A SURVEY OF THE WEYL METRICS

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1. INTRODUCTION

Hermann Weyl derived his class of metrics in 1917, just one year after Einstein had presented his now-celebrated general theory of relativity to the world. With the passing of some seventy years since that initial flurry, it is perhaps time to pause and assess what progress has been made with respect to the Weyl metrics. Exactly what is known about them, and what still remains to be done?

From about twenty-five years ago, an interest in the Weyl metrics developed, particularly as exterior solutions in astrophysical problems [1] and as possible final states of gravitational collapse [2],[3]. However, in addition to being of relevance to physics, they are also of interest simply because they present us with the rare opportunity of explicitly determining and investigating a large class of relativistic metrics.

The Weyl metrics are, in principle, all 'known' since there exists a precise algorithm for generating them from an infinite set of Newtonian potential functions. This procedure is given in Section 2. In practice, however, the global structure of only a few such solutions is well understood, and it seems that much work and new insights will be required if this situation is to change.

The member of the Weyl class which is simplest to obtain is the Curzon metric (see Section 2). Yet, despite the ease with which it is generated, its source structure and global structure remained a mystery until the papers by Scott & Szekeres [4],[5] (see also Scott [6]) appeared in 1986. Due to space considerations, their findings will not be summarised here, but nonetheless form an integral part of this subject.

Since the Schwarzschild solution belongs to the Weyl class, the question naturally arises as to how it is generated. This question is investigated and answered in Section 3, and ultimately, of course, involves a change from Weyl coordinates to the standard Schwarzschild coordinates. Finding the relationship between the two coordinate systems is facilitated by a consideration of the general form of gravitational equipotentials of the Weyl metrics.

The Schwarzschild solution is a special member of the subclass of the Weyl metrics known as the Zipoy-Voorhees metrics. These metrics form the main focus of this survey and are discussed in Sections 4,5 & 6. In Section 4, the metrics are specified and new coordinates, more suited to their geometry, are chosen to replace the original Weyl coordinates. The problem of finding sources for these metrics is discussed in Section 5, and the possibility of performing extensions is considered in Section 6.

Some general properties of the Weyl metrics are given in Section 7, and in particular, the relationship between an arbitrary Weyl metric and its generating Newtonian potential is examined. The question of how flat space is generated within this framework is fully investigated, and at the end of this section, there is a list of some related open problems.

In Section 8, a brief history of the static two-body problem of general relativity is presented, including the early controversy over the two-particle Curzon solution, as well as some much more recent developments. Section 9 gives a short description of a new mathematical approach to the stationary, axisymmetric, vacuum space-times. It is hoped that this approach will eventually offer new insight into some of the unanswered questions related to the Weyl metrics.

The survey concludes with Section 10, which consists of a small but new observation by the author regarding ring singularities occurring in Weyl metrics. Before proceeding, it only remains to point out that the aim of this survey was to be as comprehensive as possible within the given space constraints. There are, of course, certain omissions, for which the author apologizes in advance.

2. THE WEYL METRICS

Using cylindrical coordinates (r,z,φ) where $r \geqslant 0$, $z \in \mathbb{R}$ and $0 \leqslant \varphi < 2\pi$ (with $\varphi = 0$ and $\varphi = 2\pi$ identified), the static, axisymmetric, vacuum solutions of Einstein's field equations are given by the Weyl metrics [7],[8] (see also Synge [9])

$$ds^2 = -\,e^{2\lambda}\,dt^2 + e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2\,e^{-2\lambda}\,d\varphi^2 \tag{2.1}$$

where $\lambda(r,z)$ and $\nu(r,z)$ are solutions of the equations

$$\lambda_{rr} + \lambda_{zz} + r^{-1}\lambda_r = 0 \tag{2.2}$$

and

$$\nu_r = r(\lambda_r{}^2 - \lambda_z{}^2)\ , \qquad \nu_z = 2r\lambda_r\lambda_z\ . \tag{2.3}$$

If a solution λ of (2.2) is found, then (2.3) can be integrated to find ν. In fact, (2.2) is recognized as being simply the Laplace equation in cylindrical coordinates for a φ-independent function. We thus have a straightforward method of obtaining static, axisymmetric, vacuum, general relativistic fields. Namely choose an appropriate Newtonian gravitational field, and then integrate (2.3).

