1. PRELIMINARIES, NOTATION, CONVENTIONS

Even though the notation and conventions adopted here are fairly standard, slight variations that occur in the literature can cause inconvenience to the reader. So, the problem of making the whole text sufficiently self-contained is solved by placing this chapter at the beginning. None-the-less the chapter can be used as an appendix, that is, the reader may refer to it only as the need arises. To facilitate such usage, frequent references to this one are made in the subsequent chapters.

A. The need to treat real and complex vector spaces separately will only seldom arise. Therefore, the real or complex numbers will be referred to simply as numbers or scalars.

To maintain the perspicuity of the notation pertaining to vector valued functions and integrals, the multiplication by scalars of elements of a vector space will be written commutatively. That is to say, if E is a vector space, we shall write interchangeably $cx = xc$, for any scalar c and a vector $x \in E$.

By a seminorm on a vector space E is meant a function $q : E \to [0,\infty)$ such that $q(x+y) \leq q(x) + q(y)$, for every $x \in E$ and $y \in E$, and $q(cx) = |c| q(x)$, for every $x \in E$ and a number c. So, a seminorm has all the properties of a norm with the only exception that its value may be equal to zero on a non-zero element of E.

The study of spaces of individual integrable functions, rather than those of the equivalence classes of such functions, makes it convenient to consider general seminormed and not just normed spaces. To be sure, a seminormed space is a vector space together with a specified seminorm on it. A majority of concepts referring to normed spaces are with obvious modifications applicable to seminormed spaces. The occasional difficulties are caused mainly by the non-uniqueness of limits and similar objects.

So, let E be a seminormed space with the seminorm q.

A set $S \subset E$ is called bounded if $\{q(x) : x \in S\}$ is a bounded set of numbers.

A set $S \subset E$ is dense in E if, for every $x \in E$ and $\epsilon > 0$, there exists an $y \in S$ such that $q(x-y) < \epsilon$.

A sequence, $\{x_n\}_{n=1}^{\infty}$, of elements of E is said to be convergent if there exists an element x of E such that

$$\lim_{n\to\infty} q(x-x_n) = 0 .$$

In that case, x is said to be a limit of the sequence $\{x_n\}_{n=1}^{\infty}$. We write

$$x = \lim_{n\to\infty} x_n .$$

If y too is a limit of this sequence, then $q(x-y) = 0$.

A sequence, $\{x_n\}_{n=1}^{\infty}$, of elements of E is said to be Cauchy if, for every $\epsilon > 0$, there is a δ such that $q(x_n - x_n) < \epsilon$, for every $n > \delta$ and $m > \delta$.

If we want to be specific, we speak of q-bounded sets, q-convergent sequences, and so on.

The space E is said to be complete if every Cauchy sequence of its elements is convergent.

We shall reserve the term "Banach space" to denote a complete normed space. So, E is a Banach space if q is actually a norm, that is, the equality $q(x) = 0$ implies that $x = 0$, and if E is complete.

The norm of an unspecified Banach space will be mostly denoted as modulus.

A sequence, $\{x_j\}_{j=1}^{\infty}$, of elements of the seminormed space E is said to be conditionally (or simply) summable if the sequence $\{s_n\}_{n=1}^{\infty}$, where

$$s_n = \sum_{j=1}^{n} x_j$$

for every $n = 1,2,\dots,$ is convergent. If s is a limit of the sequence $\{s_n\}_{n=1}^{\infty}$, then we write

$$s = \sum_{j=1}^{\infty} x_j$$

and call the element s a sum of the sequence $\{x_j\}_{j=1}^{\infty}$.

The sequence $\{x_j\}_{j=1}^{\infty}$ is said to be unconditionally summable if, for all choices of $\epsilon_j = 0$ or 1, $j = 1,2,\ldots,$ the sequence $\{\epsilon_j x_j\}_{j=1}^{\infty}$ is conditionally summable.

The sequence $\{x_j\}_{j=1}^{\infty}$ is said to be absolutely summable if

$$\sum_{j=1}^{\infty} q(x_j) < \infty \tag{A.1}$$

and if it is summable.

The following two statements are designated as propositions with their own numbers only to give them prominence. Their proofs are of course omitted.

PROPOSITION 1.1. *The seminormed space* E *is complete if and only if every sequence,* $\{x_j\}_{j=1}^{\infty}$, *of its elements which satisfies condition* (A.1) *is summable.*

PROPOSITION 1.2. *Let* E *be a Banach space with the norm* q. *Let* H *be a dense vector subspace of* E. *Then every element,* x, *of the space* E *can be expressed as the sum of some elements,* x_j, *of* H, $j = 1,2,\ldots,$ *satisfying condition* (A.1). *Furthermore,*

$$q(x) = \inf \sum_{j=1}^{\infty} q(x_j) ,$$

where the infimum is taken over all expressions of x *as the sum of elements* x_j *of* H, $j = 1,2,\ldots,$ *satisfying* (A.1).

B. Let F be a vector space. Let $Q \subset F$; the set Q is not assumed to be a vector space.

The linear hull of Q will be denoted by $\mathrm{sim}(Q)$. That is, $x \in \mathrm{sim}(Q)$ if and only if there exist a (strictly) positive integer n, numbers c_j and elements, x_j, of Q, $j = 1,2,\ldots,n$, such that

$$x = \sum_{j=1}^{n} c_j x_j . \tag{B.1}$$

Elements of the space F that belong to $\mathrm{sim}(Q)$ are called Q-simple. This notation and terminology originated in elementary integration theory and will be mainly used in that context. (See Section D below.)

