4. SET FUNCTIONS

Given an additive set function, μ, on a semiring of sets, $\mathcal{Q}$, the problem arises naturally of finding a gauge which integrates for μ. (See Section 3A.) If there exists a finite non-negative σ-additive set function, ι, on $\mathcal{Q}$ such that $|\mu(X)| \le \iota(X)$, for every $X \in \mathcal{Q}$, then μ is said to have finite variation. In that case, ι is a gauge integrating for μ. This situation is classical.

The point of this chapter is that, even when μ does not have finite variation, there may exist gauges integrating for μ. For, there may exist a continuous, convex and increasing function, Φ, on $[0,\infty)$ such that $\Phi(0) = 0$ and a σ-additive set function $\iota : \mathcal{Q} \to [0,\infty)$ such that $\Phi(|\mu(X)|) \le \iota(X)$, for every $X \in \mathcal{Q}$. Then $|\mu(X)| \le \rho(X)$, where $\rho(X) = \varphi(\iota(X))$, for every $X \in \mathcal{Q}$, and φ is the inverse function to Φ. By Proposition 2.26, the gauge ρ is integrating.

So, we are led to the consideration of higher variations introduced by N. Wiener and L.C. Young. (See Example 4.1 in Section A below.)

A. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. Recall that, by $\Sigma = \Sigma(\mathcal{Q})$ is denoted the set of all families of pair-wise disjoint sets belonging to $\mathcal{Q}$. (See Section 1D.) An element, $\mathcal{P}$, of Σ such that its union is equal to Ω and, for every $X \in \mathcal{Q}$, the sub-family $\{Y \in \mathcal{P} : Y \cap X \neq \emptyset\}$ of $\mathcal{P}$ is finite, is called a partition. The set of all partitions is denoted by $\Pi = \Pi(\mathcal{Q})$.

Let E be a Banach space and $\mu : \mathcal{Q} \to E$ an additive set function.

Given a Young function Φ (see Section 1G), a set X from $\mathcal{Q}$ and a partition $\mathcal{P}$, let

(A.1) $$v_{\Phi}(\mu,\mathcal{P};X) = \sum_{Y \in \mathcal{P}} \Phi(|\mu(X \cap Y)|) .$$

Then, for the given Φ, X and a set of partitions $\Delta \subset \Pi$, let

(A.2) $$v_{\Phi}(\mu,\Delta;X) = \sup\{v_{\Phi}(\mu,\mathcal{P};X) : \mathcal{P} \in \Delta\} .$$

The possibility of $v_\Phi(\mu,\Delta;X) = \infty$ is admitted. We write $v_\Phi(\mu;X) = v_\Phi(\mu,\Pi;X)$, for every $X \in \mathcal{Q}$.

The set function $v_\Phi(\mu,\Delta)$, that is, $X \mapsto v_\Phi(\mu,\Delta;X)$, $X \in \mathcal{Q}$, is called the Φ-variation of the set function μ with respect to the family of partitions Δ . The set function $v_\Phi(\mu) = v_\Phi(\mu,\Pi)$ is called simply the Φ-variation of μ . If $v_\Phi(\mu,\Delta;X) < \infty$ for every $X \in \mathcal{Q}$, the set function μ is said to have finite Φ-variation with respect to the set of partitions Δ .

In the case when $\Phi(s) = s^p$, or even when $\Phi(s) = cs^p$, for some constants $c > 0$ and $p \geq 1$ and every $s \in [0,\infty)$, we shall write simply $v_p(\mu,\Delta)$ instead of $v_\Phi(\mu,\Delta)$ and speak of the p-variation instead of the Φ-variation. Similar conventions are used without explicit mention in other symbols denoting objects depending on Φ , and in the corresponding terminology. The 1-variation, $v_1(\mu,\Delta)$, of the set function μ with respect to the family of partitions Δ is called simply the variation of μ with respect to Δ and denoted by $v(\mu,\Delta)$.

Formulas (A.1) and (A.2) have meaning as they stand for arbitrary quasirings, not only multiplicative ones. For, $X \cap Z = XZ \in \mathrm{sim}(\mathcal{Q})$, whenever $X \in \mathcal{Q}$ and $Z \in \mathcal{Q}$, and so, by the convention introduced in Section 1B, $\mu(X \cap Z)$ is well-defined. However, in such wider context, useful pronouncements would require more complicated formulations and the gained generality would be of little value.

On the other hand, it is sometimes advantageous to define $v_\Phi(\mu,\mathcal{P};X)$ and $v_\Phi(\mu,\Delta;X)$ by (A.1) and (A.2), respectively, for any set X belonging to the ring, $\mathcal{R} = \mathcal{R}(\mathcal{Q})$, generated by $\mathcal{Q}$, not only for $X \in \mathcal{Q}$. This represents no difficulty because every set belonging to $\mathcal{R}$ is equal to the union of a finite family of pair-wise disjoint sets belonging to $\mathcal{Q}$.

EXAMPLE 4.1. Let a and b be real numbers such that $a < b$. Let $\Omega = (a,b]$ and $\mathcal{Q} = \{(s,t] : a \leq s \leq t \leq b\}$. Let d be a function on the interval $[a,b]$ and let

$$\mu((s,t]) = d(t) - d(s) ,$$

for any s and t such that $a \leq s \leq t \leq b$.

Although not much attention seems to have been paid to Φ-variation of additive set functions in general, there is already considerable literature devoted to this case. To be sure, the Φ-variation of the set function μ is discussed in terms of the function d. In fact, if the partition $\mathcal{P}$ is determined by the points $a = s_0 < s_1 < s_2 < \dots < s_{n-1} < s_n = b$, that is, $\mathcal{P} = \{(s_{j-1}, s_j] : j = 1,2,\dots,n\}$, then

$$v_\Phi(\mu,\mathcal{P};\Omega) = \sum_{j=1}^{n} \Phi(|d(s_j) - d(s_{j-1})|) .$$

Actually, often the function d itself is the centre of interest, because some convergence properties of the Fourier series of d can be studied using the notion of the Φ-variation; see e.g. [66].

Besides $\Delta = \Pi$, the set of all dyadic partitions is often taken for Δ, especially when $a = 0$ and $b = 1$.

The variation (that is, 1-variation) is a classical concept dating back to C. Jordan. The notion of the p-variation was introduced in this case by N. Wiener in [67]. It was subsequently studied by several authors, notably by L.C. Young, who considered, in [69], Stieltjes integration with respect to functions of finite p-variation and introduced, in [70], the notion of a function of finite Φ-variation. Spaces of functions of finite Φ-variation were studied by W. Orlicz and his collaborators, [51], [42], and by M. Bruneau, [4].

The notation and terminology are not firmly established in the literature although they seem to converge to similar ones to those adopted here.

The introduction of the set of partitions, Δ, as an additional parameter on which the Φ-variation, $v_\Phi(\mu,\Delta)$, depends, genuinely increases the generality of this notion. It is illustrated by the following classical

EXAMPLE 4.2. In the situation of Example 4.1, let $a = 0$ and $b = 1$. For every $m = 1,2,\dots,$ let $\mathcal{P}_m$ be a partition, determined by the points

$$0 = s_{m,0} < s_{m,1} < \dots < s_{m,n_m} = 1 ,$$

such that $\mathcal{P}_m \prec \mathcal{P}_{m+1}$, that is, every point $s_{m,\ell}$, $\ell = 0,1,\ldots,n_m$, is among the points determining the partition $\mathcal{P}_{m+1}$, and

$$\lim_{m\to\infty} \max\{s_{m,\ell} - s_{m,\ell-1} : \ell = 1,2,\ldots,n_m\} = 0 .$$

Let $\Delta = \{\mathcal{P}_m : m = 1,2,\ldots\}$. By a classical result of P. Lévy, [43], (see also [11], Theorem VIII.2.3) the limit

$$\lim_{m\to\infty} v_2(\mu,\mathcal{P}_m;\Omega)$$

exists for almost every, in the sense of the Wiener measure, continuous function d on $[0,1]$ and, hence, $v_2(\mu,\Delta;\Omega) < \infty$. However, $v_2(\mu,\Pi;\Omega) = \infty$. See, e.g., [64], §4.

EXAMPLE 4.3. Let $\Omega = \mathbb{R}$. Let $\mathcal{Q}$ be the family of all bounded Borel sets in Ω. Let ι be the Lebesgue measure on $\mathbb{R}$. Let $1 < p < \infty$ and let $E = L^p(\iota)$. If $X \in \mathcal{Q}$, let

$$(\mu(X))(t) = \lim_{u\to 0+} \frac{1}{\pi} \left[\int_{-\infty}^{t-u} + \int_{t+u}^{\infty} \right] \frac{X(s)}{t-s}\, ds ,$$

for every $t \in \mathbb{R}$ for which this limit exists. Then $\mu(X)$ represents an element of the space E. What is more, M. Riesz has proved, see [7], that there exists a constant, A, depending on p, such that

$$\|\mu(f)\|_{p,\iota}^{p} \leq A \int_{-\infty}^{\infty} |f(s)|^p ds$$

for every $f \in \operatorname{sim}(\mathcal{Q})$. Consequently, the resulting additive set function $\mu : \mathcal{Q} \to E$ has finite p-variation.

The Riesz estimate was extended to a wide class of kernels in Euclidean spaces of arbitrary dimension by A.P. Calderón and A. Zygmund, [7]. Accordingly, such kernels give rise to similar vector valued set functions of finite p-variation on bounded Borel sets in $\mathbb{R}^n$, $n = 1,2,\ldots$.

EXAMPLE 4.4. Let $\Omega = \mathbb{R}$ and let $\mathcal{Q}$ be the family of all bounded intervals (of all kinds) in Ω. Let

$$S_X f(s) = \int_X \hat{f}(\omega)\exp(2\pi i s\omega)d\omega ,$$

for any $s \in \mathbb{R}$, $X \in \mathcal{Q}$ and any function f on $\mathbb{R}$ integrable with respect to the Lebesgue measure, where

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(s)\exp(2\pi i s\omega)ds ,$$

for every $\omega \in \Omega$. J.L. Rubio de Francia, F.J. Ruiz and J.L. Torrea have proved, in [60], Corollary 2.4, that, for every $p \in [2,\infty)$, there exists a constant C_p such that

$$\sum_{X\in\mathcal{P}} \int_{\mathbb{R}} |S_X f(s)|^p ds \leq C_p \int_{\mathbb{R}} |f(x)|^p ds ,$$

for any such function f and every family of intervals $\mathcal{P} \in \Sigma(\mathcal{Q})$.

Consequently, if $E = L^p(\iota)$, $f \in \mathcal{L}^1 \cap \mathcal{L}^p(\iota)$, where ι is the Lebesgue measure in $\mathbb{R}$, and if, for every $X \in \mathcal{Q}$, we define $\mu(X)$ to be the element of the space E determined by the function $S_X f$, we obtain an additive set function $\mu : \mathcal{Q} \to E$ having finite p-variation.

PROPOSITION 4.5. *Let $\Delta \subset \Pi$, $\mathcal{P} \in \Delta$ and $X \in \mathcal{Q}$. Then*

$$\sum_{Y\in\mathcal{P}} v_\Phi(\mu,\Delta;X\cap Y) \leq v_\Phi(\mu,\Delta;X) ,$$

for any additive set function $\mu : \mathcal{Q} \to E$ and a Young function Φ.

Proof. It is obvious.

It is worth-while to note explicitly that, if the Young function Φ is not a multiple of the identity function on $[0,\infty)$, then the Φ-variation is not necessarily additive.

EXAMPLE 4.6. In the situation of Example 4.1, let $a = 0$, $b = 1$ and $d(s) = s$ for every $s \in [0,1]$. Then $v_2(\mu;(s,t]) = (t-s)^2$, for every s and t such that $0 \leq s \leq t \leq 1$.

B. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. Let Φ be a Young function.