An obvious choice is the gravitational field produced by a spherically symmetric mass distribution with total mass m, which is located at the origin of the cylindrical coordinate system. So

(2.4) $$\lambda = -\, m/R \quad \text{where} \quad R = \sqrt{r^2 + z^2}$$

and

(2.5) $$\nu = -\, m^2 r^2 / \, 2R^4 \quad .$$

This is the so-called Curzon metric [10]. Although generated by the Newtonian mass monopole, it is not equivalent to the Schwarzschild metric, which as is well-known (Birkhoff's Theorem [11]), is the unique spherically symmetric, vacuum solution of general relativity.

3. THE SCHWARZSCHILD SOLUTION

The Schwarzschild solution is, in fact, generated by the Newtonian potential of a constant density line mass (or rod) with total mass m and length 2m, which is located along the z-axis with mid-point at the origin. So for this important example we have

(3.1) $$\lambda = 1/2 \ln \frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m}$$

where

(3.2) $$R_1 = \left[r^2 + (z - m)^2\right]^{1/2} \quad \text{and} \quad R_2 = \left[r^2 + (z + m)^2\right]^{1/2}$$

and

(3.3) $$\nu = 1/2 \ln \frac{(R_1 + R_2)^2 - 4m^2}{4\, R_1 R_2} \quad .$$

Naïvely, one might have been tempted to make the simple coordinate transformation

(3.4) $$R = \sqrt{r^2 + z^2} \,, \qquad \tan \theta = r/z$$

from the cylindrical system (r,z,φ) used by Weyl (Weyl coordinates), to a spherical system (R,θ,φ). However, this will **not** cast the metric into the familiar Schwarzschild form, since under such a coordinate transformation the rod maps to the portion of axis specified by $\theta = 0$, $0 \leqslant R \leqslant m$ and $\theta = \pi$, $0 \leqslant R \leqslant m$. So instead of producing the customary point mass, the line mass persists.

In fact, any coordinate system which is one-to-one with the cartesian system (x,y,z) on an open neighbourhood of the rod, can be ruled out for the same reason. A different type of coordinate transformation is needed here, and the key to finding it lies in the following observation.

For the Weyl metrics (2.1), (2.2) & (2.3), the 3-metric ${}^3g_{\alpha\beta}$, ${}^3g^{\alpha\beta}$ $(\alpha,\beta = 1,2,3)$ induced on the hypersurface t = constant is given by

(3.5) $$ {}^3g_{\alpha\beta}dx^\alpha dx^\beta = e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2\,e^{-2\lambda}\,d\varphi^2 $$

where $x^1 = r$, $x^2 = z$, $x^3 = \varphi$. Now it can be shown that

(3.6) $$ {}^3g^{\alpha\beta}\; e^{\lambda}{}_{|\alpha\beta} \;=\; {}^3g^{\alpha\beta}\; {}^3\nabla_\beta {}^3\nabla_\alpha\, e^{\lambda} \;=\; 0 \quad . $$

Thus e^λ is an analogue of the Newtonian potential λ, and the surfaces on which it is constant may be thought of as gravitational equipotentials.