Let E be another vector space. A map $\mu : Q \to E$ will be called linear if

$$\sum_{j=1}^{n} c_j \mu(x_j) = 0$$

for any $n = 1,2,\ldots,$ numbers c_j and elements x_j of Q, $j = 1,2,\ldots,n$, such that

$$\sum_{j=1}^{n} c_j x_j = 0 .$$

A map $\mu : Q \to E$ is linear if and only if there exists a linear map $\tilde{\mu} : \mathrm{sim}(Q) \to E$ such that $\tilde{\mu}(x) = \mu(x)$ for every $x \in Q$. If it exists, such a linear map $\tilde{\mu}$ is unique. Therefore, following the custom, we shall not distinguish, in terminology and notation, between a linear map $\mu : Q \to E$ and the linear map on $\mathrm{sim}(Q)$ into E that extends μ.

If E is the one-dimensional vector space, that it, the space of scalars, then a linear map $\mu : Q \to E$ is called a linear function, or a linear functional. The vector space of all linear functions on the whole of F is called the algebraic dual space to F and denoted by F^*.

Assume now that E and F are seminormed spaces with the seminorms p and q, respectively. Then we can speak about the continuity of a map $\mu : F \to E$ at a point $x \in F$. To be sure, such a map is continuous at a point $x \in F$ if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $p(\mu(y) - \mu(x)) < \epsilon$ for every $y \in F$ for which $q(y-x) < \delta$.

As in the case of normed spaces, for a linear map $\mu : F \to E$, the following statements (i), (ii) and (iii), are equivalent:

(i) There is a point in F at which μ is continuous.

(ii) The map μ is continuous at every point of the space F.

(iii) There is a constant $k \geq 0$ such that $p(\mu(x)) \leq kq(x)$, for every $x \in F$.

So, it is quite unambiguous to say simply about a linear map on a (whole) vector space that it is continuous.

The vector space of all continuous linear functionals on a seminormed space F is called the continuous dual space to F, or just the dual of F, and denoted by F'.

If we define $q'(x') = \sup\{|x'(\mathrm{x})| : q(x) \leq 1\}$, for every $x' \in F'$, Then q' is a norm on F' which makes of F' a Banach space.

A sequence, $\{x_j\}_{j=1}^{\infty}$, of elements of a seminormed space F is said to be conditionally weakly summable if there exists an element s of F such that

$$\sum_{j=1}^{\infty} x'(x_j) = x'(s) ,$$

for every $x' \in F'$.

A sequence, $\{x_j\}_{j=1}^{\infty}$, of elements of a seminormed space is said to be unconditionally weakly summable if, for any choice of $\epsilon_j = 0$ or 1 , $j = 1,2,\ldots$, the sequence $\{\epsilon_j x_j\}_{j=1}^{\infty}$, is conditionally weakly summable.

PROPOSITION 1.3. *Any unconditionally weakly summable sequence of elements of a seminormed space is unconditionally summable.*

This proposition is known in the literature as the Orlicz-Pettis lemma. A special case of it appeared in the early work of W. Orlicz on trigonometric series. However, the first published proof for an arbitrary Banach space is due to B.J. Pettis, [57]. Several other proofs were invented since; see, for example, [9], Corollary 4.4 and the remarks on p.34, and [23], Lemma 3.2.1 and Theorem 3.2.3. It is a matter of a mere routine to weaken the assumptions so as to allow an arbitrary seminormed space.

We are now going to modify a classical lemma of H. Hahn, see e.g. [23], Theorem 2.7.7, about the construction of a continuous linear functional from its values on a subset of a Banach space. The modification consists in relaxing the assumptions on the functional if the norm of the given Banach space satisfies a certain, rather stringent, condition. The condition says that it is the largest norm on the space with a given restriction on the given subset. So, the resulting proposition turns out to be rather trivial. However, it applies to the usual constructions of L^1-spaces, some of their generalizations, and to the projective tensor products of pairs of Banach spaces.

PROPOSITION 1.4. *Let* F *be a Banach space with the norm* q *and let* $Q \subset F$. *Assume that* $\mathrm{sim}(Q)$ *is dense in* F *and that, for every* $x \in \mathrm{sim}(Q)$, (B.2) $$q(x) = \inf \sum_{j=1}^{n} |c_j|\, q(x_j) ,$$

where the infimum is taken over all expressions of x *in the form* (B.1) *with arbitrary* $n = 1,2,\ldots,$ *numbers* c_j *and elements* $x_j \in Q$, $j = 1,2,\ldots,n$.

Let E *be a Banach space with the norm denoted as the modulus. Let* $\mu : Q \to E$ *be a linear map such that* $|\mu(x)| \le q(x)$, *for every* $x \in Q$.

Then there exists a unique linear map $\tilde{\mu} : F \to E$ *such that* $\tilde{\mu}(x) = \mu(x)$, *for every* $x \in Q$, *and* $|\tilde{\mu}(x)| \le q(x)$, *for every* $x \in F$.

Proof. Let $\mu_1 : \mathrm{sim}(Q) \to E$ be the unique linear extension of μ . Then

$$|\mu_1(x)| = \Big| \sum_{j=1}^{n} c_j \mu(x_j) \Big| \le \sum_{j=1}^{n} |c_j|\, |\mu(x_j) \le \sum_{j=1}^{n} |c_j|\, q(x_j)$$

for every $x \in \mathrm{sim}(Q)$ and every expression of x in the form (B.1). Consequently, by the assumption, $|\mu_1(x)| \le q(x)$, for every $x \in \mathrm{sim}(Q)$. So, there exists a unique linear map $\tilde{\mu} : F \to E$ such that $\tilde{\mu}(x) = \mu_1(x)$, for every $x \in \mathrm{sim}(Q)$, and $|\tilde{\mu}(x)| \le q(x)$, for every $x \in F$.

