\|M_t(Y)\,\varphi\|^2}{\iota(\,Y)}\ \iota(X\cap Y)$$

for every $X \in \mathcal{R}_t$, putting, by convention, $\iota(X\cap Y)/\iota(\,Y) = 0$, whenever $\iota(\,Y) = 0$. By Proposition 2.13, the set function $\sigma_{\mathcal{P}}$ is an integrating gauge on $\mathcal{R}_t$, for every $\mathcal{P} \in \Gamma$. So, if P is a non-empty subset of Γ, by Proposition 2.14, the set function σ, defined by

$$\sigma(X) = \sup\{\sigma_{\mathcal{P}}(X) : \mathcal{P} \in \mathrm{P}\}\ ,$$

for every $X \in \mathcal{R}_t$, is an integrating gauge on $\mathcal{R}_t$.

By Proposition 7.8, however we choose P, the equality $\sigma(\Upsilon_t) = \|\varphi\|^2$ holds. Moreover, the gauge σ is monotonic. (See Section 2G.) We can choose P so that the inequality $\|M_t(X)\varphi\|^2 \le \sigma(X)$ holds for every $X \in \mathcal{R}_t$. To do that it suffices to take $\mathrm{P} = \Gamma$. However, much more economical choices of the set P are possible. In fact, there are countable subsets of Γ which can be chosen for such a P.

Having made such a choice of P, let $\rho(X) = (\sigma(X))^{\frac{1}{2}}$, for every $X \in \mathcal{R}_t$. Then, by Proposition 2.26, the gauge ρ integrates for the set function $M_t\varphi$.

It may be interesting to note that there does not necessarily exist a non-negative σ-additive set function, σ, on $\mathcal{R}_t$ such that the gauge $\sigma^{\frac{1}{2}}$ integrates for the set function $M_t\varphi$. In fact, we have the following proposition, in which $d = 1$, due to Brian Jefferies, which implies that $v_2(M_t\varphi,\Pi(\mathcal{R}_t);\Upsilon_t) = \infty$, for some $\varphi \in E$.

PROPOSITION 7.9. *Let $\mathcal{Q}$ be the semialgebra in the space $\mathbb{R}\times\mathbb{R}$ consisting of all sets of the form $A\times B$ with $A\in\mathcal{B}$ and $B\in\mathcal{B}$. Let $\varphi(x)=\exp(-\frac{1}{2}(1+i)x^2)$, for every $x\in\mathbb{R}$. Let $\nu(A\times B)=P(B)S(1)P(A)\varphi$, for every $A\in\mathcal{B}$ and $B\in\mathcal{B}$. Then $v_2(\nu,\Pi(\mathcal{Q});\mathbb{R}\times\mathbb{R})=\infty$.*

Proof. For any $A\in\mathcal{B}$ and $B\in\mathcal{B}$, we have

$$\|\nu(A\times B)\|^2=\frac{1}{2\pi}\int_A\int_B\int_A\bar{\varphi}(x_2)\exp(-\tfrac{1}{2}i(x_2-x_1)^2)\exp(\tfrac{1}{2}i(x_1-x_0)^2)\varphi(x_0)\,dx_0\,dx_1\,dx_2$$

$$=\frac{1}{2\pi}\int_B\left|\int_A\exp(i(\tfrac{1}{2}x_0^2-x_0x_1))\varphi(x_0)\,dx_0\right|^2dx_1=\frac{1}{2\pi}\int_B\left|\int_A\exp(-ixy-\tfrac{1}{2}x^2)\,dx\right|^2dy\,.$$

The Cauchy-Schwarz inequality implies that

$$\left|\int_B\int_A\exp(-ixy-\tfrac{1}{2}x^2)\,dx\,dy\right|^2\le\lambda(B)\int_B\left|\int_A\exp(-ixy-\tfrac{1}{2}x^2)\,dx\right|^2dy\,.$$

Now, for every n = 0,1,2,..., let $a_n=2\pi n+\pi/3$ and

$$C_n=\{(x,y):x\in\mathbb{R},\,y\in\mathbb{R},\,|xy-2\pi n|\le\pi/3\,,\ y\ge a_n\}\,.$$

Then $0\le x\le 1$ and $\cos(xy)\ge\frac{1}{2}$, for every $(x,y)\in C_n$, so

$$\left|\int_B\int_A\exp(-ixy-\tfrac{1}{2}x^2)\,dx\,dy\right|\ge\tfrac{1}{2}\exp(-\tfrac{1}{2})\lambda(A)\lambda(B)\,,$$

whenever $A\in\mathcal{B}$, $B\in\mathcal{B}$, $A\times B\subset C_n$, $n=0,1,2,\ldots$. Consequently,

$$\frac{1}{\lambda(B)}\left|\int_B\int_A\exp(-ixy-\tfrac{1}{2}x^2)\,dx\,dy\right|^2\ge\tfrac{1}{4}\exp(-1)(\lambda(A))^2\lambda(B)\,,$$

and, hence,

$$\text{(E.1)}\qquad\|\nu(A\times B)\|^2\ge\frac{\exp(-1)}{8\pi}(\lambda(A))^2\lambda(B)\,,$$

for any sets $A\in\mathcal{B}$ and $B\in\mathcal{B}$ such that $A\times B\subset C_n$, $n=0,1,2,\ldots$.

If B is sufficiently small interval centered around a point $y>a_n$, then there exists an interval, A, of length arbitrarily close to $2\pi/3y$ such that $A\times B\subset C_n$, $n=0,1,2,\ldots$. Moreover, for every $n=0,1,2,\ldots$, the set C_n contains a pair-wise disjoint family, $\mathcal{J}_n$, of such sets, $A \times B$, which can be chosen so that

$$\sum_{A \times B \in \mathcal{J}_n} (\lambda(A))^2 \lambda(B) > \int_{a_n}^{\infty} \left[\frac{2\pi}{3y}\right]^2 dy - 2^{-n} .$$

Because the sets C_n , $n = 0,1,2,\ldots,$ are pair-wise disjoint, by (E.1), the 2-variation, $v_2(\nu,\Pi(\mathcal{Q});\mathbb{R} \times \mathbb{R})$, of the set function ν is not less than

$$\frac{\exp(-1)}{8\pi} \sum_{n=0}^{\infty} \left[\int_{a_n}^{\infty} \left[\frac{2\pi}{3y}\right]^2 dy - 2^{-n} \right] = \infty .$$