For the Schwarzschild potential λ given by (3.1),

$$ e^\lambda = \text{constant} $$

$$ \Rightarrow \quad R_1 + R_2 = c \text{ (a constant)} \quad . $$

These gravitational equipotentials are, of course, just the 2-surfaces ρ = constant, where ρ is the radial coordinate normally used for the Schwarzschild metric. From the metric component

$$ g_{oo} = -\,e^{2\lambda} \,, $$

the function p(c) is readily determined to be

$$p(c) = 1/2\ (c + 2m)\ ,$$

and with a little more effort, the coordinate transformation

$$\rho = 1/2\ (R_1 + R_2 + 2m)$$

(3.7)

$$\cos\theta = 1/2m\ (R_2 - R_1)$$

is found to be the one which casts the Schwarzschild metric into its familiar form

(3.8)

$$ds^2 = -\ (1 - 2m/\rho)\ dt^2 + (1 - 2m/\rho)^{-1}d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\ d\varphi^2)\ .$$

We note that this transformation from (t,r,z,φ) coordinates to (t,ρ,θ,φ) coordinates, is a one-to-one mapping of the entire region surrounding (but not including) the line mass, onto the **exterior** Schwarzschild solution $\rho > 2m$. This is not really very surprising, since the Weyl metrics are **static**.

4. THE ZIPOY-VOORHEES METRICS

The Schwarzschild solution falls naturally into the subclass of Weyl metrics generated by the Newtonian potential of a constant density line mass (or rod) with total mass m and length 2ℓ, which is located along the z-axis with mid-point at the origin. So we have

(4.1)
$$\lambda = 1/2\ m/\ell\ \ln \frac{R_1 + R_2 - 2\ell}{R_1 + R_2 + 2\ell}$$

where

(4.2)
$$R_1 = \left[r^2 + (z - \ell)^2\right]^{1/2} \quad \text{and} \quad R_2 = \left[r^2 + (z + \ell)^2\right]^{1/2}$$

and

(4.3)
$$\nu = 1/2\ (m/\ell)^2\ \ln \frac{(R_1 + R_2)^2 - 4\ell^2}{4\ R_1R_2}\ .$$

This metric was first derived by Bach and Weyl [12], and is occasionally referred to as simply 'the metric of Bach and Weyl'. However, it has since been discussed and investigated to a varying extent by numerous authors [13],[14],[15],[16],[17],[18],[19], and is more commonly referred to as the Voorhees metric or the Zipoy-Voorhees metric after two of them.

In fact the papers of Zipoy [14] and Voorhees [18] are particularly interesting, and warrant some further discussion here. There is a common philosophy underpinning both, namely that the coordinate system chosen to express a particular Weyl metric (λ,ν), should be adapted to the symmetries of the source (or mass distribution) giving rise to the Newtonian potential λ. So for the line metrics (4.1), (4.2) & (4.3), an obvious choice is the prolate spheroidal coordinate system (u,θ) defined implicitly by

$$r = \ell \sinh u \cos\theta \qquad z = \ell \cosh u \sin\theta \quad , \tag{4.4}$$

where $u \geqslant 0$ and $-\pi/2 \leqslant \theta \leqslant \pi/2$.

If we further define the coordinate x by $x = \cosh u$, where $x \geqslant 1$, then the coordinates (x,θ) form an orthogonal system whose level curves x = constant and θ = constant are confocal ellipses and hyperbolas respectively, with foci at $r = 0$, $z = +\ell, -\ell$ $(x = 1, \theta = +\pi/2, -\pi/2)$. These coordinates are illustrated in Figure 1.

If we further define the coordinate ρ by $\rho = \ell x$, where $\rho \geqslant \ell$, then in (t,ρ,θ,φ) coordinates the metric becomes

$$\begin{aligned} ds^2 = &- e^{2\lambda}\, dt^2 + e^{2(\nu-\lambda)}(\rho^2 - \ell^2\sin^2\theta)\left[d\rho^2/(\rho^2 - \ell^2) + d\theta^2\right] \\ &+ e^{-2\lambda}\,(\rho^2 - \ell^2)\cos^2\theta\, d\varphi^2 \end{aligned} \tag{4.5}$$

where

$$(4.6) \qquad \lambda = 1/2\ m/\ell\ \ln \frac{\rho - \ell}{\rho + \ell}$$

and

$$(4.7) \qquad \nu = 1/2\ (m/\ell)^2\ \ln \frac{\rho^2 - \ell^2}{\rho^2 - \ell^2 \sin^2\theta} \quad .$$