C. Let Ξ and Υ be any non-empty sets. Let $\Omega = \Xi \times \Upsilon$ be their Cartesian product. If f is a function on Ω with values in a given Banach space and $\xi \in \Xi$, then by $f(\xi,\cdot)$ is denoted the function on Υ whose value at any point $v \in \Upsilon$ is equal to $f(\xi,v)$. Similarly, for any given $v \in \Upsilon$, by $f(\cdot,v)$ is denoted the function on Ξ whose value at any $\xi \in \Xi$ is $f(\xi,v)$.

Now, let E , F and G be vector spaces. A map $b : E \times F \to G$ is said to be bilinear if, for every $x \in E$, the map $b(x,\cdot) : F \to G$ is linear and also, for every $y \in F$, the map $b(\cdot,y) : E \to G$ is linear. If G happens to be the space of scalars, we speak of a bilinear function.

Let $B(E,F)$ be the vector space of all bilinear functions on $E \times F$. Let $B^*(E,F)$ be its algebraic dual; that is, $B^*(E,F)$ is the vector space of all linear functions on $B(E,F)$.

For each $x \in E$ and $y \in F$, let $x \otimes y$ denote the linear function on $B(E,F)$ whose value at any element, b, of $B(E,F)$ is equal to $b(x,y)$. The map $(x,y) \mapsto x \otimes y$, $x \in E$, $y \in F$, is an injection of $E \times F$ into $B^*(E,F)$; it identifies $E \times F$ with a subset of $B^*(E,F)$ which we denote by Q. The vector space, $\mathrm{sim}(Q)$, spanned by Q is denoted by $E \otimes F$ and is called the tensor product of the spaces E and F. The map $(x,y) \mapsto x \otimes y$, $x \in E$, $y \in F$, is called the canonical bilinear map of $E \times F$ into $E \otimes F$.

It is immediate that (i) $c(x \otimes y) = (cx) \otimes y = x \otimes (cy)$, for any number c and vectors $x \in E$ and $y \in F$. Also (ii) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$, for any $x_1 \in E$, $x_2 \in E$ and $y \in F$; and, similarly (iii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$, for any $x \in E$ and $y_1 \in F$ and $y_2 \in F$. So, an element, z, of $B^*(E,F)$ belongs to $E \otimes F$ if and only if there is an integer $n = 1,2,\ldots$ and vectors $x_j \in E$ and $y_j \in F$, $j = 1,2,\ldots,n$, such that

$$z = \sum_{j=1}^{n} x_j \otimes y_j . \tag{C.1}$$

Alternatively, the tensor product, $E \otimes F$, of the vector spaces E and F can be defined as the set of all formal linear combinations of the products $x \otimes y$, with $x \in E$ and $y \in F$, reduced so that the identities (i), (ii) and (iii) hold. More precisely, we define V to be the vector space whose basis is $E \times F$ and V_0 to be the subspace of V spanned by the elements of the form $(0,y)$, $(x,0)$, $(x_1+x_2,y) - (x_1,y) - (x_2,y)$, $(x,y_1+y_2) - (x,y_1) - (x,y_2)$, $(cx,y) - c(x,y)$ and $(x,cy) - c(x,y)$, with an arbitrary number c, vectors x, x_1 and x_2 in E and vectors y, y_1 and y_2 in F. Then the space $E \otimes F$ is isomorphic (as a vector space) with the quotient space V/V_0 under the linear map that associates any element $x \otimes y$ of $E \otimes F$, $x \in E$, $y \in F$, with the element $(x,y) + V_0$ of the space V/V_0.

Assume now that E and F are normed spaces with norms p and q, respectively. Let the norm r on $E \otimes F$ be defined by

$$r(z) = \inf \sum_{j=1}^{n} p(x_j) q(y_j) ,$$

for every $z \in E \otimes F$, where the infimum is taken over all expressions of z in the form

(C.1) with arbitrary $n = 1,2,\ldots,$ $x_j \in E$ and $y_j \in F$, $j = 1,2,\ldots,n$. Clearly, $r(x \otimes y) = p(x)q(y)$, for every $x \in E$ and $y \in F$. In fact, r is the largest norm on $E \otimes F$ having this property.

By $E \hat{\otimes} F$ is denoted the completion of the space $E \otimes F$ in the norm r. The Banach space $E \hat{\otimes} F$ is called the (complete) projective tensor product of the normed spaces E and F.

PROPOSITION 1.5. *For every element, z, of the complete projective tensor product, $E \hat{\otimes} F$, of the spaces E and F, there exist elements, x_j, of the space E and elements, y_j, of the space F, $j = 1,2,\ldots,$ such that*

$$\sum_{j=1}^{\infty} p(x_j)q(y_j) < \infty \tag{C.2}$$

and

$$z = \sum_{j=1}^{\infty} x_j \otimes y_j . \tag{C.3}$$

Moreover, the norm of z in the space $E \hat{\otimes} F$ is equal to the infimum of the numbers (C.2) *subject to the expression of z in the form* (C.3).

Proof. It follows directly from Proposition 1.2.

Let now G be a Banach space with the norm denoted as modulus. A bilinear map $b : E \times F \to G$ is continuous if and only if there is a constant $k \geq 0$ such that

$$|b(x,y)| \leq kp(x)q(y) , \tag{C.4}$$

for every $x \in E$ and $y \in F$.

PROPOSITION 1.6. *If $b : E \times F \to G$ is a continuous bilinear map, then there exists a unique continuous linear map $\mu : E \hat{\otimes} F \to G$ such that $\mu(x \otimes y) = b(x,y)$, for every $x \in E$ and $y \in F$. Furthermore, if* (C.4) *holds for every $z \in E$ and $y \in F$, then $|\mu(z)| \leq kr(z)$, for every $z \in E \hat{\otimes} F$.*

Proof. It follows from Proposition 1.4.

This is all that will be needed in the sequel about tensor products. For further general facts and facts concerning the relation of tensor products with vector integration, the interested reader is referred to [9], Chapter VIII.