Recall that the set $\Pi = \Pi(\mathcal{Q})$ of all partitions is directed by the relation of refinement. (See Section 1D.) We refer to the same relation when we speak of directed subsets of Π.

To avoid some trivialities, we assume that, for every finite set $\mathcal{P}_0 \in \Sigma(\mathcal{Q})$, there exists a partition $\mathcal{P} \in \Pi$ such that $\mathcal{P}_0 \subset \mathcal{P}$.

Let E be a Banach space and $\mu : \mathcal{Q} \to E$ an additive set function.

PROPOSITION 4.7. *The Φ-variation, $v_\Phi(\mu)$, of the set function μ is additive if and only if*

(B.1) $$v_\Phi(\mu;X) = \sup\{v_\Phi(\mu,\mathcal{P};X) : \mathcal{P}_0 \prec \mathcal{P} \in \Pi\},$$

for every $X \in \mathcal{Q}$ and $\mathcal{P}_0 \in \Pi$.

Proof. For any $X \in \mathcal{Q}$ and $\mathcal{P}_0 \in \Pi$,

$$\sum_{Y \in \mathcal{P}_0} v_\Phi(\mu;X\cap Y) = \sum_{Y\in\mathcal{P}_0} \sup\{v_\Phi(\mu,\mathcal{P};X\cap Y) : \mathcal{P} \in \Pi\} =$$

$$= \sum_{Y\in\mathcal{P}_0} \sup\{v_\Phi(\mu,\mathcal{P};X\cap Y) : \mathcal{P}_0 \prec \mathcal{P} \in \Pi\} =$$

$$= \sup\left\{\sum_{Y\in\mathcal{P}_0} v_\Phi(\mu,\mathcal{P};X\cap Y) : \mathcal{P}_0 \prec \mathcal{P} \in \Pi\right\} = \sup\{v_\Phi(\mu,\mathcal{P};X) : \mathcal{P}_0 \prec \mathcal{P} \in \Pi\}.$$

Therefore,

$$v_\Phi(\mu;X) = \sum_{Y\in\mathcal{P}_0} v_\Phi(\mu;X\cap Y)$$

if and only if (B.1) holds.

Let ι be a non-atomic measure in the space Ω such that every set $X \in \mathcal{Q}$ is ι-integrable. (See Section 3B.)

For a partition $\mathcal{P} \in \Pi$, the ι-mesh, $\|\mathcal{P}\|_\iota$, of $\mathcal{P}$ is defined by

$$\|\mathcal{P}\|_\iota = \sup\{\iota(X) : X \in \mathcal{P}\}.$$

Because the cardinal number of $\mathcal{P}$ may be infinite, the possibility that $\|\mathcal{P}\|_\iota = \infty$ may occur.

A set of partitions $\Delta \subset \Pi$ will be called ι-fine if,

$$\inf\{\|\mathcal{P}\|_\iota : \mathcal{P} \in \Delta\} = 0 .$$

We say that the Φ-variation, $v_\Phi(\mu,\Delta)$, of μ with respect to a set of partitions, Δ, is ι-continuous if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $v_\Phi(\mu,\Delta;X) < \epsilon$, for every set X in the ring, $\mathcal{R} = \mathcal{R}(\mathcal{Q})$, generated by $\mathcal{Q}$ such that $\iota(X) < \delta$. Recall that, by formula (A.2) in the previous section, $v_\Phi(\mu,\Delta;X)$ is indeed well-defined for any $X \in \mathcal{R}$.

Now, if $\Delta \subset \Pi$ is a directed set of partitions, then the family

(B.2) $$\mathcal{Q}_\Delta = \{\emptyset\} \cup \bigcup_{\mathcal{P}\in\Delta} \mathcal{P}$$

of all sets, X, for which there exists a partition, $\mathcal{P} \in \Delta$, such that $X \in \mathcal{P}$, augmented by $\emptyset$, is a quasiring.

PROPOSITION 4.8. *Let* Δ *be a directed set of partitions. If*

(B.3) $$v_\Phi(\mu,\Delta,X) = \lim_{\mathcal{P}\in\Delta} v_\Phi(\mu,\mathcal{P};X) ,$$

for every $X \in \mathcal{Q}$, *then the set function* $v_\Phi(\mu,\Delta)$ *is additive on the quasiring* $\mathcal{Q}_\Delta$. *If, moreover,* $v_\Phi(\mu,\Delta)$ *is* ι*-continuous then* $v_\Phi(\mu,\Delta)$ *is* σ*-additive on the whole of* $\mathcal{R}$.

Proof. The first statement is obvious. The second one follows from the fact that, for every set $X \in \mathcal{R}$ and $\epsilon > 0$, there is a set Y, which is the union of a finite family of pair-wise disjoint sets from $\mathcal{Q}_\Delta$, such that $\iota(|X-Y|) < \epsilon$.

In some cases of great interest, instead of (B.3), the formula

(B.4) $$v_\Phi(\mu,\Delta;X) = \lim_{r\to 0+} \sup\{v_\Phi(\mu,\mathcal{P};X) : \|\mathcal{P}\|_\iota < r , \mathcal{P} \in \Delta\}$$

holds for every $X \in \mathcal{Q}$. It might be expected that this formula too would imply the additivity of $v_\Phi(\mu,\Delta)$. However, this is not necessarily the case.

EXAMPLE 4.9. Let the set-up be as in Example 4.1 with $a = 0$ and $b = 1$. By a result of S.J. Taylor, [64], Theorem 1, if Φ is a Young function such that

$$2s^{-2}\Phi(s) \log\log s^{-1} \to 1 ,$$

as $s \to 0+$, then, for almost every (in the sense of the Wiener measure) continuous function d on $[0,1]$, (B.4) holds with $\Delta = \Pi$ and with the Lebesgue measure in the role of ι. On the other hand, M. Bruneau proved, [5], Théorème 1, that the set of points $t \in [0,1]$ such that

$$v_\Phi(\mu,\Pi;(0,1]) = v_\Phi(\mu,\Pi;(0,t]) + v_\Phi(\mu,\Pi;(t,1]) ,$$

for almost every continuous function d, has empty interior.

Because $v_\Phi(\mu,\Delta)$ indeed, also in interesting cases, fails to be σ-additive, it is desirable to find a σ-additive set function $\sigma : \mathcal{Q} \to [0,\infty)$ such that $v_\Phi(\mu,\Delta;X) \le \sigma(X)$, for every $X \in \mathcal{Q}$. Such a set function σ can be used together with the inverse function, to Φ, to produce a gauge integrating for μ.

EXAMPLE 4.10. Let the set-up be as in Example 4.1 with arbitrary $a \in \mathbb{R}$ and $b \in \mathbb{R}$, $a \le b$. For some $\Delta \subset \Pi$, assume that $v_\Phi(\mu,\Delta;\Omega) < \infty$. Let

$$\sigma((s,t]) = v_\Phi(\mu,\Delta;(a,t]) - v_\Phi(\mu,\Delta;(a,s])$$

for any s and t such that $a \le s \le t \le b$.

Now, if Δ is a directed set of partitions, then σ is a non-negative and additive set function on the quasiring $\mathcal{Q}_\Delta$ such that $v_\Phi(\mu,\Delta;X) \le \sigma(X)$, for every $X \in \mathcal{Q}_\Delta$. If, moreover, Δ is ι-fine, where ι is the one-dimensional Lebesgue measure, and the function d is continuous, then σ is σ-additive on the whole of $\mathcal{Q}$ and the inequality $v_\Phi(\mu,\Delta;X) \le \sigma(X)$ holds for every $X \in \mathcal{Q}$.

If $\Delta = \Pi$, then σ is σ-additive on $\mathcal{Q}$ and $v_\Phi(\mu,\Delta;X) \le \sigma(X)$, for every $X \in \mathcal{Q}$. This observation is due to L.C. Young, [71].

PROPOSITION 4.11. *Let $\mathcal{P}_n \in \Pi$ be a partition such that $\mathcal{P}_n \prec \mathcal{P}_{n+1}$, for every $n = 1,2,\ldots$, and*

$$\lim_{n\to\infty} \|\mathcal{P}_n\|_\iota = 0 .$$

Let $\Delta = \{\mathcal{P}_n : n = 1,2,\ldots\}$ and assume that $\iota(X) > 0$ for every non-empty set $X \in \mathcal{Q}_\Delta$.

Let Φ be a Young function such that μ has finite and ι-continuous Φ-variation with respect to the set of partitions Δ.

Then there exists a σ-additive set function $\sigma : \mathcal{Q} \to [0,\infty)$ such that

(B.5) $$v_\Phi(\mu,\Delta;X) \le \sigma(X)$$

for every $X \in \mathcal{Q}_\Delta$.

Proof. Let

$$\sigma_1(X) = \sum_{Y\in\mathcal{P}_1} v_\Phi(\mu,\Delta;Y)(\iota(Y))^{-1}\iota(X\cap Y)$$

for every ι-measurable set X. Then σ_1 is a measure in Ω such that

$$v_\Phi(\mu,\Delta;X) = \sigma_1(X)$$

for every $X \in \mathcal{P}_1$.

Now, if $n \ge 1$ is an integer and σ_n a measure in Ω such that

(B.6) $$v_\Phi(\mu,\Delta;X) \le \sigma_n(X)$$

for every $X \in \mathcal{P}_n$, for every set $Y \in \mathcal{P}_{n+1} \cup \{\emptyset\}$, let $w(Y)$ be a number such that $w(\emptyset) = 0$, $v_\Phi(\mu,\Delta;Y) \le w(Y)$ and

$$\sum_{Y\in\mathcal{P}_{n+1}} w(X\cap Y) = \sigma_n(X)$$

for every $X \in \mathcal{P}_n$. By (B.6) and Proposition 4.5, such numbers $w(Y)$, $Y \in \mathcal{P}_{n+1}$, do exist. Then we put

$$\sigma_{n+1}(X) = \sum_{Y\in \mathcal{P}_{n+1}} w(Y)(\iota(Y))^{-1}\iota(X\cap Y)$$

for every ι-measurable set X. This defines a measure, σ_{n+1}, in Ω such that $\sigma_{n+1}(Y)$, for every $Y \in \mathcal{P}_{n+1}$, and $\sigma_{n+1}(X) = \sigma_n(X)$, for every $X \in \mathcal{P}_n$.

So, by induction, a sequence of measures, σ_n, $n = 1,2,\ldots,$ is constructed such that, if we define

$$\sigma(X) = \lim_{n\to\infty} \sigma_n(X) ,$$

for every ι-measurable set X, we obtain a measure in Ω such that (B.5) holds for every $X \in \mathcal{Q}_\Delta$.

C. Let ι be a measure in a space Ω. Let $\mathcal{R}(\iota)$ be the family of all ι-integrable sets. (See Section 3B.) Let $\mathcal{Q}$ be a multiplicative quasiring of sets such that $\mathcal{Q} \subset \mathcal{R}(\iota)$. To avoid some trivialities, we assume that the measure ι is generated by its restriction to $\mathcal{Q}$. Let φ be a real valued, continuous, concave and strictly increasing function on $[0,\infty)$ such that $\varphi(0) = 0$. Let $\rho(X) = \varphi(\iota(X))$ for every $X \in \mathcal{Q}$. By Proposition 2.26, ρ is an integrating gauge on $\mathcal{Q}$.

The reason why we are interested in this situation is clear: If E is a Banach space, $\mu : \mathcal{Q} \to E$ an additive set function, Φ a Young function and $\Delta \subset \Pi(\mathcal{Q})$ a set of partitions such that $v_\Phi(\mu,\Delta;X) \le \iota(X)$, for every $X \in \mathcal{Q}$, then, assuming that φ is the inverse function to Φ, the gauge ρ integrates for the set function μ. (See Section 3A.)

The purpose of this and the next section is to provide some information about the space $\mathcal{L}(\rho,\mathcal{Q})$, namely to present workable sufficient conditions for a function to belong to $\mathcal{L}(\rho,\mathcal{Q})$. In this section, we discuss the relation of the spaces $\mathcal{L}(\rho,\mathcal{Q})$ and $\mathcal{L}^\Phi(\iota)$, where Φ is the inverse function to φ. (See Section 3C.)