The gravitational equipotentials e^{λ} = constant, now have the particularly simple form ρ = constant $(\rho > \ell)$, confirming that prolate spheroidal coordinates are indeed well suited to the given source. For the Schwarzschild solution $(\ell = m)$, the metric assumes its usual form (3.8) by a straightforward change into $(t,\rho',\theta',\varphi)$ coordinates, where

$$(4.8) \qquad \rho' = \rho + m \quad \text{and} \quad \theta' = \pi/2 - \theta \quad .$$

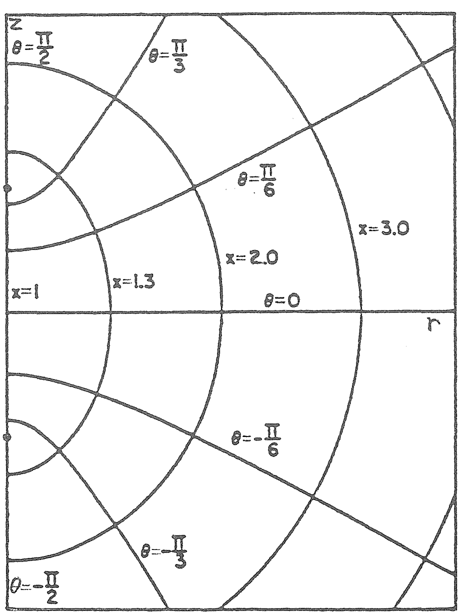

FIGURE 1. A graph showing the relationship between cylindrical coordinates (r,z) and prolate spheroidal coordinates (x,θ).

5. POSSIBLE SOURCES FOR THE ZIPOY-VOORHEES METRICS

By examining the behaviour of a particular invariant of the Riemann tensor as $x \to 1^+$, Zipoy concludes that in all but the Schwarzschild case, $x = 1$ is comprised of curvature singularities. However, this conclusion is slightly incorrect, since if $m/\ell > 2$, the invariant does in fact tend to **zero** as $x = 1$ is approached along either the positive z-axis or the negative z-axis (if $m/\ell = 2$, it tends to a finite, positive value).

The proper distance from $x_o > 1$ to $x = 1$ along the spacelike geodesics given by $\theta = 0$, t, φ constants, is found to be finite for all values of m/ℓ . Timelike geodesics given by $\theta = 0$, φ constant reach $x = 1$ in both finite coordinate time and finite proper time for all values of m/ℓ . However it is interesting to note that the circumference of the circles given by $\theta = 0$, t, x constants becomes infinite as $x \to 1^+$ for $m/\ell > 1$, and zero for $m/\ell < 1$.

Voorhees proposed the following method for determining the **geometry** of the sources for these metrics. Assuming that all the rods are of equal mass m, but have varying length 2ℓ, it is possible to determine the relationship $\bar{x}(x,\theta)$, $\bar{\theta}(x,\theta)$ between the prolate spheroidal coordinates $(\bar{x},\bar{\theta})$ used for the Schwarzschild solution, and those (x,θ) used for the solution generated by the rod of length 2ℓ .

It is then a straightforward matter to find $\rho'(x,\theta)$, $\theta'(x,\theta)$, where (ρ',θ') are the standard Schwarzschild coordinates given by (4.8). Figure 2 shows how the rod $x = 1$ transforms under this change to Schwarzschild coordinates.