D. We say that $\mathcal{K}$ is a nontrivial family of functions on a space Ω if Ω is a nonempty set and $\mathcal{K}$ is a set of scalar valued functions whose domain is Ω such that the zero function belongs to $\mathcal{K}$.

Any such nontrivial family, $\mathcal{K}$, is considered to be a subset of the vector space of all scalar valued functions on Ω. So, the symbol $\mathrm{sim}(\mathcal{K})$ has an unambiguous meaning introduced in Section B; viz., it denotes the linear hull of $\mathcal{K}$. Functions belonging to $\mathrm{sim}(\mathcal{K})$ are called $\mathcal{K}$-simple.

Clearly, $\mathcal{K}$ is a vector space if and only if $\mathrm{sim}(\mathcal{K}) = \mathcal{K}$. If $\mathcal{K}$ is a vector space whose elements are real-valued and if, with every function $f \in \mathcal{K}$, also the function $|f|$, that is, the function $\omega \mapsto |f(\omega)|$, $\omega \in \Omega$, belongs to $\mathcal{K}$, then $\mathcal{K}$ is called a vector lattice.

The notion of a $\mathcal{K}$-simple function is extended so as to permit consideration of vector valued functions. Namely, let $\mathcal{K}$ be a nontrivial family of functions on a space Ω and let E be a Banach space. By $\mathrm{sim}(\mathcal{K},E)$ is denoted the vector space spanned by all the E-valued functions cf, where $c \in E$ and $f \in K$. That is to say, $\mathrm{sim}(\mathcal{K},E)$ consists of all functions $f : \Omega \to E$ for which there exist a positive integer n, elements c_j of E and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,n$, such that

$$f = \sum_{j=1}^{n} c_j f_j .$$

Functions belonging to $\mathrm{sim}(\mathcal{K},E)$ are called $(\mathcal{K},E)$-simple.

To save subscripts and circumlocution, subsets of Ω will be identified with their characteristic functions. Accordingly, a family, $\mathcal{Q}$, of subsets of Ω is called a paving in Ω if it is a nontrivial family of functions on Ω, that is, characteristic functions of sets from $\mathcal{Q}$ a nontrivial family of functions on Ω. So, a family of subsets of Ω is a paving in Ω if it contains the empty set.

The paving $\mathcal{Q}$ is said to be multiplicative if it contains the intersection of any two of its members.

The paving $\mathcal{Q}$ in Ω is called a quasiring of sets in the space Ω if, for any sets X and Y belonging to $\mathcal{Q}$, the intersection $X \cap Y$ is equal to the union of a finite collection of pair-wise disjoint sets belonging to $\mathcal{Q}$ and also the difference $Y \setminus X$ is equal to the union of a finite collection of pair-wise disjoint sets from $\mathcal{Q}$.

The paving $\mathcal{Q}$ in Ω is called a semiring of sets in the space Ω if, for every $X \in \mathcal{Q}$ and $Y \in \mathcal{Q}$, there exist a positive integer n and pair-wise disjoint sets $Z_j \in \mathcal{Q}$, $j = 0,1,\ldots,n$, such that

$$X \cap Y = Z_0 \,, \; Y \setminus X = \bigcup_{j=1}^{n} Z_j$$

and the union

$$\bigcup_{j=0}^{k} Z_j$$

belongs to $\mathcal{Q}$, for every $k = 0,1,\ldots,n$. The notion of a semiring is due to J. von Neumann who uses the term half-ring; see [55], Definition 10.1.5. The importance of semirings will become apparent in the next section; cf., in particular, Proposition 1.9.

Every semiring is a quasiring, but it is not difficult to exhibit quasirings which are not semirings.

A quasiring of sets in Ω which contains the union of any finite collection of its members is called a ring of sets in the space Ω. A ring of sets which contains the union of any sequence of its members is called a σ-ring. A ring of sets which contains the intersection of any sequence of its members is called a δ-ring. A ring (quasiring, semiring, σ-ring) of sets in Ω which contains Ω as one of its members is called an algebra (quasialgebra, semialgebra, σ-algebra, respectively) of sets in the space Ω.

By a σ-ideal in the space Ω we understand a family of subsets of Ω that is closed under taking countable unions and subsets, that is it contains all the subsets of the union of any sequence of its elements. A family of sets with this property is in fact a σ-ideal of the Boolean algebra of all subsets of Ω ; so, this terminology represents a slight abuse of the language.

If $\mathcal{S}$ σ-algebra of sets in the space Ω , a function, f, on Ω is said to be $\mathcal{S}$-measurable, if every set of the form $\{\omega : f(\omega) \in U\}$, where U is an open set of scalars, belongs to $\mathcal{S}$.

The least σ-algebra of sets in a topological space Ω that contains all open sets is called the Borel σ-algebra in Ω ; its elements are called Borel sets. The least σ-algebra of sets in a topological space Ω that contains all sets of the form $\{\omega \in \Omega : f(\omega) \in U\}$, where f a real valued continuous function on Ω with compact support and U an open subset of $\mathbb{R}$, is called the Baire σ-algebra of sets in Ω ; its elements are called Baire sets.

If $\mathcal{Q}$ is a paving in the space Ω and $\Upsilon \subset \Omega$, $\Upsilon \neq \emptyset$, then the family $\mathcal{Q} \cap \Upsilon = \{X \cap \Upsilon : X \in \mathcal{Q}\}$ is a paving in the space Υ . If $\mathcal{Q}$ is a quasiring then so is $\mathcal{Q} \cap \Upsilon$. Similarly for a semiring, ring, algebra, σ-ring, δ-ring and σ-algebra.