PROPOSITION 4.12. *Let* $p \in [1,\infty)$ *and* $\varphi(t) = t^{1/p}$ *for every* $t \in [0,\infty)$. *Then* $\mathcal{L}(\rho,\mathcal{Q}) \subset \mathcal{L}^p(\iota)$.

Proof. Let $f \in \mathcal{L}(\rho,\mathcal{Q})$. Let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots$, such that

(C.1)
$$\sum_{j=1}^{\infty} |c_j|\rho(X_j) < \infty$$

and

(C.2)
$$f(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for every $\omega \in \Omega$ for which

(C.3)
$$\sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty .$$

Denote $f_j = c_j X_j$, for every $j = 1,2,\ldots$. Then $\|f_j\|_{p,\iota} = |c_j|(\iota(X_j))^{1/p} = |c_j|\rho(X_j)$, for every $j = 1,2,\ldots$. (See Section 3C.) So, by (C.1),

$$\sum_{j=1}^{\infty} \|f_j\|_{p,\iota} < \infty .$$

Consequently, $f \in \mathcal{L}^p(\iota)$.

The following proposition extends the above result to more general functions φ . (For the notion of a Young function, see Section 1G; for the definition of the class $\mathcal{L}^{\Phi}(\iota)$, see Section 3C.)

PROPOSITION 4.13. *Let* φ *be the inverse function to a Young function,* Φ *, and* K *a constant such that* $0 < K < \varphi(t)\varphi(t^{-1})$ *for every* $t \in (0,\infty)$ *. Then* $\mathcal{L}(\rho,\mathcal{Q}) \subset \mathcal{L}^{\Phi}(\iota)$.

Proof. First, let c be a number, X a set belonging to $\mathcal{Q}$ and $g = cX$. Assume that $c \neq 0$ and $\iota(X) > 0$. Recall that the Luxemburg norm, $\|g\|_{\Phi,\iota}$, of the function g is defined by the formula

$$\|g\|_{\Phi,\iota} = \inf\{k : k > 0 , \int_{\Omega} \Phi(k^{-1}|g(\omega)|)\iota(d\omega) \leq 1\} .$$

Hence, $\|g\|_{\Phi,\iota} = k$, where k is the number that satisfies the condition $\Phi(k^{-1}|c|)\iota(X) = 1$. It follows that $\|g\|_{\Phi,\iota} \leq K^{-1}|c|)\varphi(\iota(X)) = K^{-1}|c|\rho(X)$, where K is the constant mentioned in the statement of this proposition. This estimate is, obviously, true also if $c = 0$ or $\iota(x) = 0$.

The proof is now finished as that of Proposition 4.12. Namely, if $f \in \mathcal{L}(\rho,\mathcal{Q})$ and c_j are numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots$, satisfying (C.1), such that (C.2) holds for every $\omega \in \Omega$ for which (C.3) does, we denote $f_j = c_j X_j$, for every $j = 1,2,\ldots$. Then we use the obtained estimate of the Luxemburg norm to deduce from (C.1) that

$$\sum_{j=1}^{\infty} \|f_j\|_{\Phi,\iota} < \infty ,$$

which implies that $f \in \mathcal{L}^{\Phi}(\iota)$.

In the following proposition, no additional conditions are imposed on φ . (For the concepts used in its statement, see Section 1D.)

PROPOSITION 4.14. *If $\mathcal{Q}$ is an algebra of sets, then every bounded function measurable with respect to the σ-algebra generated by $\mathcal{Q}$ belongs to $\mathcal{L}(\rho,\mathcal{Q})$.*

Proof. Let $\mathcal{S}$ be the σ-algebra of sets generated by $\mathcal{Q}$. Because, for every set $Y \in \mathcal{S}$ and $\epsilon > 0$, there is a set $X \in \mathcal{Q}$ such that $\iota(|Y-X|) < \epsilon$ and the function φ is continuous, it is obvious that $\mathcal{S} \subset \mathcal{L}(\rho,\mathcal{Q})$. Then, by Proposition 2.7, $\mathcal{L}(q_\rho,\mathcal{S}) = \mathcal{L}(\rho,\mathcal{Q})$ and, by continuity, $q_\rho(Y) = \varphi(\iota(Y))$, for every $Y \in \mathcal{S}$. Hence, without a loss of generality, we can assume that $\mathcal{Q}$ is a σ-algebra.

Now, let f be a $\mathcal{Q}$-measurable function such that $0 \le f(\omega) \le 1$, for every $\omega \in \Omega$. Assuming that $k \ge 1$ is an integer and the sets X_j, $j = 1,2,\ldots,k-1$, are already constructed, let

$$X_k = \left\{\omega : f(\omega) - \sum_{j=1}^{k-1} 2^{-j} X_j(\omega) \ge 2^{-k}\right\} .$$

Then

$$\sum_{j=1}^{\infty} 2^{-j} \rho(X_j) \le \rho(\Omega) < \infty$$

and

$$f(\omega) = \sum_{j=1}^{\infty} 2^{-j} X_j(\omega)$$

for every $\omega \in \Omega$.

PROPOSITION 4.15. *Let* $\mathcal{Q}$ *be an algebra of sets. Let* $1 < p < q$ *and let* $\varphi(t) = t^{1/p}$, *for every* $t \geq 0$, *so that* $\rho(X) = (\iota(X))^{1/p}$, *for every* $X \in \mathcal{Q}$. *Then* $\mathcal{L}^q(\iota) \subset \mathcal{L}(\rho,\mathcal{Q})$.

Proof. Without loss of generality, we shall assume, as in the proof of Proposition 4.14, that $\mathcal{Q}$ is the family of all ι-measurable sets.

Let f be a non-negative function belonging to $\mathcal{L}^q(\iota)$. Let $X_j = \{\omega : f(\omega) \geq j\}$, for every $j = 1,2,\ldots$. Then

$$\sum_{j=1}^{\infty} q(j-1)^{q-1}\iota(X_j) \leq \sum_{j=1}^{\infty} (j^q-(j-1)^q)\iota(X_j) \leq \int_\Omega f^q d\iota < \infty ,$$

so that

$$\sum_{j=1}^{\infty} j^{q-1}\iota(X_j) < \infty .$$

By the Hölder inequality,

$$\sum_{j=1}^{\infty} \rho(X_j) = \sum_{j=1}^{\infty} (\iota(X_j))^{1/p} = \sum_{j=1}^{\infty} j^{(1-q)/p}(j^{q-1}\iota(X_j))^{1/p}$$

$$\leq \left[\sum_{j=1}^{\infty} j^{(1-q)/(p-1)}\right]^{(p-1)/p}\left[\sum_{j=1}^{\infty} j^{q-1}\iota(X_j)\right]^{1/p} < \infty ,$$

because $(q-1)/(p-1) > 1$. So, if we let

$$g(\omega) = \sum_{j=1}^{\infty} X_j(\omega) ,$$

for every $\omega \in \Omega$, then $g \in \mathcal{L}(\rho,\mathcal{Q})$.

Now, let $h = f - g$. Then $0 \leq h(\omega) \leq 1$, for every $\omega \in \Omega$. By Proposition 4.14, h belongs to $\mathcal{L}(\rho,\mathcal{Q})$ and, therefore, $f = g + h$ too belongs to $\mathcal{L}(\rho,\mathcal{Q})$.

The following examples settle some natural questions about the space $\mathcal{L}(\rho,\mathcal{Q})$. They were designed by Susumu Okada.

EXAMPLES 4.16. Let ι be the one-dimensional Lebesgue measure. Let $\Omega = (0,1]$, $\mathcal{Q} = \{(s,t] : 0 \leq s \leq t \leq 1\}$ and $\mathcal{R}$ be the algebra of sets generated by $\mathcal{Q}$. Let $1 < p$ and let $\rho(X) = (\iota(X))^{1/p}$, for every $X \in \mathcal{R}$. Then, obviously $\mathcal{L}(\rho,\mathcal{Q}) \subset \mathcal{L}(\rho,\mathcal{R})$ and, by

Proposition 4.12, $\mathcal{L}(\rho,\mathcal{R}) \subset \mathcal{L}^p(\iota)$. We wish to show that $\mathcal{L}(\rho,\mathcal{Q}) \neq \mathcal{L}(\rho,\mathcal{R})$ and $\mathcal{L}(\rho,\mathcal{R}) \neq \mathcal{L}^p(\iota)$. Let us denote, for short, $\alpha = p^{-1}$.

(i) Let us note first that there exists a constant $c_1 > 0$ such that

$$|t^{2\alpha}\cos t^{-1} - s^{2\alpha}\cos s^{-1}| \leq c_1|t-s|^{\alpha},$$

for every $s \in \Omega$ and $t \in \Omega$. Indeed, let $0 < s < t < 1$. Let $n \geq 1$ be the integer such that $(n+1)^{-1} < t \leq n^{-1}$. Assume first that $(n+2)^{-1} \leq s$ and put $u = (n+2)^{-1}$ and $v = n^{-1}$, so that $v \leq 3u$. By the Lagrange theorem,

$$|t^{2\alpha}\cos t^{-1} - s^{2\alpha}\cos s^{-1}|\,|t-s|^{-\alpha} \leq 3|t-s|^{1-\alpha}s^{2\alpha-2} \leq 3(2uv)^{1-\alpha}u^{2\alpha-2} \leq 3.6^{1-\alpha}$$

If $s < (n+2)^{-1}$, then

$$|t^{2\alpha}\cos t^{-1} - s^{2\alpha}\cos s^{-1}|\,|t-s|^{-\alpha} \leq (n^{-2\alpha}+(n+2)^{-2\alpha})((n+1)^{-1} - (n+2)^{-1})^{-\alpha} \leq 2.2^{\alpha}.$$

Integrating by parts, we then obtain that

$$\left|\int_s^t u^{2\alpha-2}\sin u^{-1}du\right| \leq |t^{2\alpha}\cos t^{-1} - s^{2\alpha}\cos t^{-1}| + \int_s^t 2\alpha u^{2\alpha-1}du \leq c|t-s|^{\alpha},$$

for some $c > 0$ and every $s \in \Omega$ and $t \in \Omega$. So, if we put $d(0) = 0$ and

$$d(t) = \lim_{s\to 0+}\int_s^t u^{2\alpha-2}\sin u^{-1}du,$$

for every $t \in (0,1]$, then d is a well-defined continuous function on $[0,1]$.

Let $\mu((s,t]) = d(t) - d(s)$, for every s and t such that $0 \leq s \leq t \leq 1$. Furthermore, given a point $s \in \Omega$, let $\mu^s(X) = \mu(X\cap(s,1])$, for every $X \in \mathcal{Q}$. We have noted that $|\mu^s(X)| < c(\iota(X))^{\alpha} = c\rho(X)$, for some $c > 0$ and every $X \in \mathcal{Q}$. Therefore, by Proposition 3.1,

$$\left|\int_s^1 f(u)u^{2\alpha-2}\sin u^{-1}du\right| = |\mu_\rho^s(f)| \leq cq_\rho(f),$$

for every $f \in \mathcal{L}(\rho,\mathcal{Q})$.

Let $g(t) = t^{1-2\alpha}\sin t^{-1}$, for every $t \in \Omega$. Then the function g does not belong to $\mathcal{L}(\rho,\mathcal{Q})$, because

$$\lim_{s\to 0+}\int_s^1 g(u)u^{2\alpha-2}\sin u^{-1}\,du = \infty .$$

None-the-less, g belongs to $\mathcal{L}(\rho,\mathcal{R})$. Indeed, if $p \geq 2$, that is, $\alpha \leq \frac{1}{2}$, this follows from Proposition 4.14. If $p < 2$, we choose a number $q \in (p,p/(2-p))$. Then $g \in \mathcal{L}^q(\iota)$ and, by Proposition 4.15, $g \in \mathcal{L}^q(\rho,\mathcal{R})$.

