5. VECTOR VALUED FUNCTIONS AND PRODUCTS

The title practically gives away the content of this chapter. We present first a Bochner-type integration theory, that is, one based on absolute summability, for Banach space valued functions. Then we consider direct products of integrating gauges along with the corresponding Fubini- and Tonelli-type theorems. These two themes are related in the formulation of the mentioned theorems; the notion of a measurable function is avoided by stating them in terms of Bochner integrability.

A. Let ρ be a gauge on a nontrivial family, $\mathcal{K}$, of scalar valued functions on a space Ω. (See Section 2A.)

Let E be a Banach space. To avoid some obvious trivialities, we assume that E contains a non-zero vector. The convention of writing interchangeably $ca = ac$, for every $c \in E$ and a scalar a, will be used throughout the chapter.

A function $f : \Omega \to E$ will be called Bochner integrable with respect to ρ, or, briefly, ρ-integrable, if there exist vectors $c_j \in E$ and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, such that

(A.1) $$\sum_{j=1}^{\infty} |c_j| \rho(f_j) < \infty$$

and

(A.2) $$f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for every $\omega \in \Omega$ for which

(A.3) $$\sum_{j=1}^{\infty} |c_j| \ |f_j(\omega)| < \infty .$$

The family of all E-valued functions on Ω, Bochner integrable with respect to ρ, is denoted by $\mathcal{L}(\rho,\mathcal{K},E)$. If the space E happens to be one-dimensional, that is, just the space of scalars, then, consistently with the notation introduced in Chapter 2, we write $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{L}(\rho,\mathcal{K},E)$.

For any function $f \in \mathcal{L}(\rho,\mathcal{K},E)$, let

$$q_\rho(f) = \inf \sum_{j=1}^{\infty} |c_j| \rho(f_j) ,$$

where the infimum is taken over all choices of the vectors $c_j \in E$ and the functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ satisfying condition (A.1), such that the equality (A.2) holds for every $\omega \in \Omega$ for which the inequality (A.3) does.

Clearly, $\mathcal{L}(\rho,\mathcal{K},E)$ is a vector space and $\mathrm{sim}(\mathcal{K},E)$ is a vector subspace of it. (See Section 1D.) Also, it is not difficult to see that q_ρ is a seminorm on $\mathcal{L}(\rho,\mathcal{K},E)$. Consequently, we can speak of q_ρ-Cauchy and q_ρ-convergent sequences of functions from $\mathcal{L}(\rho,\mathcal{K},E)$.

The ρ-equivalence class of a function $f \in \mathcal{L}(\rho,\mathcal{K},E)$ consisting of all functions $g \in \mathcal{L}(\rho,\mathcal{K})$ such that $q_\rho(f-g) = 0$, is denoted by $[f]_\rho$. The set $\{[f]_\rho : f \in \mathcal{L}(\rho,\mathcal{K},E)\}$ of all ρ-equivalence classes of functions from $\mathcal{L}(\rho,\mathcal{K},E)$ is denoted by $L(\rho,\mathcal{K},E)$. Then $L(\rho,\mathcal{K},E)$ is a normed space with respect to the linear operations induced by those of $\mathcal{L}(\rho,\mathcal{K},E)$ and the norm induced by the seminorm q_ρ . This norm is still denoted by the same symbols q_ρ .

A function $f : \Omega \to E$ is said to be ρ-null if $f \in \mathcal{L}(\rho,\mathcal{K},E)$ and $q_\rho(f) = 0$. As to the null sets, their definition remains of course the same as in Section 2B. Namely, a set $Z \subset \Omega$ is ρ-null if its characteristic function is a ρ-null element of $\mathcal{L}(\rho,\mathcal{K})$.

The introduced definitions do not differ in form from those concerning scalar valued functions given in Chapter 2. Because the space E is non-trivial, the treatment of scalar valued integrable functions presented in Sections 2A-2D is applicable practically without a change to E-valued functions. This fact was first noted by J. Mikusiński who exploited it, in [50], for his definition of Bochner integrable functions (in the usual sense). It may be useful to note explicitly that Proposition 2.2 remains valid if by a function is meant an E-valued function and if $\mathcal{L}(\rho,\mathcal{K})$ is replaced by $\mathcal{L}(\rho,\mathcal{K},E)$. It implies that a set $Z \subset \Omega$ is ρ-null if and only if there exist functions $f_j \in \mathcal{L}(\rho,\mathcal{K},E)$, $j = 1,2,\ldots,$ such that

$$\text{(A.4)} \qquad \sum_{j=1}^{\infty} q_\rho(f_j) < \infty$$

and

$$\sum_{j=1}^{\infty} |f_j(\omega)| = \infty .$$

for every $\omega \in Z$.

The following theorem, which is analogous to Theorem 2.3, is singled out because of its central importance. Its proof is of course omitted.