If $\mathcal{Q}$ is a quasiring of sets in the space Ω then every $\mathcal{Q}$-simple function has an expression

$$f = \sum_{j=1}^{n} c_j X_j , \tag{D.1}$$

where the n is a positive integer, the c_j are numbers and the X_j are pair-wise disjoint sets belonging to $\mathcal{Q}$, $j = 1,2,\ldots,n$. The family, $\mathcal{R}$, of all sets belonging to $\operatorname{sim}(\mathcal{Q})$, that is, sets whose characteristic functions re $\mathcal{Q}$-simple, is the ring of sets generated by $\mathcal{Q}$. So, every element of $\mathcal{R}$ is equal to the union of a finite collection of pair-wise disjoint sets from $\mathcal{Q}$.

Let $\mathcal{Q}$ be an arbitrary paving in the space Ω . By $\Sigma(\mathcal{Q})$ will be denoted the set of all families of pair-wise disjoint non-empty sets belonging to $\mathcal{Q}$.

A family of sets, $\mathcal{P}$, belonging to $\Sigma(\mathcal{Q})$ is called a $\mathcal{Q}$-partition (of Ω), if the union of all sets that belong to $\mathcal{P}$ is equal to Ω and, for every $X \in \mathcal{Q}$, the sub-family,

$$\{Y \in \mathcal{P} : Y \cap X \neq \emptyset\} ,$$

of $\mathcal{P}$ consisting of sets having non-empty intersection with X, is finite. The set of all $\mathcal{Q}$-partitions is denoted by $\Pi(\mathcal{Q})$.

Let $\mathcal{P}_1 \in \Pi(\mathcal{Q})$ and $\mathcal{P}_2 \in \Pi(\mathcal{Q})$. If, for every set $Y \in \mathcal{P}_2$, there exists a (necessarily unique) set $X \in \mathcal{P}_1$ such that $Y \subset X$, we say that the partition $\mathcal{P}_2$ is a refinement of the partition $\mathcal{P}_1$ and write $\mathcal{P}_1 \prec \mathcal{P}_2$.

We say that a set $\Gamma \subset \Pi(\mathcal{Q})$ is directed (by the relation of refinement) if, for every $\mathcal{P}_1 \in \Gamma$ and $\mathcal{P}_2 \in \Gamma$, there exists a partition $\mathcal{P}_3 \in \Gamma$ such that $\mathcal{P}_1 \prec \mathcal{P}_3$ and $\mathcal{P}_2 \prec \mathcal{P}_3$.

If $\mathcal{Q}$ is a multiplicative quasiring, then the set, $\Pi(\mathcal{Q})$, of all partitions is directed.

If $\mathcal{Q}$ is an arbitrary paving and Γ is a directed subset of $\Pi(\mathcal{Q})$, then the paving

$$\mathcal{Q}_\Gamma = \{\emptyset\} \cup \bigcup_{\mathcal{P}\in\Gamma} \mathcal{P},$$

to which belong the empty set and all the sets forming the partitions belonging to Γ, is a multiplicative quasiring of sets.

E. Let E be a vector space.

If $\mathcal{K}$ is a nontrivial family of functions on a given space and $\mu : \mathcal{K} \to E$ a map, the question whether the map μ is linear or not has a meaning. Indeed, the notion of a linear map was introduced in Section B. If $\mathcal{K}$ satisfies some additional hypotheses, then it may be possible to simplify the condition of linearity. It is obviously so when $\mathcal{K}$ happens to be a vector space. Less obvious simplifications are possible for some kinds of pavings.

An E-valued map whose domain is a paving is usually called an E-valued set function. The real or complex valued set functions are referred to simply as set functions, and so are E-valued set functions whenever the space E is specified otherwise or irrelevant.

Let $\mathcal{Q}$ be a paving in a space Ω and $\mu : \mathcal{Q} \to E$ a set function. Let n be a positive integer. The set function μ is said to be n-additive if

$$\mu(X) = \sum_{j=1}^{n} \mu(X_j)$$

for any set $X \in \mathcal{Q}$ and pair-wise disjoint sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,n$, such that

$$X = \bigcup_{j=1}^{n} X_j .$$

If μ is n-additive, for every $n = 1,2,\ldots,$ we say that it is additive.

PROPOSITION 1.7. *If $\mathcal{Q}$ is a quasiring of sets, then a set function $\mu : \mathcal{Q} \to E$ is linear if and only if it is additive.*

Proof. Any linear set function is additive. So, let $\mathcal{Q}$ be a quasiring of sets and $\mu : \mathcal{Q} \to E$ an additive set function. If a function $f \in \mathrm{sim}(\mathcal{Q})$ is expressed in the form (D.1), let

$$\tilde{\mu}(f) = \sum_{j=1}^{n} c_j \mu(X_j) .$$

The additivity of μ implies that this definition is unambiguous. It is then straightforward that the resulting map $\tilde{\mu} : \mathrm{sim}(\mathcal{Q}) \to E$ is linear and that $\tilde{\mu}(X) = \mu(X)$ for every $X \in \mathcal{Q}$.

This proposition implies that, if $\mathcal{Q}$ is a quasiring of sets and $\mathcal{R}$ is the ring of sets generated by $\mathcal{Q}$, then any additive set function $\mu : \mathcal{Q} \to E$ has a unique additive extension on the whole of $\mathcal{R}$; that is, there exists a unique additive set function $\tilde{\mu} : \mathcal{R} \to E$ such that $\tilde{\mu}(X) = \mu(X)$, for $X \in \mathcal{Q}$.

If $\mathcal{Q}$ happens to be a semiring, then the condition of linearity can be simplified still further.

PROPOSITION 1.8. *If $\mathcal{Q}$ is a semiring of sets, then a set function $\mu : \mathcal{Q} \to E$ is additive if and only if it is 2-additive.*

Proof. Let $\mathcal{Q}$ be a semiring of sets and $\mu : \mathcal{Q} \to E$ a 2-additive set function. As μ is trivially 1-additive, for an inductive proof, assume that $k \geq 1$ is an integer and that μ is k-additive.