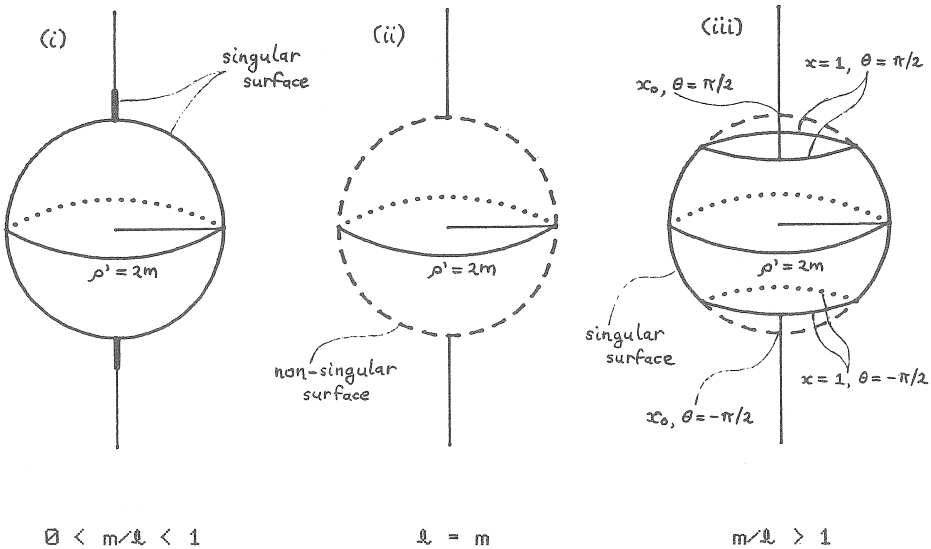

FIGURE 2. The rod of mass m and length 2ℓ $(x = 1)$ depicted in Schwarzschild coordinates (ρ', θ').

It is noted that for solutions with $m/\ell > 1$, the singular region $(x = 1)$ does not cover the entire surface $\rho' = 2m$, and indeed no curvature singularity is encountered along the axis of symmetry as $\rho' \to 2m^+$. However, consider the spacelike geodesic which in (x,θ) coordinates is given by t = constant, $\theta = \pi/2$, and extends from $x = 1$, $\theta = \pi/2$ out to $x = +\infty$, $\theta = \pi/2$.

In Schwarzschild coordinates it lies along the axis of symmetry $\theta' = 0$, and extends from $\rho' = 2m$, $\theta' = 0$ out to $\rho' = +\infty$, $\theta' = 0$. But the point $(\rho' = 2m,\ \theta' = 0)$ corresponds to the point $(x_0 > 1,\ \theta = \pi/2)$ in (x,θ) coordinates. So what happens to the piece of geodesic lying between x_0 and $x = 1$?

The answer is, that it maps **onto** the cap which is missing from the top of the sphere $\rho' = 2m$ in Figure 2 (iii)! This is an undesirable feature, and since there are further problems associated with these source representations, we conclude that the method of Voorhees yields only a very rough approximation to the source structure. The true geometry of the sources for the Zipoy-Voorhees metrics remains an open problem.

6. POSSIBLE EXTENSIONS OF THE ZIPOY-VOORHEES METRICS

In a more recent paper by Papadopoulos, Stewart & Witten [20], it is pointed out that the Zipoy-Voorhees metrics form the static limit of the Tomimatsu-Sato family of solutions [21],[22]. It is also noted that apart from the Schwarzschild solution (ℓ = m), all metrics in the class are of Petrov type D on the axis of symmetry, and type I (or general) elsewhere. But perhaps the major revelation of the paper, concerns the 'north pole' $x = 1$, $\theta = \pi/2$ and 'south pole' $x = 1$, $\theta = -\pi/2$ in solutions with $m/\ell \geqslant 2$.

In keeping with the spirit of Zipoy and Voorhees, the metric is expressed in prolate spheroidal coordinates. Then using a complex null tetrad $(m,\bar{m},\ell,k)$, the Weyl tetrad components Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4 are calculated ($\Psi_1 = 0$ & $\Psi_3 = 0$). For solutions with $m/\ell \geqslant 2$, Ψ_0, Ψ_2 & Ψ_4 are infinite along $x = 1$, $-\pi/2 < \theta < \pi/2$, confirming that the rod $x = 1$ minus its endpoints (or poles) is indeed comprised of curvature singularities. However, the value of each of Ψ_0, Ψ_2 & Ψ_4 at the north and south poles, is found to vary according to the direction of approach to the pole.

