Space-time Splitting Theorems

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In his well-known problem section appearing in [Y1], S.-T. Yau posed the problem of establishing a Lorentzian analogue of the Cheeger-Gromoll [CG] splitting theorem of Riemannian geometry. Although stated in [Y1] as a problem in pure Lorentzian geometry, there is a physical, as well as mathematical, motivation for considering this problem [Y2]. The purpose of this paper is to discuss the physical motivation, and to survey the progress that has been made on this and related problems. We will also briefly describe some of the methods which have been employed to attack such problems.

1. Rigid Singularity Theorems.

The space-time splitting problem has its physical origins in the classical Hawking-Penrose singularity theory of general relativity. As is well-known, classical general relativity predicts under quite general circumstances the occurrence of singularities in space-time. Mathematically, this occurrence is evidenced by the existence of incomplete timelike or null geodesics. For the purpose of motivation, we consider here a simple but, in many respects, typical singularity theorem.

<u>Theorem</u>. Let M be a space-time which satisfies the following:

(1) M is "spatially closed"; i.e., M contains a compact Cauchy surface S;

(2) M obeys the "strong energy condition", Ric(X,X) = $R_{ij}X^iX^j \geq 0$, for all timelike (and null) vectors X; and

(3) M obeys the "generic condition"; i.e., along each inextendible timelike (and null) geodesic γ, there is at least one point at which

$$X_{[c}R_{a]bc[d}X_{f]}X^bX^c \neq 0 \ ,$$

where X = γ' is the tangent to γ and R is the Riemann curvature tensor.

Then M is timelike (or null) geodesically incomplete.

Recall that, physically, the generic condition says that there exists a nonzero tidal acceleration at some point along the world line of each freely falling observer and photon. Also, from well-known results in causal theory, the assumption that M contains a compact Cauchy surface is equivalent to the following: M is globally hyperbolic and contains a compact spacelike hypersurface.

It will be useful for later discussion to briefly review the proof of the singularity theorem. The key step in the proof is the construction of a timelike (or null) <u>line</u>, by which we mean an inextendible timelike (or null) geodesic which is maximal "in the large", i.e., which, for every pair of points on it, gives a maximum for the length among all causal curves joining these points. Consider a sequence of points $\{q_n\}$ in the future of S which extends to future infinity, and a sequence $\{p_n\}$ in the past of S which extends to past infinity, such that $p_n \ll q_n$ for all n. Let γ_n be a maximal timelike segment from p_n to q_n. Since each γ_n meets S, and S is compact, the γ_n's accumulate to an inextendible causal curve γ. The maximality of the γ_n's implies that γ is a timelike, or possibly null, line; γ may be null because, roughly speaking, the γ_n's may be turning null as $n \to \infty$. To complete the proof, one shows that γ must be incomplete: if it were complete then the curvature assumptions would imply that γ has a pair of conjugate points, which would contradict its maximality.

The strong energy condition has an overall focusing effect on congruences of null and timelike geodesics. However, since this condition involves a weak inequality, it does not guarantee any strict focusing. The generic condition insures that there is some real focusing taking place. It is typical of all the standard singularity theorems that there be some curvature object which obeys a strict inequality (and, hence, which guarantees some strict focusing). The point of view espoused by Yau is that conditions like the generic condition used to establish singularity theorems should, in some sense, be unnecessary. This is not to say that the singularity theorem considered above is correct without condition (3). In fact, there is a very simple counterexample: consider the flat spatially closed space-time cylinder $(\mathbb{R} \times S^1, -dt^2 \oplus d\theta^2)$. It satisfies all the hypotheses of the singularity theorem except the generic condition, but is geodesically complete. However, this counterexample is quite exceptional. It is _metrically_ a product; the time factor is split off from the spatial factor, and the spatial factor is unchanging in time. In particular, this example is static ($\partial/\partial t$ is a Killing field). What one would like to show is that if only _weak_ curvature inequalities are assumed to hold (e.g., if one drops assumption (3) above) then either space-time is singular (as before), or else it is extremely special and, hence, unphysical. This aim is partially motivated by a well-known program in global Riemanniann (positive definite) geometry (see e.g., [CE]). In Riemannian geometry, there are numerous examples of results which show that, although a certain conclusion becomes false when one relaxes the curvature conditions of strict inequality to weak inequality, the conclusion can fail only under very special circumstances. Results of this type are referred to as _rigidity theorems_. Thus, in the present context, the aim is to study the rigidity of the singularity theorems.