Let $X \in \mathcal{Q}$ and let $X_i \in \mathcal{Q}$, $i = 0,1,2,\ldots,k$, be pair-wise disjoint sets whose union is equal to X. By the definition of a semiring there exist a natural number m and pair-wise disjoint sets $Z_j \in \mathcal{Q}$, $j = 0,1,2,\ldots,m$, such that $Z_0 = X \cap X_0 = X_0$,

$$\bigcup_{i=1}^{k} X_i = X \setminus X_0 = \bigcup_{j=1}^{m} Z_j$$

and, for every $l = 0,1,2,\ldots,m$, the set

$$W_l = \bigcup_{j=0}^{l} Z_j$$

belongs to $\mathcal{Q}$. Then, clearly, $W_0 = X_0$, $W_l = W_{l-1} \cup Z_l$ and $W_{l-1} \cap Z_l = \emptyset$, for every $l = 1,2,\ldots,m$, and $W_m = X$.

Now, by the 2-additivity of μ, we have $\mu(W_l) = \mu(W_{l-1}) + \mu(Z_l)$, for every $l = 1,2,\ldots,m$. Therefore, $\mu(W_1) = \mu(W_0) + \mu(Z_1) = \mu(Z_0) + \mu(Z_1)$; $\mu(W_2) = \mu(W_1) + \mu(Z_2) = \mu(Z_0) + \mu(Z_1) + \mu(Z_2)$; and so on. Hence, by finite induction ending at $l = m$,

$$\text{(E.1)} \qquad \mu(X) = \mu(W_m) = \sum_{j=0}^{m} \mu(Z_j).$$

Furthermore, for any $i = 1,2,\ldots,k$, we have $X_i \cap W_0 = X_i \cap X_0 = \emptyset$, $X_i \cap W_l = (X_i \cap W_{l-1}) \cup (X_i \cap Z_l)$ and $(X_i \cap W_{l-1}) \cap (X_i \cap Z_l) = \emptyset$, for every $l = 1,2,\ldots,m-1$, and $X_i \cap W_m = X_i \cap X = X_i$. Therefore, $\mu(X_i \cap W_l) = \mu(X_i \cap W_{l-1}) + \mu(X_i \cap Z_l)$, for every $l = 1,2,\ldots,m$, and, hence, by finite induction,

$$\text{(E.2)} \qquad \mu(X_i) = \mu(X_i \cap W_m) = \sum_{j=1}^{m} \mu(X_i \cap Z_j)$$

for every $i = 1,2,\ldots,k$.

On the other hand,

$$Z_j = \bigcup_{i=1}^{k} (X_i \cap Z_j)$$

for every $j = 1,2,\ldots,k$, and the sets $X_i \cap Z_j$, $i = 1,2,\ldots,k$, are pair-wise disjoint. Hence,

$$\text{(E.3)} \qquad \mu(Z_j) = \sum_{i=1}^{k} \mu(X_i \cap Z_j)\,,$$

for every $j = 1,2,\ldots,m$, because, by the assumption, the set function μ is k-additive.

So, by (E.1), (E.2) and (E.3),

$$\mu(X) = \sum_{j=0}^{m} \mu(Z_j) = \mu(Z_0) + \sum_{j=1}^{m} \mu(Z_j) =$$

$$= \mu(X_0) + \sum_{j=1}^{m} \sum_{i=1}^{k} \mu(X_i \cap Z_j) = \mu(X_0) + \sum_{i=1}^{k} \sum_{j=1}^{m} \mu(X_i \cap Z_j) =$$

$$= \mu(X_0) + \sum_{i=1}^{k} \mu(X_i) = \sum_{i=0}^{k} \mu(X_i)\,.$$

That is, μ is $(k+1)$-additive.

It may be interesting to note that this proposition does not hold for quasirings instead of semirings.

EXAMPLE 1.9. Let $\Omega = \{1,2,3\}$ and let $\mathcal{Q} = \{\emptyset,\{1\},\{2\},\{3\},\Omega\}$. Then $\mathcal{Q}$ is a quasiring of sets in the space Ω. Let $\mu(\emptyset) = 0$, $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\Omega) = 1$. Then obviously, $\mu(X) = \mu(Y) + \mu(Z)$, for any sets X, Y and Z belonging to $\mathcal{Q}$, such that $Y \cap Z = \emptyset$ and $X = Y \cup Z$. However, μ is not additive.

The surprisingly nontrivial Proposition 1.8 expresses a property of semirings that makes them preferable to quasirings. It is due to J. von Neumann, [55], Theorem 10.1.12; see also [19], Exercise 5 in §7. However, some naturally occurring pavings in torus-like spaces are only quasirings.

F. Let $\mathcal{Q}$ be a paving in a space Ω. Let E be a normed vector space. A set function $\mu : \mathcal{Q} \to E$ is said to be σ-additive if

$$\mu(X) = \sum_{j=1}^{\infty} \mu(X_j)$$

for any set $X \in \mathcal{Q}$ and pair-wise disjoint sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$, such that

$$X = \bigcup_{j=1}^{\infty} X_j .$$

PROPOSITION 1.10. *Let $\mathcal{Q}$ be a quasiring of sets and $\mathcal{R}$ the ring generated by $\mathcal{Q}$. Let $\mu : \mathcal{Q} \to E$ be an additive set function and $\tilde{\mu} : \mathcal{R} \to E$ its additive extension.*

The set function $\tilde{\mu}$ is σ-additive if and only if μ is σ-additive.

Proof. It follows directly from the fact that every set in $\mathcal{R}$ can be written as the union of a finite collection of a pair-wise disjoint sets from $\mathcal{Q}$.