Consequently, $\mathcal{L}(\rho,\mathcal{Q}) \neq \mathcal{L}(\rho,\mathcal{R})$.

(ii) To show that $\mathcal{L}(\rho,\mathcal{R}) \neq \mathcal{L}^p(\iota)$, let $h(t) = t^{-\alpha}|\log t|^{-1}$, for $t \in (0,\frac{1}{2}]$, and $h(t) = 0$, for $t \in (\frac{1}{2},1]$. Then $h \in \mathcal{L}^p(\iota)$. However, the function h does not belong to $\mathcal{L}(\nu,\mathcal{R}) = \mathcal{L}(\nu)$, where

$$\nu(X) = \alpha\int_X u^{\alpha-1}\,du ,$$

for every $X \in \mathcal{Q}$. Using the fact that every set in $\mathcal{R}$ is the union of a finite collection of pair-wise disjoint intervals belonging to $\mathcal{Q}$, we can prove that $\nu(X) \leq \rho(X)$, for every $X \in \mathcal{R}$. Therefore, the function h does not belong to $\mathcal{L}(\rho,\mathcal{R})$ either.

D. We maintain the notation of Section C.

A function f on Ω will be called $\mathcal{Q}$-locally ι-integrable if it is integrable with respect to ι on every set belonging to $\mathcal{Q}$, that is, if $Xf \in \mathcal{L}(\iota)$ for every $X \in \mathcal{Q}$.

Now, assuming that f is a $\mathcal{Q}$-locally ι-integrable function, let

$$M_\iota(f,X) = \frac{1}{\iota(X)}\int_X f\,d\iota$$

for every set $X \in \mathcal{Q}$ such that $\iota(X) > 0$, and $M_\iota(f,X) = 0$ for every set X such that $\iota(X) = 0$. If $\iota(X) > 0$, then the number $M_\iota(f,X)$ is the mean value of the function f on the set X with respect to the measure ι.

Furthermore, if $\mathcal{P} \in \Pi(\mathcal{Q})$ is a partition, let

$$M_\iota(f,\mathcal{P}) = \sum_{X\in\mathcal{P}} M_\iota(f,X)X .$$

So, $M_\iota(f,\mathcal{P})$ is a function on Ω, constant on every set belonging to $\mathcal{P}$, having the same mean value as the function f on every set $X \in \mathcal{P}$ such that $\iota(X) > 0$.

Let ψ be a real valued, continuous, and strictly increasing function on $[0,\infty)$ such that $\psi(0) = 0$.

We shall say that a function f on Ω satisfies the ψ-Hölder condition with respect to the quasiring $\mathcal{Q}$ and the measure ι if

$$|f(\omega) - f(v)| \leq \psi(\iota(X)),$$

for every set $X \in \mathcal{Q}$ and any points $\omega \in X$ and $v \in X$.

PROPOSITION 4.17. *Let f be a ι-measurable function satisfying the ψ-Hölder condition with respect to $\mathcal{Q}$ and ι. Then f is $\mathcal{Q}$-locally ι-integrable.*

Let $\mathcal{P}_n \in \Pi(\mathcal{Q})$ be a partition such that $\mathcal{P}_n \prec \mathcal{P}_{n+1}$, for every $n = 0,1,2,\ldots$, and $\|\mathcal{P}_n\|_\iota \to 0$ as $n \to \infty$. If

$$\sum_{X\in\mathcal{P}_0} M_\iota(f,X)\varphi(\iota(X)) + \sum_{j=1}^{\infty} \sum_{Z\in\mathcal{P}_{j-1}} \psi(\iota(Z)) \sum_{Y\in\mathcal{P}_j} \varphi(\iota(Y\cap Z)) < \infty,$$

then $f \in \mathcal{L}(\rho,\mathcal{Q})$.

Proof. The first statement is clear, because the function f is bounded on every set belonging to $\mathcal{Q}$.

Let $f_0 = M_\iota(f,\mathcal{P}_0)$ and

$$f_j = M_\iota(f - M_\iota(f,\mathcal{P}_{j-1}),\mathcal{P}_j),$$

for every $j = 1,2,\ldots$. Then

$$\sum_{j=0}^{n} f_j = M_\iota(f,\mathcal{P}_n),$$

for $n = 0,1,2,\ldots$. Now,

$$q_\rho(f_0) \leq \sum_{X\in\mathcal{P}_0} M_\iota(f,X)\rho(X).$$

(See Section 2A.) Furthermore, for every $j = 1,2,\ldots$,

$$M_\iota(f - M_\iota(f,\mathcal{P}_{j-1}), Y) = M_\iota(f - M_\iota(f,Z), Y),$$

for any $Y \in \mathcal{P}_j$, where Z is the set belonging to $\mathcal{P}_{j-1}$ such that $Y \subset Z$. Then $|f(\omega) - M_\iota(f,Z)|| \leq \psi(\iota(Z))$, for every $\omega \in Z$, and, hence,

$$|M_\iota(f-M_\iota(f,Z),Y)| \leq \psi(\iota(Z)) .$$

Consequently,

$$q_\rho(f_j) \leq \sum_{Y\in\mathcal{P}_j} |M_\iota(f-M_\iota(f,\mathcal{P}_{j-1}),Y)|\rho(Y) \leq \sum_{Z\in\mathcal{P}_{j-1}} \psi(\iota(Z)) \sum_{Y\in\mathcal{P}_j} \varphi(\iota(Y\cap Z)) ,$$

for every $j = 1,2,\ldots$. So, Proposition 2.1 applies.

COROLLARY 4.18. *Let* $\Omega \in \mathcal{Q}$, $\mathcal{P}_0 = \{\Omega\}$, $\mathcal{P}_j \prec \mathcal{P}_{j+1}$, $\iota(X) = \|\mathcal{P}_j\|_\iota = (\frac{1}{2})^j\iota(\Omega)$, *for every* $X \in \mathcal{P}_j$ *and* $j = 0,1,2,\ldots,$ *and* $\Delta = \{\mathcal{P}_j : j = 0,1,2,\ldots\}$.

If f *is an* ι*-measurable function satisfying the* ψ*-Hölder condition with respect to the quasiring* $\mathcal{Q}_\Delta$ *and the measure* ι, *and*

(D.1)
$$\int_0^1 \frac{\varphi(t)\psi(t)}{t^2}\,\mathrm{d}t < \infty ,$$

then $f \in \mathcal{L}(\rho,\mathcal{Q})$.

Proof. Let $\alpha = \iota(\Omega)$. Because the functions φ and ψ are increasing,

$$\sum_{j=1}^{\infty} \sum_{Z\in\mathcal{P}_{j-1}} \psi(\iota(Z)) \sum_{Y\in\mathcal{P}_j} \varphi(\iota(Y\cap Z)) = \sum_{j=1}^{\infty} 2^j\psi(2^{1-j}\alpha)\varphi(2^{-j}\alpha) \leq$$

$$\leq \sum_{j=1}^{\infty} 2^{2j}\psi(2^{1-j}\alpha)\varphi(2^{1-j}\alpha)2^{-j} \leq 4\int_0^1 \frac{\varphi(\alpha t)\psi(\alpha t)}{t^2}\,\mathrm{d}t = 4\alpha\int_0^\alpha \frac{\varphi(t)\psi(t)}{t^2}\,\mathrm{d}t .$$

COROLLARY 4.19. *Let* $\Omega = (a,b]$ *with* $a \in \mathbb{R}$, $b \in \mathbb{R}$ *and* $a < b$. *Let* $\mathcal{Q} = \{(s,t] : a \leq s \leq t \leq b\}$. *Let* d *be a function on* $[a,b]$ *such that*

$$|d(t) - d(s)| \leq \varphi(t-s) ,$$

and let

$$\mu((s,t]) = d(t) - d(s) \text{ and } \rho((s,t]) = \varphi(t-s) ,$$

for every s *and* t *such that* $a \leq s \leq t \leq b$. *Then* ρ *is a gauge integrating for the additive set function* μ.

If, moreover, f *is a function on* Ω *such that*

$$|f(t) - f(s)| \leq \psi(|t-s|),$$

for any $s \in \Omega$ *and* $t \in \Omega$, *and* (D.1) *holds, then* $f \in \mathcal{L}(\rho,\mathcal{Q})$.

Condition (D.1) is satisfied, in particular, when $\varphi(t) = c_1 t^{1/p}$ and $\psi(t) = c_2 t^{1/q}$, for every $t \geq 0$, where $c_1 > 0$, $c_2 > 0$ and $p^{-1} + q^{-1} > 1$.

E. In some sense the notion of an additive set function with finite p-variation is analogous to the notion of a (point) function locally belonging to an L^p space. The analogy reverses the extension of these notions though, because, if $p < q$, to have finite p-variation is a more restrictive condition than one to have finite q-variation. In this section, we introduce additive set functions which are analogous to functions locally belonging to an L^∞ space.

Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. (See Section 1D.) Let E be a Banach space. Let $\mu : \mathcal{Q} \to E$ be an additive set function.

For any set $X \in \mathcal{Q}$, let

(E.1) $$v_\infty(\mu;X) = \sup\{|\mu(X \cap Z)| : Z \in \mathcal{Q}\}.$$

The possibility $v_\infty(\mu;X) = \infty$ is admitted.

The set function μ will be called locally bounded if $v_\infty(\mu;X) < \infty$, for every $X \in \mathcal{Q}$.

A wealth of locally bounded additive set functions do not have finite Φ-variation for any Young function Φ is provided in Chapter 6. Here is a simple example of such a set function.

EXAMPLE 4.20. Let Ω and $\mathcal{Q}$ be as in Corollary 4.19. Let E be the Banach space of all bounded Borel measurable functions on Ω with the sup-norm. For every $X \in \mathcal{Q}$, let $\mu(X)$ be the characteristic function of X considered as an element of the space E. Then $\mu : \mathcal{Q} \to E$ is an additive set function such that $v_\infty(\mu;X) = 1$, but $v_\Phi(\mu;X) = \infty$, for every $X \in \mathcal{Q}$, $X \neq \emptyset$, no matter what the Young function Φ.

The set function μ will be called indeficient if it is locally bounded and the gauge, ρ, defined on $\mathcal{Q}$ by

(E.2) $$\rho(X) = v_\infty(\mu;X) ,$$

for every $X \in \mathcal{Q}$, is integrating. (See Section 2D.)

So, if the set function μ is indeficient then this gauge integrates for it. (See Section 3A.)

PROPOSITION 4.21. *The set function* μ *is indeficient if and only if it is locally bounded and*

(E.3) $$\sum_{j=1}^{\infty} c_j\mu(X_j) = 0 ,$$

for any numbers c_j *and sets* $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,$ *such that*

(E.4) $$\sum_{j=1}^{\infty} |c_j|\, v_\infty(\mu;X_j) < \infty$$

and

(E.5) $$\sum_{j=1}^{\infty} c_jX_j(\omega) = 0$$

for every $\omega \in \Omega$ *such that*

(E.6) $$\sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty .$$

Proof. Let us show first that, if the condition is satisfied, then the gauge, ρ, defined by (E.2) is integrating. Let $X \in \mathcal{Q}$. Let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots,$ satisfying condition (E.4), such that

(E.7) $$X(\omega) = \sum_{j=1}^{\infty} c_jX_j(\omega)$$

for every ω satisfying the inequality (E.6). Let $\epsilon > 0$ and let $Z \in \mathcal{Q}$ be a set such that $\rho_\mu(X) < |\mu(X\cap Z)| + \epsilon$. Because

$$\lim_{n\to\infty} |\mu(X\cap Z) - \sum_{j=1}^{n} c_j\mu(X_j\cap Z)| = 0 ,$$

the inequality

$$\rho(X) - \epsilon < |\mu(X\cap Z)| \le \sum_{j=1}^{\infty} |c_j|\ |\mu(X_j\cap Z)| \le \sum_{j=1}^{\infty} |c_j|\rho(X_j)$$

holds. So, by Proposition 2.7, the gauge ρ is integrating.