THEOREM 5.1. *Let the functions* $f_j \in \mathcal{L}(\rho,\mathcal{K},E)$, $j = 1,2,\ldots,$ *satisfy condition* (A.4). *Then*

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty$$

for ρ-*almost every* $\omega \in \Omega$. *Furthermore, if* $f : \Omega \to E$ *is a function such that*

$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

for ρ-*almost every* $\omega \in \Omega$, *then* $f \in \mathcal{L}(\rho,\mathcal{K},E)$ *and*

$$\lim_{n\to\infty} q_\rho\left[f - \sum_{j=1}^{n} f_j\right] = 0 .$$

Among the implications of this theorem is that $L(\rho,\mathcal{K},E)$ is a Banach space.

Also, it may seem that the natural seminorm of the space $\mathcal{L}(\rho,\mathcal{K},E)$ should be denoted more accurately by $q_{\rho,E}$ rather than simply by q_ρ . For if E is a subspace of a Banach space F and $f \in \mathcal{L}(\rho,\mathcal{K},E)$, then also $f \in \mathcal{L}(\rho,\mathcal{K},F)$ and $q_{\rho,F}(f) \le q_{\rho,E}(f)$. However, Theorem 5.1 implies that actually $q_{\rho,F}(f) = q_{\rho,E}(f)$, and so, the simpler notation suffices.

The introduced notions are useful perhaps only if the gauge ρ is integrating. (See Section 2D.) The proof of the following straightforward proposition is omitted.

PROPOSITION 5.2. *The gauge* ρ *is integrating if and only if* $|c|\rho(f) = q_\rho(cf)$, *for every function* $f \in \mathcal{K}$ *and a vector* $c \in E$.

B. We maintain the notation of the previous section.

Let us start with two observations. It seems that we would obtain a larger class of Bochner integrable functions if we called Bochner integrable all functions belonging to $\mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K}),E)$ rather than those belonging to $\mathcal{L}(\rho,\mathcal{K},E)$, that is, if we admitted for f_j, $j = 1,2,\dots,$ in (A.2), any functions from $\mathcal{L}(\rho,\mathcal{K})$ and not merely from $\mathcal{K}$. However, it is not the case, because Theorem 5.1 clearly implies that $\mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K}),E) = \mathcal{L}(\rho,\mathcal{K},E)$. This is an extension of the last statement of Proposition 2.7.

More interestingly, we can look at $L(\rho,\mathcal{K},E)$ as the projective tensor product of the spaces E and $L(\rho,\mathcal{K})$. (See Section 1C.) Formally, we have the following

PROPOSITION 5.3. *There is a unique isometric isomorphism of the projective tensor product,* $E\hat{\otimes}L(\rho,\mathcal{K})$, *of the spaces* E *and* $L(\rho,\mathcal{K})$ *onto the space* $L(\rho,\mathcal{K},E)$, *that maps the tensor product,* $c\otimes[f]_\rho$, *of any element,* c, *of* E *and element,* $[f]_\rho$, *of* $L(\rho,\mathcal{K})$ *to the element* $[cf]_\rho$ *of the space* $L(\rho,\mathcal{K},E)$.

Proof. Every element, z, of the projective tensor product $E\hat{\otimes}L(\rho,\mathcal{K})$ can be written in the form

(B.1)
$$z = \sum_{j=1}^{\infty} c_j \otimes [f_j]_\rho ,$$

where the vectors $c_j \in E$ and the functions $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,\dots,$ satisfy the condition

(B.2)
$$\sum_{j=1}^{\infty} |c_j|\, q_\rho(f_j) < \infty .$$

Moreover, the norm of z in the space $E\hat{\otimes}L(\rho,\mathcal{K})$ is equal to the infimum of the numbers (B.2) subject to the equality (B.1). By Theorem 5.1, any function, f, on Ω, such that (A.2) holds for every $\omega \in \Omega$ for which (A.3) does, belongs to $\mathcal{L}(\rho,\mathcal{K},E)$ and its seminorm, $q_\rho(f)$, is equal to the norm of z. Therefore, if we let correspond to z the element, $[f]_\rho$, of the space $L(\rho,\mathcal{K},E)$ determined by any such function f, we obtain an unambiguously defined map of $E\hat{\otimes}L(\rho,\mathcal{K})$ into $L(\rho,\mathcal{K},E)$. Clearly, this map is a linear isometry. Because, however, every element of the space $L(\rho,\mathcal{K},E)$ is the image of an element of $E\hat{\otimes}L(\rho,\mathcal{K})$, this map is an isometric isomorphism of the spaces $E\hat{\otimes}L(\rho,\mathcal{K})$ and $L(\rho,\mathcal{K},E)$ which, for any $c \in E$ and $f \in \mathcal{L}(\rho,\mathcal{K})$, maps $c\otimes[f]_\rho$ to $[cf]_\rho$.