Demonstration of the σ-additivity of a given set function may not be a simple matter, not even if the set function is scalar valued. In fact, the problem of σ-additivity of vector valued set functions is often reduced, *via* the Orlicz-Pettis lemma, say, to the problem of σ-additivity of some scalar valued set functions and even positive real valued ones. The basic source of positive σ-additive set functions is the theorem of A.D. Alexandrov; see [14], Theorem III.5.13 and the remarks in Section III.15 (p.233), and also [55], Theorem 10.1.20. Because of its importance, we present here an elementary proof of an extended and, at the same time, simplified version of this theorem.

A paving $\mathcal{C}$ is called compact if

(F.1) $$\bigcap_{n=1}^{\infty} C_n \neq \emptyset$$

for any sets $C_n \in \mathcal{C}$, $n = 1,2,\ldots$, such that

(F.2) $$\bigcap_{n=1}^{k} C_n \neq \emptyset ,$$

for every $k = 1,2,\ldots$.

More appropriately, instead of "compact", we should have used - as some authors actually do - the term "semicompact" or "sequentially compact". The proof of the following lemma is taken from [56], Lemma I.6.1.

LEMMA 1.11. *Let C be a compact paving. Let $\mathcal{D}$ be a paving whose elements are the unions of finite collections of sets from C. Then the paving $\mathcal{D}$ too is compact.*

Proof. Let $D_n \in \mathcal{D}$, $n = 1,2,\ldots$, be sets such that

$$\bigcap_{n=1}^{k} D_n \neq \emptyset \tag{F.3}$$

for every $k = 1,2,\ldots$. The proof will be accomplished if we show that the intersection of all the sets D_n, $n = 1,2,..$, is not empty.

For every $n = 1,2,\ldots$, let m_n be a natural number and C_n^j, $j = 1,2,\ldots,m_n$, sets from C such that

$$D_n = \bigcup_{j=1}^{m_n} C_n^j .$$

Let $M_n = \{1,2,\ldots,m_n\}$, for every $n = 1,2,\ldots$. Let J be the set of all sequences $\iota = \{\iota_n\}_{n=1}^{\infty}$ such that $\iota_n \in M_n$ for every $n = 1,2,\ldots$. Finally, for every $k = 1,2,\ldots$, let J_k be the set of all sequences $\iota \in J$ such that

$$\bigcap_{n=1}^{k} C_n^{\iota_n} \neq \emptyset . \tag{F.4}$$

It then follows immediately that,

(i) if $\iota \in J_k$, $\kappa \in J$ and $\kappa_n = \iota_n$, for every $n = 1,2,\ldots,k$, then $\kappa \in J_k$;

(ii) if p and q are natural numbers such that $p \leq q$, then $J_q \subset J_p$.

Moreover, by the distributive law,

$$\bigcap_{n=1}^{k} D_n = \bigcup_{\iota \in J} \left[\bigcap_{n=1}^{k} C_n^{\iota_n} \right] .$$

Therefore, by (F.3), (F.4) holds for at least one $\iota \in J$. So,

(iii) $J_k \neq \emptyset$ for every $k = 1,2,\ldots$.

Our next aim is to prove that there exists a sequence $\iota \in J$ which belongs to J_k for every $k = 1,2,\ldots$. Such a sequence is constructed inductively.

First, using (iii), for every $k = 1,2,\ldots,$ we fix an element ι^k of the set J_k.

(I) Because the first terms, ι_1^k, of the sequences ι^k, $k = 1,2,\ldots,$ all belong to the finite set M_1, there exist an element ι_1 of M_1 such that the set, S_1, of all natural numbers k for which $\iota_1^k = \iota_1$, is infinite.

(II) Assume that p is a natural number and that for every $n = 1,2,\ldots,p$, ι_n is an element of M_n such that the set, S_p, of all natural numbers k such that $\iota_n = \iota_n^k$, for every $n = 1,2,\ldots,p$, is infinite. Because the $(p+1)$-st terms, ι_{p+1}^k, of the sequences ι^k, $k = 1,2,\ldots,$ belong to the finite set M_{p+1}, there exists an element ι_{p+1} of M_{p+1} such that the set S_{p+1}, of those elements k of the set S_p for which $\iota_{p+1} = \iota_{p+1}^k$, is infinite. Then $\iota_n = \iota_n^k$, for every $n = 1,2,\ldots,p$, $p+1$, whenever $k \in S_{p+1}$.

So, a sequence $\iota = \{\iota_n\}_{n=1}^{\infty}$ is constructed such that, for every $p = 1,2,\ldots,$ the set S_p of natural numbers k such that $\iota_n = \iota_n^k$, for every $n = 1,2,\ldots,p$, is infinite. Consequently, for every natural number p, there exists a natural number $q \geq p$ such that $\iota_n = \iota_n^q$, for every $n = 1,2,\ldots,p$. But then, by (ii), $\iota^q \in J_p$. Hence, by (i), $\iota \in J_p$. Because the constructed sequence, ι, belongs to J_k, (F.4) holds for every $k = 1,2,\ldots$. Consequently,

$$\bigcap_{n=1}^{\infty} C_n^{\iota_n} \neq \emptyset ,$$

because the paving $\mathcal{C}$ is compact, and the intersection of the sets D_n, $n = 1,2,\ldots,$ cannot be empty either.

Let μ be a non-negative real valued additive set function on $\mathcal{Q}$ and $\mathcal{C}$ a paving in Ω. The set function μ is said to be $\mathcal{C}$-regular if, for every $X \in \mathcal{Q}$ and every $\epsilon > 0$, there exist a set $C \in \mathcal{C}$ and a set $Y \in \mathcal{Q}$ such that

$$Y \subset C \subset X \text{ and } \mu(X) - \mu(Y) < \epsilon .$$

PROPOSITION 1.12. *Let $\mathcal{Q}$ be a quasiring of sets and $\mathcal{C}$ a compact paving in the space Ω. Any $\mathcal{C}$-regular non-negative real valued additive set function on $\mathcal{Q}$ is σ-additive.*

Proof. Let μ be such a set function. Without a loss of generality we will assume that $\mathcal{Q}$ is a ring of sets. For, if it is not the case, let $\tilde{\mu}$ be the additive extension of μ on the ring, $\mathcal{R}$, generated by $\mathcal{Q}$, and $\mathcal{D}$ the paving consisting of the unions of all finite collections of sets from $\mathcal{C}$. Then $\tilde{\mu}$ is, obviously, $\mathcal{D}$-regular, because every set from $\mathcal{R}$ is the union of a finite collection of sets from $\mathcal{Q}$, and, by Lemma 1.11, the paving $\mathcal{D}$ is compact.