The north and south poles are thus the locations of **directional singularities**. In an attempt to unwrap this directional behaviour, a polar-type coordinate system based on the north pole is introduced. However, the attempt is unsuccessful, because the coordinate transformation maps the pole to a point. To successfully unwrap the directionality, it will certainly be necessary to use a coordinate transformation mapping the pole to a higher-dimensional surface.

It can be shown that timelike geodesics lying along the axis of symmetry ($x > 1$, $\theta = \pi/2$) reach the north pole in finite proper time. Since no curvature singularity is encountered there, it is argued that an extension of the space-time is necessary. As a first step towards providing one, an extension of the <u>2-dimensional 'space-time'</u> spanned by the time coordinate t and the axis of symmetry ($x > 1$, $\theta = \pi/2$) is successfully performed.

If m/ℓ ($\geqslant 2$) is an integer, the extension is analytic. If $n < m/\ell < n + 1$, where n is an integer ($n \geqslant 2$), the extension is C^n – an analytic extension is not possible in such cases. An extension of the full 4-dimensional space-time through the north pole (or likewise the south pole), has yet to be found. It is clear however, that the ability to perform such an extension, will be intimately tied to the ability to unwrap the directionality which is present at the poles.

7. SOME GENERAL PROPERTIES OF THE WEYL METRICS

From the preceeding discussion of the Zipoy-Voorhees metrics, and our earlier comments regarding the Curzon and Schwarzschild metrics, it is apparent that:

In general, there is no correspondence between the geometry of the source for a Weyl metric, and the geometry of the Newtonian source from which it is generated.

Is it at least true then, that every Weyl metric (2.1) is generated by a <u>unique</u> Newtonian potential $\lambda(r,z)$? At a superficial level, the answer to this question is, of course, 'yes'. Since the Weyl metric coefficient g_{oo} is $-e^{2\lambda}$, it is clear that two different Newtonian potentials $\lambda_1(r,z)$ and $\lambda_2(r,z)$, will certainly generate Weyl metrics which look different.

There is a possibility however, that if the second Weyl metric is expressed in a different coordinate system $(\bar{t},\bar{r},\bar{z},\bar{\varphi})$, it could assume the same form as the first metric still expressed in Weyl coordinates. That is, different λ_1 and λ_2 might generate the same metric simply expressed in different coordinates. But does this actually happen in practice ?

The answer lies in an interesting paper by Gautreau & Hoffman [23]. They set themselves the task of finding all Newtonian potentials $\lambda(r,z)$ which generate flat space. Obviously $\lambda = 0$ is one such potential, giving rise as it does to flat space expressed in cylindrical coordinates

(7.1) $$ds^2 = -dt^2 + dr^2 + dz^2 + r^2 d\varphi^2 \quad .$$

Now the Newtonian potential of a constant density line mass of infinite extent, lying along the entire z-axis, is given by

(7.2) $$\lambda = 2\sigma \ln r \ ,$$

where $\sigma > 0$ is the mass per unit length. If the Riemann tensor components are calculated for this subclass of the Weyl metrics, it is readily seen that they all vanish for the case $\sigma = 1/2$ $(r > 0)$. So $\lambda = \ln r$ is another potential which generates flat space.

It can be shown that there are precisely two other such potentials, namely

(7.3) $$\lambda = 1/2 \ln \left[\sqrt{r^2 + z^2} + z\right] , \quad \lambda = 1/2 \ln \left[\sqrt{r^2 + z^2} - z\right] .$$

They correspond to the Newtonian potentials of semi-infinite line masses of constant density 1/2, lying along the entire positive z-axis and entire negative z-axis respectively. So with four distinct Newtonian potentials which generate flat space, we conclude that:

There is not a strict 1-1 correspondence between the Weyl metrics and their generating Newtonian potentials $\lambda(r,z)$.

However, some questions naturally arise here. For instance, how special is the case of flat space in this context ? In other words, is it true, in general, that a Weyl metric is generated by more than one potential ? If not, then what is the class of exceptions ? Also, can a strict 1-1 correspondence be obtained by restricting the generating Newtonian potentials, to those corresponding to mass distributions of finite extent ? At the present time, all of these questions remain unanswered.