The spaces $L(\rho,\mathcal{K})$ and $L(\rho,\mathcal{K},E)$ cannot be replaced, in this proposition, by $\mathcal{L}(\rho,\mathcal{K})$ and $\mathcal{L}(\rho,\mathcal{K},E)$, respectively, even if the notion of a tensor product were extended to seminormed spaces. For, any E-valued function that vanishes ρ-almost everywhere belongs to $\mathcal{L}(\rho,\mathcal{K},E)$. Consequently, the space $\mathcal{L}(\rho,\mathcal{K},E)$ may contain functions that cannot be canonically specified by a sequence of vectors from E and a sequence of functions from $\mathcal{L}(\rho,\mathcal{K})$.

Now, let F and G be Banach spaces and let $b : E\times F \to G$ be a continuous bilinear map. (See Section 1C.) Let $\mu : \mathcal{K} \to F$ be an additive map. Assume that the gauge ρ integrates for the map μ. (See Section 3A.)

PROPOSITION 5.4. *There exists a unique continuous linear map,* $\mu_{\rho,b} : \mathcal{L}(\rho,\mathcal{K},E) \to G$, *such that*

$$\mu_{\rho,b}(cf) = b(c,\mu(f)), \tag{B.3}$$

for any vector $c \in E$ *and a function* $f \in \mathcal{K}$.

Proof. By the basic property of projective tensor products, there exists a unique continuous linear map, $\ell : E\hat{\otimes}L(\rho,\mathcal{K}) \to G$, such that $\ell(c\otimes[f]_\rho) = b(c,\mu(f))$, for every $c \in E$ and $f \in \mathcal{L}(\rho,\mathcal{K})$. Because the vector space spanned by $\{[f]_\rho : f \in \mathcal{K}\}$ is dense in $L(\rho,\mathcal{K})$, ℓ is the unique continuous linear map from $E\hat{\otimes}L(\rho,\mathcal{K})$ to G, such that $\ell(c\otimes[f]_\rho) = b(c,\mu(f))$, for every $c \in E$ and $f \in \mathcal{K}$. Now, for every $f \in \mathcal{L}(\rho,\mathcal{K},E)$, let $\mu_{\rho,b}(f) = \ell(z)$, where z is the element of the space $E\hat{\otimes}L(\rho,\mathcal{K})$ such that the element $[f]_\rho$ of $L(\rho,\mathcal{K},E)$ is the image of z under the isomorphism of Proposition 5.3. By the definition of the space $L(\rho,\mathcal{K},E)$ and Proposition 5.3, this defines a unique continuous linear map, $\mu_{\rho,b} : \mathcal{L}(\rho,\mathcal{K},E) \to G$, such that (B.3) holds for every $c \in E$ and every $f \in \mathcal{K}$.

Under the assumptions of this proposition, we write

$$\int_\Omega b(f,\mathrm{d}_\rho\mu) = \int_\Omega b(f(\omega),\mu(\mathrm{d}_\rho\omega)) = \mu_{\rho,b}(f)\ ,$$

for every function $f \in \mathcal{L}(\rho,\mathcal{K},E)$. Of course, if a different notation is used for the bilinear map b, then it is also used in the symbol for integral. So, for example, if we write $b(x,y) = xy$, for any $x \in E$ and $y \in F$, using simple juxtaposition, then we also write

$$\int_\Omega f\mathrm{d}_\rho\mu = \mu_{\rho,b}(f)\ ,$$

for every $f \in \mathcal{L}(\rho,\mathcal{K},E)$. Or, if the function f is F-valued and the map μ is E-valued, we denote the integral by

$$\int_\Omega b(\mathrm{d}_\rho\mu,f) = \int_\Omega b(\mu(\mathrm{d}_\rho\omega),f(\omega))\ .$$

C. Let the space Ω be equal to the Cartesian product of the spaces Ξ and Υ. That is, $\Omega = \Xi\times\Upsilon$.

If g is a function on Ξ and h a function on Υ, then $f = g\otimes h$ will stand for the function on Ω such that $f(\omega) = g(\xi)h(v)$, for every $\omega = (\xi,v)$ with $\xi \in \Xi$ and $v \in \Upsilon$.

Let $\mathcal{G}$ be a nontrivial family of functions on the space Ξ and $\mathcal{H}$ a nontrivial family of functions on the space Υ. Let $\mathcal{K} = \{g\otimes h : g \in \mathcal{G}\ ,\ h \in \mathcal{H}\}$.

Let σ be a gauge on $\mathcal{G}$ and τ a gauge on $\mathcal{H}$. By $\rho = \sigma\otimes\tau$ is denoted the gauge on $\mathcal{K}$ such that

$$\rho(f) = \sigma(g)\tau(h)\ ,$$

for any function $f = g\otimes h$ with $g \in \mathcal{G}$ and $h \in \mathcal{H}$. The gauge ρ is called the direct product of the gauges σ and τ.