Conversely, assume that μ is indeficient. That is, $v_\infty(\mu;X) < \infty$ for each $X \in \mathcal{Q}$ and the gauge (E.2) integrates for μ. So, if c_j are numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,...,$ satisfying (E.4), such that (E.5) holds for every $\omega \in \Omega$ for which (E.6) does, then, by Proposition 2.1,

$$\lim_{n\to\infty} q_\rho\left[\sum_{j=1}^{n} c_jX_j\right] = 0 .$$

Because

$$\left|\sum_{j=1}^{n} c_j\mu(X_j)\right| \le cq_\rho\left[\sum_{j=1}^{n} c_jX_j\right] ,$$

for some number $c \ge 0$ and every $n = 1,2,...,$ (E.3) follows.

The following proposition is a simple means for producing examples: it helps us to prove the indeficiency of some additive set functions which arise in connection with classical improper integrals and are not σ-additive.

PROPOSITION 4.22. *Let the set function* $\mu : \mathcal{Q} \to E$ *be locally bounded. Let* $\Omega_n \in \mathcal{Q}$ *be sets such that* $\Omega_n \subset \Omega_{n+1}$ *and the restriction of* μ *to the quasiring* $\mathcal{Q} \cap \Omega_n$ *is indeficient, for every* $n = 1,2,...,$ *and that*

$$\lim_{n\to\infty} |\mu(X) - \mu(X\cap\Omega_n)| = 0 ,$$

for every $X \in \mathcal{Q}$.

Then the set function μ *is indeficient.*

Proof. Let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,...,$ satisfying condition (E.4) such that the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does.

Let $\epsilon > 0$. Let J be a positive integer such that

$$\sum_{j=J+1}^{\infty} |c_j| v_{\infty}(\mu;X_j) < \epsilon .$$

Let m be a positive integer such that

$$\left| \sum_{j=1}^{J} c_j\mu(X_j) - \sum_{j=1}^{J} c_j\mu(X_j \cap \Omega_m) \right| < \epsilon .$$

Let N be a positive integer such that

$$\left| \sum_{j=1}^{n} c_j\mu(X_j \cap \Omega_m) \right| < \epsilon$$

for every $n > N$. Such an integer N exists because, by the assumption, the restriction of μ to $\mathcal{Q} \cap \Omega_m$ is indeficient. Then

$$\left| \sum_{j=1}^{n} c_j\mu(X_j) \right| \le \left| \sum_{j=1}^{n} c_j\mu(X_j \cap \Omega_m) \right| + \left| \sum_{j=1}^{n} c_j\mu(X_j) - \sum_{j=1}^{n} c_j\mu(X_j \cap \Omega_m) \right| \le$$

$$\le \epsilon + \left| \sum_{j=1}^{J} c_j\mu(X_j) - \sum_{j=1}^{J} c_j\mu(X_j \cap \Omega_m) \right| + \left| \sum_{j=J+1}^{n} c_j\mu(X_j) - \sum_{j=J+1}^{n} c_j\mu(X_j \cap \Omega_m) \right| \le$$

$$\le 2\epsilon + 2 \sum_{j=J+1}^{\infty} |c_j| v_{\infty}(\mu;X_j) < 4\epsilon ,$$

for every $n > \max\{J,N\}$. Hence, by Proposition 4.21, the set function μ is indeficient.

EXAMPLES 4.23. (i) A non-negative real valued additive set function on a quasiring of sets is indeficient if and only if it is σ-additive. This follows from Proposition 2.13 and Proposition 4.21. However, the argument establishing Proposition 2.13 can be simplified for the purpose of proving the indeficiency of such a set function directly.

So, let ι be a non-negative real valued additive set function on the quasiring $\mathcal{Q}$. Then $v_{\infty}(\iota,X) = \iota(X)$, for every $X \in \mathcal{Q}$.

If ι is not σ-additive, then, obviously, it is not an integrating gauge. Let us assume, therefore, that ι is σ-additive. We want to prove that

$$\text{(E.8)} \qquad \iota(X) \leq \sum_{j=1}^{\infty} |c_j| \iota(X_j)$$

for any set $X \in \mathcal{Q}$, numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$, such that the equality (E.7) holds for every $\omega \in \Omega$ for which the inequality (E.6) does. Let $\epsilon > 0$ and, for every $n = 1,2,\ldots$, let Z_n be the set of those points $\omega \in X$ for which

$$\sum_{j=1}^{n} |c_j| X_j(\omega) > 1 - \epsilon .$$

Then $Z_n \in \mathrm{sim}(\mathcal{Q})$, $Z_n \subset Z_{n+1}$ and

$$\sum_{j=1}^{n} |c_j| \iota(X_j) \geq \sum_{j=1}^{n} |c_j| \iota(X_j \cap Z_n) \geq (1-\epsilon)\iota(Z_n) ,$$

for every $n = 1,2,\ldots$. Because ι is σ-additive on the ring of sets whose characteristic functions belong to $\mathrm{sim}(\mathcal{Q})$, and the union of the sets Z_n, $n = 1,2,\ldots$, is equal to X, there is an integer $n \geq 1$ such that $\iota(Z_n) > \iota(X) - \epsilon$. Hence,

$$\sum_{j=1}^{\infty} |c_j| \iota(X_j) \geq (1-\epsilon)(\iota(X)-\epsilon)$$

for every $\epsilon > 0$, and the inequality (E.8) follows. By Proposition 2.7, the gauge $X \mapsto v_{\infty}(\iota;X) = \iota(X)$ is integrating and, hence, ι is indeficient.

(ii) Let $\mathcal{Q}$ be a ring of sets and let μ be a locally bounded real valued σ-additive set function on $\mathcal{Q}$. Then μ is indeficient.

In fact, let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ. So μ^+ and μ^- are non-negative σ-additive set functions on $\mathcal{Q}$ such that $\rho_\mu(X) \leq \mu^+(X) + \mu^-(X)$ and $\mu^+(X) \leq \rho_\mu(X)$, $\mu^-(X) \leq \rho_\mu(X)$, for every $X \in \mathcal{Q}$. Hence, the indeficiency of μ follows from that of μ^+ and μ^- by Proposition 4.21.

(iii) Let $\mathcal{Q}$ be a ring of sets and let μ be a locally bounded complex valued σ-additive set function on $\mathcal{Q}$. Then μ is indeficient. This follows from (ii) by considering the real and imaginary parts of μ .

(iv) Let $\Omega = \{1,2,\ldots\}$ be the set of all positive integers. Let $\mathcal{Q}$ be the family of all intervals in Ω , that is, intersections of Ω with intervals of the real-line. Let E be a Banach space and let $\{a_j\}_{j=1}^{\infty}$ be a conditionally summable sequence of its elements. Let

$$\mu(X) = \lim_{n\to\infty} \sum_{j=1}^{n} X(j)a_j$$

for every $X \in \mathcal{Q}$.

If we choose $\Omega_n = \{1,2,\ldots,n\}$, for $n = 1,2,\ldots,$ in Proposition 4.22, we deduce easily that the set function μ is indeficient.

(v) Let $\Omega = \mathbb{R}$ and let $\mathcal{Q}$ be the family of all (bounded and unbounded) intervals of the real-line. Let $s \neq 0$ be a real number and let

$$\mu(X) = \lim_{u\to\infty} \int_{-u}^{u} X(t)\exp(ist^2)dt$$

for every $X \in \mathcal{Q}$. Then μ is an indeficient additive set function on $\mathcal{Q}$.

In fact, let $\Omega_n = (-n,n)$, for every $n = 1,2,\ldots$. The restriction of μ to $\mathcal{Q}\cap\Omega_n$ is indeficient for every $n = 1,2,\ldots$. This can be seen by considering the real and imaginary parts of μ separately and noting that each Ω_n can be divided into a finite number of intervals such that in each of them $\mathrm{Re}\mu$ and $\mathrm{Im}\mu$ are of constant sign. Proposition 4.22 then applies.

If the set function $\mu : \mathcal{Q} \to E$ is indeficient then the gauge ρ , defined by (E.1) and (E.2), integrates for μ . However, this is not necessarily the only gauge which integrates for μ . For example, if μ has finite and σ-additive variation it might be convenient to let the variation integrate for μ . But the resulting spaces of integrable functions could be very different even if E is just the space of scalars.

EXAMPLE 4.24. Let Ω and $\mathcal{Q}$ be as in Example 4.23(iv). Let

$$\mu(X) = \sum_{j\in X} (-1)^j j^{-2}$$

for every $X \in \mathcal{Q}$. Then μ has finite and σ-additive variation, $v(\mu)$, and, by Example 4.23(ii), it is indeficient.

Let $e(\omega) = \omega$, for every $\omega \in \Omega$. Then

$$e(\omega) = \sum_{j=1}^{\infty} X_j(\omega)$$

for every $\omega \in \Omega$, where $X_j = \{j,j+1,...\}$ for every $j = 1,2,...$. Because

$$\rho(X_j) = v_\infty(\mu;X_j) = \sup\{|\mu(X_j \cap Z)| : Z \in \mathcal{Q}\} = j^{-2},$$

for every $j = 1,2,...,$ the function e belongs to $\mathcal{L}(\rho,\mathcal{Q})$.

On the other hand, a function f belongs to $\mathcal{L}(v(\mu),\mathcal{Q})$ if and only if

$$\sum_{j=1}^{\infty} |f(j)|j^{-2} < \infty .$$

F. Roughly speaking, indeficiency is preserved by closed rather than continuous maps.

Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. Let E be a Banach space.

Let A be an index set and, for every $\alpha \in \mathrm{A}$, let F_α be a Banach space and $T_\alpha : E \to F_\alpha$ a continuous linear map. We say that the family of maps $\{T_\alpha : \alpha \in \mathrm{A}\}$ separates the points of the space E if the equality $T_\alpha(x) = 0$, for some $x \in E$ and every $\alpha \in \mathrm{A}$, implies that $x = 0$.

For every $\alpha \in \mathrm{A}$, let $\nu_\alpha : \mathcal{Q} \to F_\alpha$ be a locally bounded additive set function. The family of set functions $\{\nu_\alpha : \alpha \in \mathrm{A}\}$ is said to be collectively indeficient if

$$\text{(F.1)} \qquad \sum_{j=1}^{\infty} c_j \nu_\alpha(X_j) = 0 ,$$

for every $\alpha \in \mathrm{A}$, whenever c_j are numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,...,$ such that

$$\text{(F.2)} \qquad \sum_{j=1}^{\infty} |c_j| v_\infty(\sigma_\alpha;X_j) < \infty ,$$

for every $\alpha \in A$, and the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does.

By Proposition 4.21, if each set function ν_α, $\alpha \in A$, is indeficient, then the family $\{\nu_\alpha : \alpha \in A\}$ is collectively indeficient.

PROPOSITION 4.25. *Let $\mu : \mathcal{Q} \to E$ be a locally bounded additive set function. Let $\nu_\alpha = T_\alpha \circ \mu$, for every $\alpha \in A$.*

If the family of maps $\{T_\alpha : \alpha \in A\}$ separates points of the space E and the family of set functions $\{\nu_\alpha : \alpha \in A\}$ is collectively indeficient, then the set function μ is indeficient.

Proof. Let us note first that the local boundedness of μ and the boundedness of T_α imply that each set function ν_α, $\alpha \in A$, is locally bounded.

Let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots$, satisfying condition (E.4), such that the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does. Let

$$x = \sum_{j=1}^{\infty} c_j \mu(X_j) .$$

Condition (E.4) and the continuity of T_α imply that (F.2) holds for every $\alpha \in A$. Consequently, (F.1) holds for every $\alpha \in A$, because the family of set functions $\{\nu_\alpha : \alpha \in A\}$ is collectively indeficient. So, by the continuity of T_α, the equality

$$T_\alpha(x) = T_\alpha\left[\sum_{j=1}^{\infty} c_j \mu(X_j)\right] = \sum_{j=1}^{\infty} c_j \nu_\alpha(X_j) = 0$$

holds for every $\alpha \in A$. Then $x = 0$, that is, (E.3) holds, because the family of maps $\{T_\alpha : \alpha \in A\}$ separates points of the space E. So, by Proposition 4.21, the set function μ is indeficient.