8. THE TWO-PARTICLE CURZON SOLUTION

No survey of the Weyl metrics would be complete without mentioning the two-particle Curzon solution, which as the name suggests, was first found by Curzon [24] (and later by Silberstein [25]). This solution is generated by the Newtonian potential $\lambda(r,z)$ of two particles (point masses) of mass m_1 and m_2 . Obviously, for the mass configuration to be axisymmetric, the two particles must both lie along the z-axis at z_1 and z_2 respectively ($z_1 < z_2$). So we have

(8.1) $$\lambda = - m_1/\rho_1 - m_2/\rho_2$$

where

(8.2) $$\rho_1 = \left[r^2 + (z - z_1)^2\right]^{1/2} \quad \text{and} \quad \rho_2 = \left[r^2 + (z - z_2)^2\right]^{1/2}$$

and

$$\nu = -1/2\, r^2 \left[m_1{}^2/\rho_1{}^4 + m_2{}^2/\rho_2{}^4\right]$$

(8.3) $$+ \frac{2\, m_1 m_2}{(z_2 - z_1)^2}\left[\frac{r^2 + (z - z_1)(z - z_2)}{\rho_1\rho_2} - 1\right] .$$

Silberstein claimed that the existence of a <u>static</u> solution consisting only of two point masses surrounded by vacuum, indicated the incorrectness of the general theory of relativity. Two masses at rest in vacuum should gravitate! Einstein [26] countered that the two-particle solution is <u>not</u> a purely **vacuum** solution, and provided the following argument. Consider a small circle given by t = constant, z = constant ($z_1 < z < z_2$), r = constant, where r is small. If we take the circumference C and radius R of this circle, it is found that in the limit as $R \to 0^+$,

$$C/R \;\to\; 2\pi\, e^{-\nu} \qquad \text{where} \qquad \nu = \nu(0,z) \;\; .$$

Now for $z_1 < z < z_2$, $\nu(0,z) \neq 0$ and so C/R does not approach 2π as $R \to 0^+$. Hence the space-time violates the condition of <u>elementary flatness</u> on the section of axis between the two particles, suggesting the existence of a "strut". This would explain the static nature of the solution.

However, in 1968, some thirty-two years after Einstein's paper on this subject, Szekeres [27] demonstrated that static, two-body solutions **do** exist in general relativity. In his solutions, at least one of the two point masses is endowed with a multipole mass structure, which allows equilibrium to be achieved **without** the need for an intervening strut. The simplest example is that of a pure mass monopole (a Curzon particle) balanced by a mass monopole-dipole, where the mass of each particle (as represented by the monopole moment) is positive.

Another major contribution to this subject came quite recently, in 1982. Using a technique to generate stationary solutions from static ones, Dietz and Hoenselaers [28] obtained from the two-particle Curzon solution, a stationary, axisymmetric solution representing two particles precisely balanced by their spin-spin interaction. Their solution is also a purely vacuum solution, with no strut required.

The source structure for the two-particle Curzon solution is still unknown. From Section 2, we know that although the Curzon metric (2.4) & (2.5) is generated by the Newtonian mass monopole, the Curzon solution is not the unique, spherically symmetric, vacuum solution of general relativity. The source for the Curzon solution is a ring singularity with finite radius and infinite circumference, and the space-time has a double-sheeted topology inside the ring.

So without further investigation, there is no reason to expect that the source for the two-particle Curzon solution simply consists of two point masses joined by a strut. The source structure is probably considerably more complicated, and the space-time may even be extendible. A first step towards resolving these issues would presumably be to look for directional behaviour at the two particle locations : $(r = 0, z_1)$ and $(r = 0, z_2)$.

9. RECENT MATHEMATICAL DEVELOPMENTS

In a recent paper by Woodhouse & Mason [29], the ideas presented in an earlier paper by Ward [30] are developed into a geometric correspondence between the stationary, axisymmetric vacuum space-times and particular complex analytic objects - holomorphic vector bundles on a non-Hausdorff Riemann surface (twistor space).