PROPOSITION 5.5. *If the gauges* σ *and* τ *are both integrating, then their direct product,* $\rho = \sigma\otimes\tau$, *too is integrating.*

Proof. Let $f = g\otimes h$, $g \in \mathcal{G}$, $h \in \mathcal{H}$. Let c_j be numbers and $f_j \in \mathcal{K}$ functions, $f_j = g_j\otimes h_j$, $g_j \in \mathcal{G}$, $h_j \in \mathcal{H}$, $j = 1,2,\dots$, such that

$$\text{(C.1)} \qquad \sum_{j=1}^{\infty} |c_j|\rho(f_j) < \infty$$

and

$$f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j||f_j(\omega)| < \infty .$$

Let ℓ be a continuous linear functional of norm not greater than one on the space $L(\sigma,\mathcal{G})$ such that $|\ell([g])_\sigma| = q_\sigma(g) = \sigma(g)$, and m a continuous linear functional of norm not greater than one on the space $L(\tau,\mathcal{H})$ such that $|m(h)| = q_\tau([h]_\tau) = \tau(h)$.

Let $u = \ell(g)h$ and $u_j = c_j\ell(g_j)h_j$, for every $j = 1,2,\dots$. Then $q_\tau(u_j) = |c_j||\ell(g_j)|\tau(h_j) \le |c_j|\sigma(g_j)\tau(h_j) = |c_j|\rho(f_j)$, for every $j = 1,2,\dots$, and, by (C.1),

$$\text{(C.2)} \qquad \sum_{j=1}^{\infty} q_\tau(u_j) < \infty .$$

Now, for every $v \in \Upsilon$, such that

$$\text{(C.3)} \qquad \sum_{j=1}^{\infty} |c_j|\sigma(g_j)|h_j(v)| < \infty ,$$

let Ξ_v be the set of all points $\xi \in \Xi$ such that

$$\sum_{j=1}^{\infty} |c_j||g_j(\xi)||h_j(v)| < \infty .$$

By (5.10) and Proposition 2.2, the set $\Xi\backslash\Xi_v$ is σ-null, and, by Theorem 2.3,

$$\lim_{n\to\infty} q_\sigma\left[h(v)g - \sum_{j=1}^{n} c_j h_j(v) g_j\right] = 0 .$$

Hence, by the continuity of the functional ℓ,

$$\sum_{j=1}^{\infty} u_j(v) = u(v)$$

for every $v \in \Upsilon$ such that (C.3) holds. However, by (C.2) and Theorem 2.3, (C.3) holds for τ-almost every $v \in \Upsilon$. Therefore, by (C.2) and Theorem 2.3,

$$\lim_{n\to\infty} q_\tau\left[u - \sum_{j=1}^{n} u_j\right] = 0 .$$

So, by the continuity of the functional m,

$$\ell(g)m(h) = m(u) = \sum_{j=1}^{\infty} m(u_j) = \sum_{j=1}^{\infty} c_j\ell(g_j)m(h_j) .$$

Consequently,

$$\rho(f) = \sigma(g)\tau(h) = |\ell(g)m(h)| \le \sum_{j=1}^{\infty} |c_j|\rho(f_j) ,$$

because $|\ell(g_j)m(h_j)| \le q_\sigma(g_j)q_\tau(h_j) = \sigma(g_j)\tau(h_j) = \rho(f_j)$, for every $j = 1,2,\dots$. Hence, by Proposition 2.7, the gauge ρ is integrating.

In many situations, for example when $\mathcal{G}$ and $\mathcal{H}$ are quasirings of sets and σ and τ non-negative σ-additive set functions or $\mathcal{G}$ and $\mathcal{H}$ are vector spaces and σ and τ seminorms, a simpler direct proof of this proposition, avoiding the duality considerations, can be given. The proof presented here was suggested by Brian Jefferies.

D. Let Ξ, Υ, Ω, $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ have the same meaning as in Section C .

PROPOSITION 5.6. *Let σ be an integrating gauge on $\mathcal{G}$ and τ an integrating gauge on $\mathcal{H}$ and let $\rho = \sigma\otimes\tau$ be their direct product.*

If $g \in \mathcal{L}(\sigma,\mathcal{G})$ and $h \in \mathcal{L}(\tau,\mathcal{H})$, then the function $f = g\otimes h$ is ρ-integrable and $q_\rho(f) = q_\sigma(g)q_\tau(h)$.