COROLLARY 4.26. *Let $\mu : \mathcal{Q} \to E$ be a locally bounded additive set function.*

If the family of functionals $x' \in E'$, such that the scalar valued set function $x' \circ \mu$ is indeficient, separates points of the space E, then the set function μ is indeficient.

EXAMPLE 4.27. Let E be a Banach space. Let $\mathcal{Q}$ be a ring of sets in a space Ω and let $\mu : \mathcal{Q} \to E$ be a locally bounded additive set function. By Corollary 4.26 and Example 4.23(iii), if the set of functionals $x' \in E'$, such that the set function $x' \circ \mu$ is σ-additive, separates the space E, then the set function μ is indeficient. In particular, a locally bounded σ-additive set function $\mu : \mathcal{Q} \to E$ is indeficient. This fact opens another way to integration 'with respect to vector measures'.

So, let $\mu : \mathcal{Q} \to E$ be a locally bounded σ-additive set function. Let $\rho_\mu(X) = v_\infty(\mu;X)$, for every $X \in \mathcal{Q}$. Let ρ be the seminorm on $\mathrm{sim}(\mathcal{Q})$ defined by

$$\rho(f) = \sup\{v(x' \circ \mu, |f|) : x' \in E', |x'| \leq 1\},$$

for every $f \in \mathrm{sim}(\mathcal{Q})$. Then $\rho_\mu(X) \leq \rho(X) \leq C\rho_\mu(X)$, for some $C \geq 1$ and every $X \in \mathcal{Q}$. (See Proposition 3.13.) Therefore, $\mathcal{L}(\rho,\mathcal{Q}) = \mathcal{L}(\rho_\mu,\mathcal{Q})$. But of course $\mathcal{L}(\rho,\mathcal{Q}) \subset \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$ and the inclusion may be strict.

In fact, let $\Omega = \{1,2,...\}$ be the set of all positive integers and let $E = c_0$ be the space of all scalar valued sequences tending to 0 equipped with the usual sup norm. Let $\mathcal{Q}$ be the family of all subsets of Ω. For every $X \in \mathcal{Q}$, let

$$\mu(X) = \sum_{j \in X} j^{-1} e_j,$$

where e_j, $j = 1,2,...,$ are the elements of the standard base of the space c_0. Let

$$f = \sum_{j=2}^{\infty} j(\log j)^{-1}\{j\}.$$

The function f is $v(x' \circ \mu)$-integrable, for every $x' \in E'$. (See Section 3F and/or Section A of this chapter.) Moreover, if

$$\nu(X) = \sum_{j \in X, j \geq 2} (\log j)^{-1} e_j,$$

for every $X \in \mathcal{Q}$, then

$$(x' \circ \nu))(X) = \int_\Omega fX \mathrm{d}(x' \circ \mu),$$

for every $x' \in E'$ and $X \in \mathcal{Q}$. Hence, by Proposition 3.13, the function f belongs to $\mathcal{L}(\rho,\text{sim}(\mathcal{Q}))$.

On the other hand, the function f does not belong to $\mathcal{L}(\rho_\mu,\mathcal{Q})$. In fact, let

$$\lambda(X) = \sum_{j \in X}^{\infty} j^{-2} ,$$

for every $X \in \mathcal{Q}$. Then $\lambda(X) \leq 2\rho_\mu(X)$, for every $X \in \mathcal{Q}$. Therefore, $\mathcal{L}(\rho_\mu,\mathcal{Q}) \subset \mathcal{L}(\lambda,\mathcal{Q})$. Because f does not belong to $\mathcal{L}(\lambda,\mathcal{Q})$, it does not belong to $\mathcal{L}(\rho_\mu,\mathcal{Q})$ either.

EXAMPLE 4.28. Let $\Omega = (0,1]$ and let $\mathcal{Q}$ be the semiring of all intervals $X = (s,t]$ such that $0 \leq s \leq t \leq 1$. Let c be the space of all convergent sequences $x = \{x_n\}_{n=1}^{\infty}$ of scalars equipped with the standard sup norm. Let d be a continuous scalar valued function in the interval $[0,1]$ and let $\nu((s,t]) = d(t) - d(s)$ for every s and t such that $0 < s < t < 1$. Let ι be the one-dimensional Lebesgue measure. Given an integer $n \geq 1$, let $Z_j = ((j-1)n^{-1},jn^{-1}]$ for every $j = 1,2,\ldots,n$, and let

$$\mu_n(X) = \sum_{j=1}^{n} n\iota(X \cap Z_j)\nu(Z_j)$$

for every $X \in \mathcal{Q}$. Finally, let $\mu(X) = \{\mu_n(X)\}_{n=1}^{\infty}$ for every $X \in \mathcal{Q}$. This defines an additive set function $\mu : \mathcal{Q} \to c$.

The set function μ is locally bounded. Furthermore, by Proposition 2.23, each component of μ is indeficient because it is the direct sum of a finite collection of multiples of the Lebesgue measure. Since the coordinate functionals separate the space c, by Corollary 4.26, the set function μ is indeficient.

If the set function $\mu : \mathcal{Q} \to E$ is indeficient, then the set functions $x' \circ \mu$, $x' \in E'$, are not necessarily all indeficient.

EXAMPLE 4.29. Let Ω and $\mathcal{Q}$ be as in Example 4.28. Let E be the closure of $\text{sim}(\mathcal{Q})$ in the space of bounded functions on Ω equipped with the sup norm. For every $X \in \mathcal{Q}$, let $\mu(X) = X$, interpreted as an element of the space E.

To see that μ is indeficient, let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,...$, satisfying condition (E.4), such that (E.5) holds for every $\omega \in \Omega$ for which (E.6) does. Then of course

$$\sum_{j=1}^{\infty} c_j \mu(X_j) = \sum_{j=1}^{\infty} c_j X_j = 0$$

in the space E.

On the other hand, let

$$x'(x) = \lim_{\omega \to 0+} x(\omega)$$

for every $X \in E$. Then $x' \in E'$ and $x' \circ \mu$ is scalar valued additive set function which is not indeficient.

G. Proposition 4.25 and its consequence, Corollary 4.26, are only effective when the space E is infinite-dimensional. However, we describe now a device which makes it possible, at least in principle, to use these propositions also on scalar valued set functions.

Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. We assume that $\mathcal{Q}$ is directed upwards by inclusion. That is, the union of any finite collection of sets from $\mathcal{Q}$ is contained in a set belonging to $\mathcal{Q}$.

Let E be a Banach space. Let $\mathrm{BV}^{\infty}(\mathcal{Q},E)$ be the set of all bounded additive set functions $\xi : \mathcal{Q} \to E$. Then $\mathrm{BV}^{\infty}(\mathcal{Q},E)$ is a vector space with respect to the natural (set-wise) operations. Let

$$V_{\infty}(\xi) = \sup\{ |\xi(X)| : X \in \mathcal{Q}\}$$

for every $\xi \in \mathrm{BV}^{\infty}(\mathcal{Q},E)$. Then $\xi \mapsto V_{\infty}(\xi)$, $\xi \in \mathrm{BV}^{\infty}(\mathcal{Q},E)$, is a norm which makes of $\mathrm{BV}^{\infty}(\mathcal{Q},E)$ a Banach space.

Let $\mu : \mathcal{Q} \to E$ be a locally bounded additive set function. For every $f \in \mathrm{sim}(\mathcal{Q})$, let $f\mu$ be the element of $\mathrm{BV}^{\infty}(\mathcal{Q},E)$ such that $(f\mu)(X) = \mu(fX)$, for every $X \in \mathcal{Q}$. It is straightforward that the set function $f\mu$ so defined is indeed an element of $\mathrm{BV}^{\infty}(\mathcal{Q},E)$.

PROPOSITION 4.30. *Let $\mu : \mathcal{Q} \to E$ be a locally bounded additive set function. Let $\hat{\mu} : \mathcal{Q} \to \mathrm{BV}^{\infty}(\mathcal{Q},E)$ be the set function defined by $\hat{\mu}(X) = X\mu$, for every $X \in \mathcal{Q}$.*

Then μ is indeficient if and only if $\hat{\mu}$ is indeficient.

Proof. The set function $\hat{\mu}$ is obviously additive and locally bounded.

Now, if $\hat{\mu}$ is indeficient then it follows easily from Proposition 4.21 that μ is indeficient because

$$v_{\infty}(\hat{\mu};X) = \sup\{V_{\infty}(\hat{\mu}(X\cap Z)) : Z \in \mathcal{Q}\} = \sup\{|\mu(X\cap Z)| : Z \in \mathcal{Q}\} = v_{\infty}(\mu;X),$$

for every $X \in \mathcal{Q}$. The multiplicativity of $\mathcal{Q}$ is used.

Conversely, let μ be indeficient. Again, Proposition 4.21 implies that $\hat{\mu}$ is indeficient. Indeed, let c_j be some numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots$, such that

$$\text{(G.1)} \qquad \sum_{j=1}^{\infty} |c_j| \, v_{\infty}(\hat{\mu};X_j) < \infty$$

and the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does. Then

$$\lim_{n\to\infty} \left| \sum_{j=1}^{n} c_j \mu(X_j \cap Z) \right| = 0,$$

for every $Z \in \mathcal{Q}$, by the indeficiency of μ. But then

$$\lim_{n\to\infty} V_{\infty}\left[\sum_{j=1}^{n} c_j \hat{\mu}(X_j) \right] = 0.$$

For a locally bounded additive set function $\mu : \mathcal{Q} \to E$, let $\mathrm{BV}^{\infty}(\mu,\mathcal{Q},E)$ be the closure of the space $\{f\mu : f \in \mathrm{sim}(\mathcal{Q})\}$ in $\mathrm{BV}^{\infty}(\mathcal{Q},E)$.

PROPOSITION 4.31. *Let $\mathcal{P}$ and $\mathcal{Q}$ be multiplicative quasirings of sets in the space Ω such that $\mathcal{Q} \subset \mathcal{P}$. Let E and F be Banach spaces and $\mu : \mathcal{Q} \to E$ and $\nu : \mathcal{P} \to F$ locally bounded additive set functions. Assume that ν is indeficient and that there exists an injective continuous linear map $T : \mathrm{BV}^{\infty}(\mu,\mathcal{Q},E) \to \mathrm{BV}^{\infty}(\nu,\mathcal{P},F)$ such that $T(X\mu) = X\nu$, for every $X \in \mathcal{Q}$. Then the set function μ is indeficient.*

Proof. Let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots,$ satisfying condition (G.1), such that the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does. Then the sequence $\{c_j X_j \mu\}_{j=1}^{\infty}$ is absolutely summable in the space $\mathrm{BV}^{\infty}(\mu,\mathcal{Q},E)$; let ξ be its sum. Because the map T is linear and continuous and $X_j \nu = T(X_j \mu)$, for every $j = 1,2,\ldots,$ the sequence $\{c_j X_j \nu\}_{j=1}^{\infty}$ is absolutely summable in the space $\mathrm{BV}^{\infty}(\nu,\mathcal{P},E)$. By the indeficiency of ν and Proposition 4.30, the sum of the sequence $\{c_j X_j \nu\}_{j=1}^{\infty}$ is the zero-element of the space $\mathrm{BV}^{\infty}(\nu,\mathcal{P},E)$. Then $T(\xi) = 0$, because the map T is continuous, and then $\xi = 0$, because T is injective. Hence, by Proposition 4.30, the set function μ is indeficient.

The use of Proposition 4.31 is mainly in that it gives a sufficient condition for the preservation of indeficiency in passing to a sub-quasiring.

H. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω . Let E be a normed space and $\mu : \mathcal{Q} \to E$ an indeficient additive set function. Let the gauge ρ be defined by (E.1) and (E.2), for every $X \in \mathcal{Q}$. Then of course the gauge ρ integrates for the set function μ . But the usefulness of ρ is thereby not exhausted; the gauge ρ integrates possibly for many other, not necessarily indeficient, additive set functions on $\mathcal{Q}$. For instance, it does integrate for every set function of the form $T \circ \mu$, where T is a continuous map from E into another Banach space.

EXAMPLE 4.32. Let us adopt the notation of Example 4.28. Because

$$\nu(X) = \lim_{n\to\infty} \mu_n(X) ,$$

for every $X \in \mathcal{Q}$, and the limit is a continuous linear functional on the space c , the gauge ρ integrates for the scalar valued set function ν .

Such a gauge integrating for the set function ν is especially interesting if ν does not have finite variation in any interval.

EXAMPLE 4.33. Let $E = L^2(\mathbb{R})$. Let $S(0) = I$ be the identity operator on the space E . For $t \neq 0$, let $S(t)$ be the operator on E such that

$$(S(t)\varphi)(x) = \frac{1}{\sqrt{2\pi i t}} \int_{\mathbb{R}} \exp\left[-\frac{(x-y)^2}{2i\,t}\right] \varphi(y)\mathrm{d}y$$

for every $\varphi \in L^1 \cap L^2(\mathbb{R})$. It is well-known that by this a unitary operator $S(t) : E \to E$ is defined and that the resulting one-parameter family of operators $t \mapsto S(t)$, $t \in (-\infty,\infty)$, is a unitary group.

For a Borel set B in $\mathbb{R}$, let $P(B)$ be the operator of point-wise multiplication by the characteristic function of B on the space E.

Let $t > 0$ be fixed and let Ω be the set of all continuous functions (paths) $\omega : [0,t] \to \mathbb{R}$. Let $\mathcal{Q}$ be the family of all sets

(H.1) $$X = \{\omega \in \Omega : \omega(t_j) \in B_j\, ,\, j = 1,2,\ldots,n\}\, ,$$

for arbitrary $n = 1,2,\ldots,$ $0 \le t_1 < t_2 < \ldots < t_{n-1} < t_n \le t$ and Borel sets B_j in $\mathbb{R}$, $j = 1,2,\ldots,n$.

Let φ be a non-zero element of the space E. Let

$$\nu(X) = S(t-t_n)P(B_n)S(t_n-t_{n-1})P(B_{n-1}) \ldots P(B_2)S(t_2-t_1)P(B_1)\varphi$$

for any set $X \in \mathcal{Q}$ written in the form (H.1).

Then $\nu : \mathcal{Q} \to E$ is an additive set function which has infinite variation on every set $X \in \mathcal{Q}$. A gauge integrating for ν can be constructed in a similar manner as a gauge for the set function of Example 4.28.

Indeed, let $\not{h}_n$ be partitions of the real-line into finite numbers of intervals such that $\not{h}_{n+1}$ is a refinement of $\not{h}_n$, $n = 1,2,\ldots$. For every $n = 1,2,\ldots,$ let $\mathcal{P}_n$ be the family of all sets $X \in \mathcal{Q}$, which can be written in the form

$$X = \{\omega : \omega(j/2^n) \in B_j,\ j = 1,2,\ldots,2^n\}\, ,$$

where the sets B_j, depending on X, belong to $\not{h}_n$, $j = 1,2,\ldots,2^n$. Then $\mathcal{P}_n \in \Pi(\mathcal{Q})$ are partitions such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, for every $n = 1,2,\ldots$. Let ι be the Wiener measure in Ω with unit variance per unit of time and with the standard normal initial distribution, say. That is, ι is the measure such that

$$\iota(X) = \left[(2\pi)^{n+1} \prod_{j=1}^{n} (t_j - t_{j-1})\right]^{-\frac{1}{2}} \times$$

$$\int_{B_n} \int_{B_{n-1}} \cdots \int_{B_1} \int_{\mathbb{R}} \exp\left[-\frac{x_0^2}{2} - \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}\right] dx_0 dx_1 \dots dx_{n-1} dx_n ,$$

for every set, X, of the form (H.1), where we put $t_0 = 0$. Assume that the partitions $\mathcal{P}_n$ are so chosen that, for every $n = 1,2,\ldots$, there is a number $m_n > 0$ such that $\iota(X) = m_n$ for every $X \in \mathcal{P}_n$ and that $m_n \to 0$ as $n \to \infty$. Partitions of Ω similar to $\mathcal{P}_n$ were used by N. Wiener in the first constructions of the measure named after him; see, for example, [68].

Now, given an integer $n \geq 1$, let

$$\mu_n(X) = m_n^{-1} \sum_{Z \in \mathcal{P}_n} \iota(Z \cap X)\nu(Z)$$

for every $X \in \mathcal{Q}$. Then $\mu_n : \mathcal{Q} \to E$ is an indeficient additive set function.

Let c_E be the space of all convergent sequences of elements of the space E equipped with the usual sup norm. Let $\mu : \mathcal{Q} \to c_E$ be the set functions such that $\mu(X) = \{\mu_n(X)\}_{n=1}^{\infty}$, for every $X \in \mathcal{Q}$. Let $F_n = E$ and let $T_n : c_E \to F_n$ be the n-th coordinate map, for every $n = 1,2,\ldots$. The set functions $T_n \circ \mu : \mathcal{Q} \to F_n$ are then indeficient because $T_n \circ \mu = \mu_n$, $n = 1,2,\ldots$. Therefore, by Proposition 4.25, the set function μ is indeficient.

Because

$$\nu(X) = \lim_{n \to \infty} \mu_n(X) ,$$

for every $X \in \mathcal{Q}$, and the limit is a continuous linear map from the space c_E onto E, the gauge ρ, defined by (E.1) and (E.2) for every $X \in \mathcal{Q}$, integrates for ν.

J. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω directed upward by inclusion. (See Section G.) Let $\Delta \subset \Pi(\mathcal{Q})$ be a set of partitions. Let E be a Banach space.

Given a Young function, Φ, the family of all additive set functions $\xi : \mathcal{Q} \to E$

such that

$$\sup\{v_{\Phi}(\xi,\Delta;X) : X \in \mathcal{Q}\} < \infty ,$$

will be denoted by $\mathrm{BV}^{\Phi}(\Delta,E)$. We shall write $\mathrm{BV}^{\Phi}(\mathcal{Q},E) = \mathrm{BV}^{\Phi}(\Pi,E)$.

These notions are useful mainly in the case when $\mathcal{Q}$ is a quasialgebra, that is, $\Omega \in \mathcal{Q}$. In that case, the definitions can be simplified somewhat.

PROPOSITION 4.34. *If* Φ *is a Young function, then* $\mathrm{BV}^1(\Delta,E) \subset \mathrm{BV}^{\Phi}(\Delta,E) \subset \mathrm{BV}^{\infty}(\mathcal{Q},E)$, *for any set of partitions* $\Delta \subset \Pi$.

If Φ *and* Ψ *are Young functions for which there exist numbers* $a > 0$ *and* $k > 0$ *such that* $\Psi(s) \le k\Phi(s)$, *for every* $s \in [0,a]$, *then* $\mathrm{BV}^{\Phi}(\Delta,E) \subset \mathrm{BV}^{\Psi}(\Delta,E)$, *for any set of partitions* $\Delta \subset \Pi$.

Proof. The first statement is obvious. The second one is analogous to the statement 1.15 in [51]. For its proof, let us note first that, if the condition is satisfied, then, for every $b > 0$, there is a constant $\ell > 0$ such that $\Psi(s) \le \ell\Phi(s)$, for every $s \in [0,b]$. In fact, if $a \le s \le b$, then

$$\Phi(s) \ge \frac{\Psi(a)}{k\Phi(a)}\,\Phi(s) = \frac{1}{k}\,\frac{\Psi(a)}{\Phi(a)}\,\frac{\Phi(s)}{\Psi(s)}\,\Psi(s) \ge \frac{1}{k}\,\frac{\Psi(a)}{\Psi(b)}\,\Psi(s) .$$

So, let us assume that $\xi \in \mathrm{BV}^{\Phi}(\Delta,E)$. Then there exists a $b > 0$ such that $|\xi(X \cap Y)| \le b$ for every set $X \in \mathcal{Q}$ and every set Y belonging to some $\mathcal{P} \in \Delta$. Consequently, $\Psi(|\xi(X\cap Y)| \le \ell\Phi(|\xi(X\cap Y)|)$ and $v_{\Psi}(\xi,\Delta;X) \le \ell v_{\Phi}(\xi,\Delta;X)$.

The second part of this proposition has a converse: If Ω and $\mathcal{Q}$ are as in Example 5.28 and $\mathrm{BV}^{\Phi}(\mathcal{Q},\mathbb{R}) \subset \mathrm{BV}^{\Psi}(\mathcal{Q},\mathbb{R})$, then there exist numbers $a > 0$ and $k > 0$ such that $\Psi(s) \le k\Phi(s)$ for every $s \in [0,a]$. Cf. statement 1.15 in [51].

The sets $\mathrm{BV}^1(\Delta,E)$ and $\mathrm{BV}^{\infty}(\Delta,E)$ are, obviously, vector spaces with respect to the natural operations. The following proposition says that, if the Young function, Φ , satisfies condition (Δ_2) for small values of the argument (see Section 1G), then also $\mathrm{BV}^{\Phi}(\Delta,E)$ is a vector space. It is analogous to statement 1.13 in [51] and so, its proof too is analogous.

PROPOSITION 4.35. *If the Young function,* Φ , *satisfies the condition* (Δ_2) *for small values of the argument, then* $\mathrm{BV}^{\Phi}(\Delta,E)$ *is a vector space under the natural operations.*

Proof. Assume that $k > 0$ and $a > 0$ are numbers such that $\Phi(2s) \leq k\Phi(s)$ for every $s \in [0,a]$. Then, for every $b > 0$, there is an $\ell(b) \geq 1$ such that $\Phi(2s) \leq \ell(b)\Phi(s)$ for every $s \in [0,\ell(b)]$. In fact, if $\frac{1}{2}a \leq s \leq b$, then

$$\Phi(s) \geq \frac{\Phi(a)}{k\Phi(\frac{1}{2}a)}\,\Phi(s) = \frac{1}{k}\,\frac{\Phi(a)}{\Phi(\frac{1}{2}a)}\,\frac{\Phi(s)}{\Phi(\frac{1}{2}a)}\,\Phi(2s) \geq \frac{1}{k}\,\frac{\Phi(a)}{\Phi(2b)}\,\Phi(2s) .$$

Now, if $\xi \in \mathrm{BV}^{\Phi}(\Delta,E)$ and $\eta \in \mathrm{BV}^{\Phi}(\Delta,E)$, there exists a $b > 0$ such that $|\xi(X \cap Y)| \leq b$ and $|\eta(X \cap Y)| \leq b$, for every $X \in \mathcal{Q}$ and every set Y belonging to any partition from Δ . Consequently,

$$v_{\Phi}(\xi+\eta,\Delta;X) \leq \ell(b)(v_{\Phi}(\xi,\Delta;X) + v_{\Phi}(\eta,\Delta;X)) ,$$

for every $X \in \mathcal{Q}$. If, further, c is a number, let m be the least positive integer such that $|c| \leq 2^m$. Then

$$v_{\Phi}(c\xi,\Delta;X) \leq (\ell(2^{m-1}b))^m v_{\Phi}(\xi,\Delta;X) ,$$

for every $X \in \mathcal{Q}$.

For every $\xi \in \mathrm{BV}^1(\Delta,E)$, let

$$V_1(\xi,\Delta) = \sup\{v_1(\xi,\Delta;X) : X \in \mathcal{Q}\}.$$

Then the functional $\xi \mapsto V_1(\xi,\Delta)$, $\xi \in \mathrm{BV}^1(\Delta,E)$, is a norm making the space $\mathrm{BV}^1(\Delta,E)$ complete.