So, let $X_n \in \mathcal{Q}$ be sets such that $X_n \supset X_{n+1}$ and $\mu(X_n) \geq \alpha$, for some $\alpha > 0$, $n = 1,2,\ldots$. Let $C_n \in \mathcal{C}$ and $Y_n \in \mathcal{Q}$ be sets such that

$$Y_n \subset C_n \subset X_n \text{ and } \mu(X_n) - \mu(Y_n) < 2^{-n}\alpha,$$

$n = 1,2,\ldots$. Let

$$Z_k = \bigcap_{n=1}^{k} Y_n,$$

for every $k = 1,2,\ldots$. Then, by the assumption that $\mathcal{Q}$ is a ring, $Z_k \in \mathcal{Q}$, and

$$\mu(X_k) - \mu(Z_k) \leq \sum_{n=1}^{k} (\mu(X_n) - \mu(Y_n)) < \alpha,$$

so that $Z_k \neq \emptyset$ and (F.2) holds for every $k = 1,2,\ldots$. By the compactness of $\mathcal{C}$, (F.1) holds, and, consequently,

$$\bigcap_{n=1}^{\infty} X_n \neq \emptyset,$$

which implies the σ-additivity of μ, because $\mathcal{Q}$ is a ring of sets.

G. By a Young function we shall understand a real valued function, Φ, on the interval $[0,\infty)$ that is continuous, strictly increasing and convex and satisfies the conditions

$$\lim_{t \to 0+} \frac{\Phi(t)}{t} = 0 \text{ and } \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.$$

It follows that $\Phi(0) = 0$ and $\Phi(t) > 0$ for $t > 0$.

Proofs of the following two propositions can be found in [38], I.1.5 and I.2.2, respectively.

PROPOSITION 1.13. *A function, Φ, on $[0,\infty)$ is a Young function if and only if there exists a non-decreasing function, φ, on $[0,\infty)$ such that $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$, $\varphi(s) \to \infty$ as $s \to \infty$, and*

$$\Phi(t) = \int_0^t \varphi(s) ds , \tag{G.1}$$

for every $t \geq 0$. Moreover, if φ is right-continuous at every point of the interval $[0,\infty)$, then it is unique.

The Young function, Φ, is said to satisfy condition (Δ_2) for large values of the argument if there exist numbers $k > 0$ and $a \geq 0$ such that

$$\Phi(2t) \leq k\Phi(t) , \tag{G.2}$$

for every $t \in [a,\infty)$.

The Young function, Φ, is said to satisfy condition (Δ_2) for small values of the argument if there exist numbers $k > 0$ and $a > 0$ such that (G.2) holds for every $t \in [0,a]$.

If a Young function satisfies condition (Δ_2) for small and also for large values of the argument, we say that it satisfies condition (Δ_2).

Let Φ be a Young function. The function Ψ defined by

$$\Psi(t) = \sup\{st - \Phi(s) : s \geq 0\} ,$$

for every $t \geq 0$, that is, the Legendre transform of Φ, is called the function complementary to Φ.

PROPOSITION 1.14. *Let Φ be a Young function and let φ be the right-continuous function in $[0,\infty)$ such that* (G.1) *holds for every $t \geq 0$. Let*

$$\psi(t) = \sup\{s : \varphi(s) \leq t\}$$

for every $t \in [0,\infty)$. Then the function Ψ, complementary to Φ, is given by

$$\Psi(t) = \int_0^t \psi(s)\mathrm{d}s \,,$$

for every $t \geq 0$.

The function Ψ , complementary to a Young function, Φ , is again a Young function and the function complementary to Ψ is Φ . If Φ and Ψ is a pair of mutually complementary Young functions, then the inequality, called the Young inequality,

$$st \leq \Phi(s) + \Psi(t)$$

holds for every $s \geq 0$ and $t \geq 0$.

Given a Young function, Φ , and an integer $n \geq 1$, let

$$M_\Phi(x) = \sum_{j=1}^{n} \Phi(|x_j|) \,,$$

for every vector $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{C}^n$.

The following proposition is known; its proof can be found, for example, in [51], 3.32. It is of course a special case of an inequality valid in general Orlicz spaces. (See Section 3C below.)

PROPOSITION 1.15. *For every vector* $x \in \mathbb{C}^n$, *let*

$$\|x\|_\Phi = \inf\{k : k>0 \,,\, M_\Phi(k^{-1}x) \leq 1\}$$

and, if $x = (x_1, x_2, \ldots, x_n)$,

$$\|x\|_\Phi^0 = \sup\{|\sum_{j=1}^{n} x_j y_j| : y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n \,,\, M_\Psi(y) \leq 1\} \,,$$

where Ψ *is the function complementary to* Φ .

Then the functions $x \mapsto \|x\|_\Phi$ *and* $x \mapsto \|x\|_\Phi^0$, $x \in \mathbb{C}^n$, *are norms on* $\mathbb{C}^n$, *each making of* $\mathbb{C}^n$ *a Banach space, such that*

$$\|x\|_\Phi \leq \|x\|_\Phi^0 \leq 2\|x\|_\Phi \,,$$

for every $x \in \mathbb{C}^n$.