For every function $f \in \mathcal{L}(\rho,\mathcal{K})$, *there exist functions* $g_j \in \mathcal{L}(\sigma,\mathcal{G})$ *and* $h_j \in \mathcal{L}(\tau,\mathcal{H})$, $j = 1,2,\ldots$, *such that*

$$\sum_{j=1}^{\infty} q_\sigma(g_j)q_\tau(h_j) < \infty \tag{D.1}$$

and

$$f(\xi,v) = \sum_{j=1}^{\infty} g_j(\xi)h_j(v) , \tag{D.2}$$

for every $\xi \in \Xi$ *and* $v \in \Upsilon$ *such that*

$$\sum_{j=1}^{\infty} |g_j(\xi)h_j(v)| < \infty . \tag{D.3}$$

Furthermore,

$$\lim_{n\to\infty} q_\rho\left[f - \sum_{j=1}^{n} g_j \otimes h_j\right] = 0 . \tag{D.4}$$

Conversely, if f *is a function on* Ω *for which there exists functions* $g_j \in \mathcal{L}(\sigma,\mathcal{G})$ *and* $h_j \in \mathcal{L}(\tau,\mathcal{H})$, $j = 1,2,\ldots$, *satisfying condition* (D.1), *such that the equality* (D.2) *holds for every* $\xi \in \Xi$ *and* $v \in \Upsilon$ *for which the inequality* (D.3) *does, then* $f \in \mathcal{L}(\rho,\mathcal{K})$.

Proof. By a straightforward application of Proposition 2.1, if $g \in \mathcal{G}$, $h \in \mathcal{L}(\tau,\mathcal{K})$ and $f = g \otimes h$, then $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = \sigma(g)q_\tau(h)$. By a second application of Proposition 2.1, if $g \in \mathcal{L}(\sigma,\mathcal{G})$, $h \in \mathcal{L}(\tau,\mathcal{K})$, then $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = q_\sigma(g)q_\tau(h)$.

If $f \in \mathcal{L}(\rho,\mathcal{K})$, then such functions $g_j \in \mathcal{L}(\sigma,\mathcal{G})$ and $h_j \in \mathcal{L}(\tau,\mathcal{H})$, $j = 1,2,\ldots$, as claimed exist trivially because $\mathcal{G} \subset \mathcal{L}(\sigma,\mathcal{G})$, $\mathcal{H} \subset \mathcal{L}(\tau,\mathcal{H})$, $q_\sigma(g) = \sigma(g)$, for $g \in \mathcal{G}$; and $q_\tau(h) = \tau(h)$, for $h \in \mathcal{H}$. The equality (D.4) follows by Proposition 2.1. Conversely, if such functions g_j and h_j do exist, then, as we have just noted, the functions $f_j = g_j \otimes h_j$ belong to $\mathcal{L}(\rho,\mathcal{K})$, for every $j = 1,2,\ldots$, and, hence, by Proposition 2.1, $f \in \mathcal{L}(\rho,\mathcal{K})$.

COROLLARY 5.7. *There is a canonical isometric isomorphism of the space* $L(\rho,\mathcal{K})$ *onto the projective tensor product of the spaces* $L(\sigma,\mathcal{G})$ *and* $L(\tau,\mathcal{H})$.

Proof. Every element, t, of the projective tensor product of the spaces $L(\sigma,\mathcal{G})$ and $L(\tau,\mathcal{H})$ can be written in the form

$$t = \sum_{j=1}^{\infty} [g_j]_\sigma \otimes [h_j]_\tau ,$$

where the functions $g_j \in \mathcal{L}(\sigma,\mathcal{G})$ and $h_j \in \mathcal{L}(\tau,\mathcal{H})$, $j = 1,2,\ldots$, satisfy condition (D.1). Moreover, the infimum of the numbers (D.1) over all such representations of t is equal to the projective tensor product norm of the element t.

PROPOSITION 5.8. *Let E, F and G be Banach spaces and let $b : F\times G \to E$ be a continuous bilinear map. Let σ be a gauge integrating for an additive map $\nu : \mathcal{G} \to F$ and τ a gauge integrating for an additive map $\lambda : \mathcal{H} \to G$. Let $\rho = \sigma\otimes\tau$ be their direct product. Let $\mu(f) = b(\nu(g),\lambda(h))$, for every $f = g\otimes h$ such that $g \in \mathcal{G}$ and $h \in \mathcal{H}$.*

Then $\mu : \mathcal{K} \to E$ is an additive map and the gauge ρ integrates for μ.

Proof. By Proposition 3.1, there exist a unique continuous linear map $\nu_\sigma : \mathcal{L}(\sigma,\mathcal{G}) \to F$ that extends ν and a unique continuous linear map $\lambda_\tau : \mathcal{L}(\tau,\mathcal{H}) \to G$ that extends λ. Let $\ell : L(\sigma,\mathcal{G}) \hat{\otimes} L(\tau,\mathcal{H}) \to E$ be the continuous linear map such that $\ell([g]_\sigma \otimes [h]_\tau) = b(\nu_\sigma(f),\lambda_\tau(h))$, for every $g \in \mathcal{L}(\sigma,\mathcal{G})$ and $h \in \mathcal{L}(\tau,\mathcal{H})$. Now, given a function $f \in \mathcal{L}(\rho,\mathcal{K})$, let t be the element of the tensor product $L(\sigma,\mathcal{G}) \hat{\otimes} L(\tau,\mathcal{H})$ that corresponds to the element $[f]_\rho$ of the space $L(\rho,\mathcal{K})$ under the isomorphism of Corollary 5.7 and let $\mu_\rho(f) = \ell(t)$. This defines a continuous linear map $\mu_\rho : \mathcal{L}(\rho,\mathcal{K}) \to E$ such that $\mu_\rho(f) = \mu(f)$, whenever $f = g\otimes h$ with $g \in \mathcal{G}$ and $h \in \mathcal{H}$. So, the map $\mu : \mathcal{K} \to E$ is indeed additive and the gauge ρ integrates for it.