If the Young function, Φ , satisfies condition (Δ_2) for small values of the argument (see Section 1G), then a norm still can be introduced in the space $\mathrm{BV}^{\Phi}(\Delta,E)$. It can be naturally done in at least two ways. Thus let

$$V_{\Phi}(\xi,\Delta) = \inf\{k > 0 : v_{\Phi}(k^{-1}\xi,\Delta;X) \leq 1 \ , \ X \in \mathcal{Q}\} ,$$

for every $\xi \in \mathrm{BV}^{\Phi}(\Delta,E)$. Secondly, given a set function $\xi \in \mathrm{BV}^{\Phi}(\Delta,E)$ and a partition $\mathcal{P} \in \Delta$, let

$$V^0_\Phi(\xi,\Delta) = \sup\left\{ \sum_{Y\in\mathcal{P}} \beta(X\cap Y)\,|\mu(X\cap Y)| : \beta\in B_{X\mathcal{P}},\ \mathcal{P}\in\Delta,\ X\in\mathcal{Q}\right\},$$

where $B_{X\mathcal{P}}$ is the set of all functions

$$\beta : \{X\cap Y:\ Y\in\mathcal{P}\} \to [0,\infty)$$

such that

$$\sum_{Y\in\mathcal{P}} \Psi(\beta(X\cap Y)) \le 1\,,$$

and Ψ is the Young function complementary to Φ . (See Section 1G.)

By analogy with the usual terminology in Orlicz spaces, the functional $\xi \mapsto V_\Phi(\xi,\Delta)$, $\xi\in \mathrm{BV}^\Phi(\Delta,E)$, will be called the Luxemburg norm and the functional $\xi\mapsto V^0_\Phi(\xi,\Delta)$, $\xi\in\mathrm{BV}^\Phi(\Delta,E)$, the Orlicz norm. It turns out that these functionals are indeed norms on the space $\mathrm{BV}^\Phi(\Delta,E)$ and they are equivalent.

PROPOSITION 4.36. *Assume that the Young function* Φ *satisfies conditions* (0) , (∞) *and* (Δ_2) *for small values of the argument. Then the functionals* $V_\Phi(\cdot,\Delta)$ *and* $V^0_\Phi(\cdot,\Delta)$ *are norms on the space* $\mathrm{BV}^\Phi(\Delta,E)$ *such that*

(J.1)
$$V_\Phi(\xi,\Delta) \le V^0_\Phi(\xi,\Delta) \le 2V_\Phi(\xi,\Delta)$$

for every $\xi\in\mathrm{BV}^\Phi(\Delta,E)$. *The space* $\mathrm{BV}^\Phi(\Delta,E)$ *is complete in each of these norms.*

Proof. The inequalities (J.1) follow directly from the definitions of the functionals $V_\Phi(\cdot,\Delta)$ and $V^0_\Phi(\cdot,\Delta)$ and from Proposition 1.15. We omit the proofs that these functionals are indeed norms and of the completeness of the space $\mathrm{BV}^\Phi(\Delta,E)$.

Let us note that, if $1 < p < \infty$ and $\Phi(s) = s^p$, for every $s\in[0,\infty)$, then

$$V_p(\xi,\Delta) = \left[\sup\left\{ \sum_{Y\in\mathcal{P}} |\xi(X\cap Y)|^p : \mathcal{P}\in\Delta,\ X\in\mathcal{Q}\right\}\right]^{1/p},$$

for every $\xi\in\mathrm{BV}^p(\Delta,E)$.

K. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω which is directed upward by inclusion and E a Banach space. Let $\Delta \subset \Pi = \Pi(\mathcal{Q})$ be a set of partitions and let Φ be a Young function satisfying condition (Δ_2) for small values of the argument. (See Section 1G.)

Let us note first that, if the additive set function $\mu : \mathcal{Q} \to E$ has finite Φ-variation with respect to the set of partitions Δ and f is a $\mathcal{Q}$-simple function, then $f\mu \in \mathrm{BV}^{\Phi}(\Delta,E)$. Now, assuming that μ is such a set function, the closure of the vector space $\{f\mu : f \in \mathrm{sim}(\mathcal{Q})\}$ in $\mathrm{BV}^{\Phi}(\Delta,E)$ will be denoted by $\mathrm{BV}^{\Phi}(\Delta,\mu)$. Then $\mathrm{BV}^{\Phi}(\Delta,\mu)$ is a Banach space, being a closed subspace of $\mathrm{BV}^{\Phi}(\Delta,E)$. Again, we write $\mathrm{BV}^{\Phi}(\mathcal{Q},\mu) = \mathrm{BV}^{\Phi}(\Pi,\mu)$.

If ι is a real valued positive σ-additive set function on $\mathcal{Q}$, then

$$(K.1) \qquad V_1(f\iota,\Pi) = \int_{\Omega} |f| \, d\iota$$

for every $f \in \mathrm{sim}(\mathcal{Q})$. Therefore, the elements of the space $\mathrm{BV}^{1}(\mathcal{Q},\iota)$ are canonically associated with ι-integrable functions, or, more accurately, with the equivalence classes of such functions. In other words, the space $\mathrm{BV}^{1}(\mathcal{Q},\iota)$ is identified with $L^1(\iota)$.

In this section, those set functions, $\mu : \mathcal{Q} \to E$, are isolated for which an analogous identification of $\mathrm{BV}^{\Phi}(\Delta,\mu)$ with a space of (equivalence classes of) functions on Ω is possible. The definition is immediate.

An additive set function $\mu : \mathcal{Q} \to E$ will be called (Φ,Δ)-closable if it has finite Φ-variation with respect to the set of partitions Δ and the seminorm $\rho = \rho_{\mu,\Phi,\Delta}$ on $\mathrm{sim}(\mathcal{Q})$, defined by

$$\rho(f) = V_{\Phi}(f\mu,\Delta)$$

for every $f \in \mathrm{sim}(\mathcal{Q})$, is integrating. In that case, we write $\mathcal{L}(\mu,\Phi,\Delta) = \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$ and $\|\cdot\|_{\mu,\Phi,\Delta} = \rho_{\mu,\Phi,\Delta} = \rho = q_{\rho}$. Also, $\mathcal{L}(\mu,\Phi,\mathcal{Q}) = \mathcal{L}(\mu,\Phi,\Pi)$.

Because $\mathrm{sim}(\mathcal{Q})$ is dense in $\mathcal{L}(\mu,\Phi,\Delta)$, for every $f \in \mathcal{L}(\mu,\Phi,\Delta)$, there is a unique element $\nu_f \in \mathrm{BV}^{\Phi}(\Delta,\mu)$ such that $\nu_f = f\mu$ for $f \in \mathrm{sim}(\mathcal{Q})$ and the map $f \mapsto \nu_f$, from $\mathcal{L}(\mu,\Phi,\Delta)$ onto $\mathrm{BV}^{\Phi}(\Delta,\mu)$, is continuous. We write, of course,

$f\mu = \nu_f$, for every $f \in \mathcal{L}(\mu,\Phi,\Delta)$, and call $f\mu$ the indefinite integral of the function f with respect to μ .

To introduce an interesting class of (Φ,Δ)-closable additive set functions, we adopt the following definition. An additive set function $\mu : \mathcal{Q} \to E$ will be called Φ-scattered if the set function $X \mapsto \Phi(|\mu(X)|)$, $X \in \mathcal{Q}$, is σ-additive.

This notion originates from the case when E is a Hilbert space and for any disjoint sets, X and Y, belonging to $\mathcal{Q}$, the values $\mu(X)$ and $\mu(Y)$ are orthogonal. Such a set function is called orthogonally scattered. It is immediate that, if μ is an orthogonally scattered set function, then the set function $X \mapsto |\mu(X)|^2$, $X \in \mathcal{Q}$, is additive and, if E is a real Hilbert space, then also the converse is true. Since, however, the converse is not necessarily true in a complex Hilbert space and σ-additivity is built in the notion of a 2-scattered set function, which is convenient for the purpose of this example, we keep the notions of an orthogonally scattered and a 2-scattered set function distinct. For a systematic treatment of orthogonally scattered additive set functions, see [49].

PROPOSITION 4.37. *Assume that the Young function Φ satisfies condition (Δ_2) . Let $\mu : \mathcal{Q} \to E$ be a Φ-scattered additive set function. Denote $\iota(X) = \Phi(|\mu(X)|)$ for every $X \in \mathcal{Q}$. Assume that the measure generated by the set function ι is σ-finite. Then the set function μ is (Φ,Π)-closable, $\mathcal{L}(\mu,\Phi,\mathcal{Q}) = \mathcal{L}^{\Phi}(\iota)$ and*

(K.2)
$$\mathrm{V}^0_{\Phi}(f\mu;\Pi) = \|f\|^0_{\iota,\Phi} ,$$

for every $f \in \mathcal{L}(\mu,\Phi,\mathcal{Q})$.

Proof. First we prove (K.2) for $f \in \mathrm{sim}(\mathcal{Q})$. So, let

$$f = \sum_{j=1}^{n} c_j X_j$$

with an arbitrary $n = 1,2,\ldots,$ numbers c_j and pairwise disjoint sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,n$. Let Ψ be the Young function complementary to Φ . Then

$$\|f\|^0_{\iota,\Phi} = \sup\left\{ \int_\Omega fg\,d\iota : g \in \mathrm{sim}(\mathcal{Q}) \,,\, \int_\Omega \Psi(|g|)d\iota \le 1 \right\}$$

and

$$V^0_\Phi(f\mu;\Pi) = \sup\left\{ \sum_{Y\in\mathcal{P}} |(f\mu)(Y)|\beta(Y) : \beta \in \mathrm{B}_\mathcal{P} \,,\, \mathcal{P} \in \Pi \right\} ,$$

where $\mathrm{B}_\mathcal{P}$ is the family of all functions $\beta : \mathcal{P} \to [0,\infty)$ such that

$$\sum_{Y\in\mathcal{P}} \Psi(\beta(Y)) \le 1 \,.$$

Because $V^0_\Phi(\cdot,\Pi)$ is a norm in the space $\mathrm{BV}^\Phi(\mu,\mathcal{Q})$, it suffices to calculate the supremum over partitions $\mathcal{P} \in \Pi$ such that every set X_j, $j = 1,2,\ldots,n$, is equal to the union of some elements of $\mathcal{P}$. Furthermore, it suffices to take $\beta \in \mathrm{B}_\mathcal{P}$ such that $\beta(Y) = 0$, whenever $Y \cap X_j = \emptyset$ for each $j = 1,2,\ldots,n$. Then, given such a β, we put

$$g = \sum_{Y\in\mathcal{P},\,\iota(Y)\neq 0} \Psi^{-1}\left[\frac{\Psi(\beta(Y))}{\iota(Y)}\right] Y \,.$$

Because, in calculating $\|f\|^0_{\iota,\Phi}$, it suffices to take those functions $g \in \mathrm{sim}(\mathcal{Q})$ which are obtained in this manner, the equality (K.2) is indeed true.

The equality (K.2) is analogous to, or a generalization of, (K.1). It implies that the set function μ is σ-additive, (Φ,Π)-closable and that $\mathcal{L}(\mu,\Phi,\mathcal{Q}) = \mathcal{L}^\Phi(\iota)$.

It seems difficult to prove the (Φ,Δ)-closability of set functions which are not in a sense equivalent to Φ-scattered ones. None-the-less, the norms V_Φ and V^0_Φ could still be helpful. For, if the additive set function $\mu : \mathcal{Q} \to E$ has finite Φ-variation, then the gauge ρ, defined by

$$\rho(X) = V_\Phi(X\mu,\Delta) \,,$$

for every $X \in \mathcal{Q}$, is usually very sub-additive (see Section 2J) and so, in many cases, Proposition 2.25 applies. Then this gauge can be used instead of the one studied in Section C.