As a result, the solutions to the Ernst equations on space-time can be described in terms of certain free holomorphic functions on regions in the Riemann sphere (or on parts of the twistor space).

The paper discusses the effect of the action of the Geroch group on these free holomorphic functions, and also the conditions on them implied by global properties such as axis regularity and asymptotic flatness. Unfortunately, the construction is, at present, tied to the use of Weyl coordinates, so that aspects of the singularity/source structure and global structure which are obscured by the use of Weyl coordinates, are difficult to address in this new framework also.

Nevertheless, the construction is geometric, and it should therefore be possible to articulate it independently of the choice of such coordinates. The study of singularities would then perhaps be reducible to the study of singularities of holomorphic functions. However, further work needs to be done before these ideas are able to contribute to the study of the singularities occurring in the Weyl metrics.

10. RING SINGULARITIES

Perhaps the most appropriate way to conclude a survey, is to add a small, but new observation on the given subject - in this case the Weyl metrics. This particular observation will concern ring singularities (that is, rings comprised of curvature singularities), occurring in the hypersurfaces t = constant of the Weyl space-times. These rings are known to be a common feature throughout the entire Weyl class.

That the Weyl metrics should exhibit singularities in the form of rings is not really very surprising, since all metrics in the class are axisymmetric. So if a curvature singularity occurs at the point (r_o, z_o, φ_o) in the hypersurface t = constant $(r_o \neq 0)$, then a curvature singularity occurs at every point (r_o, z_o, φ), where $0 \leqslant \varphi < 2\pi$. In other words, (r_o, z_o) is a ring singularity.

What past investigators have found rather more surprising, is that these rings may have an infinite circumference. If, in addition, the ring singularity can be reached from the axis of symmetry via a finite number of spacelike geodesics, each having finite proper length, then we are, indeed, confronted by a highly counter-intuitive phenomenon – namely, a ring having finite radius, but infinite circumference!

The Curzon metric provides us with the most well-known example. Although, in Weyl coordinates, the Curzon singularity appears as a point (at R = 0) exhibiting highly directional behaviour, a change to the new coordinates constructed by Scott & Szekeres, unwraps the point to include, amongst other things, a ring singularity with finite radius and infinite circumference.

When past investigators of Weyl metrics have happened across an example of this phenomenon, they have tended to regard it as an exceptional case. However, the simple argument which follows, will indicate that for ring singularities with finite radius occurring in Weyl metrics, the generic case is that the circumference of the ring is **infinite**, not finite. We note that the standard Weyl coordinates (t,r,z,φ) will be used in what follows, although the argument could proceed equally well in any other coordinate system (t,x,y,φ).

Suppose that in a Weyl space-time, a curvature singularity occurs at the point $\mathbf{p} = (t_o, r_o \neq 0, z_o, \varphi_o)$. We assume that $\mathbf{p}$ can be reached from the axis of symmetry by a C^0 curve γ, which consists of a finite number of spacelike geodesics, each having finite proper length. So (r_o, z_o), $0 \leqslant \varphi < 2\pi$, is a ring singularity with finite radius, which occurs in every hypersurface t = constant.

Now it will generally be true, that $\lambda = -\infty$ at the curvature singularity $\mathbf{p}$. This means that $\lambda \to -\infty$ as $\mathbf{p}$ is approached from any direction. So if we consider the circle given by (r = constant, z_o), where $0 < r < r_o$, then its circumference C_r is found to be

$$C_r = 2\pi r\, e^{-\lambda} , \tag{10.1}$$

and it is readily seen that as $r \to r_o$, $C_r \to +\infty$.

The details have been omitted here, but the argument can be made rigorous. We note that the fact that the ring singularity has a finite radius is **not** used to show that it has an infinite circumference. In other words, a ring singularity with an infinite radius would also have an infinite circumference, but this is not very surprising after all. It only remains to find a physical explanation of this strange phenomenon. **How** can a ring singularity with finite radius have an infinite circumference ?

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