EXAMPLE 5.9. Let E, F and G be Banach spaces, $b : F\times G \to E$ a continuous bilinear map. Let $\mathcal{Q}$ and $\mathcal{R}$ be σ-algebras of sets in the spaces Ξ and Υ, respectively. Let $\nu : \mathcal{Q} \to F$ and $\lambda : \mathcal{R} \to G$ be σ-additive set functions. Let $\mu(X\times Y) = b(\nu(X),\lambda(Y))$, for every $X \in \mathcal{Q}$ and $Y \in \mathcal{R}$. It is known that μ is not necessarily a σ-additive set function on the semialgebra $\mathcal{P} = \{X\times Y : X \in \mathcal{Q}, Y \in \mathcal{R}\}$; not even if $F = G$ is a Hilbert space, E is the space of scalars and b is the inner product in F. Cf. [13] and [58]. However, if

$$\sigma(g) = \sup\{v(y' \circ \nu, |g|) : y' \in F', \|y'\| \le 1\} ,$$

for every $g \in \mathrm{sim}(\mathcal{Q})$, and

$$\tau(h) = \sup\{v(z' \circ \lambda, |h|) : z' \in G', \|z'\| \le 1\} ,$$

for every $h \in \mathrm{sim}(\mathcal{R})$, then σ is a seminorm on $\mathrm{sim}(\mathcal{Q})$ integrating for ν and τ a seminorm on $\mathrm{sim}(\mathcal{R})$ integrating for λ. (See Section 3F, formula (F.2).) Therefore, $\rho = \sigma \otimes \tau$ is a gauge on the family of functions $\mathcal{K} = \{g \otimes h : g \in \mathrm{sim}(\mathcal{Q}) , \quad h \in \mathrm{sim}(\mathcal{R})\}$ which integrates for μ.

E. Let Ξ , Υ , Ω , $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ have the same meaning as in Section C .

Given a function f on Ω and a point $\xi \in \Xi$, by $f(\xi,\cdot)$ is denoted the function $v \mapsto f(\xi,v)$, $v \in \Upsilon$. The meaning of $f(\cdot,v)$, for a given $v \in \Upsilon$, is analogous.

PROPOSITION 5.10. *Let σ be an integrating gauge on $\mathcal{G}$ and τ an integrating gauge on $\mathcal{H}$. Let $\rho = \sigma \otimes \tau$. Let $f \in \mathcal{L}(\rho,\mathcal{K})$.*

Then, for σ-almost every $\xi \in \Xi$, the function $f(\xi,\cdot)$ is τ-integrable. Furthermore, if φ is an $L(\tau,\mathcal{H})$-valued function on Ξ such that $\varphi(\xi) = [f(\xi,\cdot)]_\tau$, for σ-almost every $\xi \in \Xi$, then the function φ is Bochner integrable with respect to σ and $q_\sigma(\varphi) = q_\rho(f)$.

Similarly, for τ-almost every $v \in \Upsilon$, the function $f(\cdot,v)$ is σ-integrable. Furthermore, if ψ is an $L(\sigma,\mathcal{G})$-valued function on Υ such that $\psi(v) = [f(\cdot,v)]_\sigma$, for τ-almost every $v \in \Upsilon$, then the function ψ is Bochner integrable with respect to τ and $q_\tau(\psi) = q_\rho(f)$.

Proof. Let c_j be numbers and $g_j \in \mathcal{G}$ and $h_j \in \mathcal{H}$ functions, $j = 1,2,\ldots,$ such that

$$\sum_{j=1}^{\infty} |c_j| \sigma(g_j)\tau(h_j) < \infty \tag{E.1}$$

and

(E.2) $$f(\xi,v) = \sum_{j=1}^{\infty} c_j g_j(\xi) h_j(v)$$

for every $\xi \in \Xi$ and $v \in \Upsilon$ for which

(E.3) $$\sum_{j=1}^{\infty} |c_j| |g_j(\xi)| |h_j(v)| < \infty .$$

By Proposition 2.2,

$$\sum_{j=1}^{\infty} |c_j| |g_j(\xi)| \tau(h_j) < \infty$$

for σ-almost every $\xi \in \Xi$. Furthermore, if $\xi \in \Xi$ is a point such that (E.3) holds, then $f(\xi,\cdot) \in \mathcal{L}(\tau,\mathcal{X})$, because (E.1) holds for every $v \in \Upsilon$ for which (E.2) does.

Now, let $\varphi_j(\xi) = c_j g_j(\xi)[h_j]_\tau$, for every $\xi \in \Xi$. Then $\varphi_j \in \mathcal{L}(\sigma,\mathcal{G},L(\tau,\mathcal{X}))$ and $q_\sigma(\varphi_j) = |c_j| q_\sigma(g_j) q_\tau(h_j) = |c_j| \sigma(g_j)\tau(h_j)$, for every $j = 1,2,\dots$. Let φ_0 be an $L(\tau,\mathcal{X})$-valued function on Ξ such that

$$\varphi_0(\xi) = \sum_{j=1}^{\infty} \varphi_j(\xi)$$

for every $\xi \in \Xi$ for which

$$\sum_{j=1}^{\infty} q_\tau(\varphi_j(\xi)) = \sum_{j=1}^{\infty} |c_j| |g_j(\xi)| \tau(h_j) < \infty .$$

Then, by Theorem 5.1, $\varphi_0 \in \mathcal{L}(\sigma,\mathcal{G},L(\tau,\mathcal{X}))$. So, if $\varphi(\xi) = \varphi_0(\xi)$ for σ-almost every $\xi \in \Xi$, then φ too belongs to $\mathcal{L}(\alpha,\mathcal{G},L(\tau,\mathcal{X}))$. The equality $q_\sigma(\varphi) = q_\rho(f)$ follows from the definition of the seminorm q_σ on $\mathcal{L}(\sigma,\mathcal{G},L(\tau,\mathcal{X}))$ and that of the seminorm q_ρ on $\mathcal{L}(\rho,\mathcal{K})$.

This proposition already contains all the ingredients necessary to state the following theorem of Fubini type.

THEOREM 5.11. *Let E, F and G be Banach spaces and $b : F\times G \to E$ a continuous bilinear map. Let σ be a gauge on $\mathcal{G}$ integrating for an additive map $\nu : \mathcal{G} \to F$ and τ a gauge on $\mathcal{X}$ integrating for an additive map $\lambda : \mathcal{X} \to G$. Let*

$\rho = \sigma \otimes \tau$. Let $\mu(f) = b(\nu(g), \lambda(h))$, for every function $f = g \otimes h$ such that $g \in \mathcal{G}$ and $h \in \mathcal{H}$.

If $f \in \mathcal{L}(\rho, \mathcal{K})$, then

(E.4)
$$\int_\Omega f(\omega)\mu(\mathrm{d}_\rho\omega) = \int_\Xi b\left[\nu(\mathrm{d}_\sigma\xi)\,,\,\int_\Upsilon f(\xi,v)\lambda(\mathrm{d}_\tau v)\right] = $$
$$= \int_\Upsilon b\left[\int_\Xi f(\xi,v)\nu(\mathrm{d}_\gamma\xi)\,,\,\lambda(\mathrm{d}_\tau v)\right].$$

Proof. The existence of the integrals follows from Proposition 5.10. The equalities (E.4) are obviously true if $f = g \otimes h$ with $g \in \mathcal{L}(\sigma, \mathcal{G})$ and $h \in \mathcal{L}(\tau, \mathcal{H})$. Furthermore, all terms are linear in f. Propositions 5.4 and 5.10 imply that all terms of (E.4) depend continuously on f. Because the algebraic tensor product $L(\sigma,\mathcal{G}) \otimes L(\tau,\mathcal{H})$ is isomorphic to a dense subspace of $L(\rho,\mathcal{K})$, the equalities (E.4) are valid for every $f \in \mathcal{L}(\rho,\mathcal{K})$.

F. We still maintain the notation of the previous section and assume that σ is an integrating gauge on $\mathcal{G}$ and τ an integrating gauge on $\mathcal{H}$. So, $\rho = \sigma \otimes \tau$ is an integrating gauge on $\mathcal{K}$.

We prove a converse to Proposition 5.10, which is a Tonelli-type theorem only under some additional assumptions.

ASSUMPTION 5.12. *Let $Z \subset \Omega$ and let X be the set of all points $\xi \in \Xi$ such that the set $\{v \in \Upsilon : (\xi, v) \in Z\}$ is not τ-null. If the set X is σ-null, then the set Z is ρ-null.*

The following proposition gives a convenient sufficient condition for Assumption 5.12 to be satisfied.

PROPOSITION 5.13. *If there exists a function $h \in \mathcal{L}(\tau, \mathcal{H})$ such that $h(v) \neq 0$, for every $v \in \Upsilon$, then Assumption 5.12 is satisfied.*

Proof. Let $Z \subset \Omega$ be a set and let X be the set of all points $\xi \in \Xi$ such that the set $\{v \in \Upsilon : (\xi, v) \in Z\}$ is not τ-null. If the set X is σ-null, then, by Proposition 2.2, there exist numbers c_j and functions $g_j \in \mathcal{G}$, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} |c_j| \sigma(g_j) < \infty$$

but

$$\sum_{j=1}^{\infty} |c_j| |g_j(\xi)| = \infty$$

for every $\xi \in \Xi$. Let $h \in \mathcal{L}(\tau,\mathcal{Y})$ be a function such that $h(v) \neq 0$, for every $v \in \Upsilon$, and let $f_j(\omega) = c_j g_j(\xi) h(v)$, for every $\omega = (\xi, v)$, $\xi \in \Xi$, $v \in \Upsilon$, so that $q_\rho(f_j) = |c_j| \sigma(g_j) q_\tau(h)$, for every $j = 1,2,\ldots$. Then

$$\sum_{j=1}^{\infty} q_\rho(f_j) < \infty,$$

but

$$\sum_{j=1}^{\infty} |f_j(\omega)| = \infty$$

for every $\omega \in Z$. By Proposition 2.2, the set Z is ρ-null.

PROPOSITION 5.14. *Let Assumption 5.12 be satisfied. Let f be a function on Ω such that, for σ-almost every $\xi \in \Xi$, the function $f(\xi,\cdot)$ is τ-integrable and, if φ is an $L(\tau,\mathcal{Y})$-valued function on Ξ such that $\varphi(\xi) = [f(\xi,\cdot)]_\tau$, for σ-almost every $\xi \in \Xi$, then the function φ is Bochner integrable with respect to σ.*

Then $f \in \mathcal{L}(\rho,\mathcal{X})$.

Proof. Let $g_j \in \mathcal{G}$ and $h_j \in \mathcal{L}(\tau,\mathcal{Y})$, $j = 1,2,\ldots$, be functions such that

$$\sum_{j=1}^{\infty} q_\rho(g_j \otimes h_j) = \sum_{j=1}^{\infty} \sigma(g_j) q_\tau(h_j) < \infty$$

and

$$\varphi(\xi) = \sum_{j=1}^{\infty} g_j(\xi)[h_j]_\tau,$$

in the sense of convergence in the space $L(\tau,\mathcal{Y})$, for every $\xi \in \Xi$ for which

$$\text{(F.1)} \qquad \sum_{j=1}^{\infty} |g_j(\xi)| q_\tau(h_j) < \infty.$$

By a modification of the function φ and/or f on a set of points $\xi \in \Xi$ which is negligible with respect to σ, we can achieve that $\varphi(\xi) = [f(\xi,\cdot)]_{\tau}$, for every point $\xi \in \Xi$ for which the inequality (F.1) holds. Then, given such a ξ, the equality

$$f(\xi,v) = \sum_{j=1}^{\infty} g_j(\xi)h_j(v) \tag{F.2}$$

holds for τ-almost every $(\xi,v) \in \Omega$. Therefore, by Theorem 2.3, $f \in \mathcal{L}(\rho,\mathcal{K})$.

If Assumption 5.12 is not satisfied, then the conclusion of this proposition does not necessarily hold.

EXAMPLE 5.15. Let $\Xi = (0,1]$, $\mathcal{Q} = \{(s,t] : 0 \leq s \leq t \leq 1\}$ and let ι be the Lebesgue measure on $\mathcal{Q}$. Let $\Upsilon = (0,1]$, let $\mathcal{R}$ be the family of all finite subsets of Υ and, for every $Y \in \mathcal{R}$, let $\kappa(Y)$ be the number of elements in Y.

Let f be the characteristic function of the set $\{1\}\times\Upsilon$. Then, for ι-almost every $\xi \in \Xi$, $f(\xi,\cdot)$ is the zero-function on Υ, but the function f does not belong to $\mathcal{L}(\iota\otimes\kappa,\mathcal{K})$, where $\mathcal{K} = \{X\times Y : X \in \mathcal{Q}, Y \in \mathcal{R}\}$.

Obviously, the roles of the spaces Ξ and Υ, and of the structures they carry, are not symmetric in Assumption 5.12, Proposition 5.13 and Proposition 5.14. Although, it is quite clear how to formulate analogous assumptions and propositions with these roles interchanged, for the record we formulate the analogies of Assumption 5.12 and Proposition 5.14.

ASSUMPTION 5.16. *Let $Z \subset \Omega$ and let Y be the set of all points $v \in \Upsilon$ such that the set $\{\xi \in \Xi : (\xi,v) \in Z\}$ is not σ-null. If the set Y is τ-null, then the set Z is ρ-null.*

PROPOSITION 5.17. *Let Assumption 5.16 be satisfied. Let f be a function on Ω such that, for τ-almost every $v \in \Upsilon$, the function $f(\cdot,v)$ is σ-integrable and, if ψ is an $L(\sigma,\mathcal{G})$-valued function on Υ such that $\psi(v) = [f(\cdot,v)]_{\sigma}$, for τ-almost every $v \in \Upsilon$, then the function ψ is Bochner integrable with respect to τ.*

Then $f \in \mathcal{L}(\rho,\mathcal{K})$.