In view of the preceding discussion, one is led to consider the following conjecture (which is stated as Conjecture 2 in [B3]).

Conjecture. Let M be a space-time which satisfies the following:

(1) M contains a compact Cauchy surface; and

(2) M obeys the strong energy condition, $Ric(X,X) \geq 0$, for all timelike vectors X.

Then either M is timelike geodesically incomplete, or else M splits isometrically into the metric product ($\mathbb{R} \times V$, $-dt^2 \oplus h$), where (V,h) is a compact Riemannian manifold; in particular M is static.

We remark that in the case M is a vacuum (i.e., Ricci flat), four dimensional space-time, if M splits as above then it is necessarily flat. Bartnik has coined the phrase "cosmological space-time" for a space-time satisfying conditions (1) and (2) of the conjecture. Thus, the conjecture asserts that a cosmological space-time either is singular or splits. If proved, the conjecture should be viewed as a (rigid) singularity theorem, since the exceptional possibility that space-time splits can be ruled out as unphysical.

Restricted versions of this conjecture have been considered in the past ([A], [GR]). In the early 60's Avez [A] considered the so-called "time-periodic" case for four-dimensional vacuum space-times. In fact, he gave a proof in this case, but his proof contains a well-known error (which is propagated in [GR]; see [MT]).

The conclusion of the conjecture can be formulated as follows: if M is tgc (timelike geodesically complete) then M splits. In [G1] we gave a proof of the conjecture assuming an additional completeness type assumption.

<u>Theorem 1 [G1]</u>. Let M be a cosmological space-time. If M is tgc and has no observer horizons then M splits as in the conjecture.

By definition, M has no observer horizons, provided that for every inextendible timelike curve γ, $I^-(\gamma) = I^+(\gamma) = M$. Frequently, a nontrivial observer horizon ($\partial I^{\pm}(\gamma) \neq \emptyset$) signals the occurrence of a singularity (consider the flat space-time cylinder of finite height). However, as de Sitter space illustrates, nontrivial observer horizons can also arise in space-times which expand at an accelerating rate. Of course, in the case of de Sitter space, this acceleration is fueled by <u>negative</u> Ricci curvature. In general, the strong energy condition has an overall decelerating effect on expansion. Thus, the theorem above may be interpreted as a singularity theorem of sorts: unless it splits, a cosmological space-time either is singular or else has a nontrivial observer horizon, in which case we at least <u>suspect</u> that it's singular.

To eliminate the no observer horizon condition from Theorem 1, one might try prove that, for a cosmological space-time, timelike geodesic completeness implies there are no observer horizons. Recall that the Ricci curvature of a unit timelike vector can be written as <u>minus</u> the sum of tidal accelerations or, in geometric terms, as <u>minus</u> the sum of <u>timelike</u> sectional curvatures (i.e., sectional curvatures of timelike planes). The most stringent curvature requirement consistent with gravity being attractive is that all timelike sectional curvatures be nonpositive. The Friedmann cosmological models (and sufficiently small perturbations of them) and anti-de Sitter space are examples of space-times having nonpositive timelike sectional curvatures.

Let M be a space-time with compact Cauchy surface. In the case that M has nonpositive timelike sectional curvtures, it can be shown, using a result of Harris [H2], that timelike geodesic completeness implies there are no observer horizons. Consequently, one obtains the following corollary to Theorem 1.

Corollary 2. Let M be a space-time with compact Cauchy surface and nonpositive timelike sectional curvatures. Then either M is timelike geodesically incomplete or M splits.

We wish to say a few words about the proof of Theorem 1 given in [G1]. The key step in the proof is to establish the existence of a compact, maximal (mean curvature zero) hypersurface in M. Gerhardt [GC] and Bartnik [B1] have given proofs of the existence of compact maximal hypersurfaces under the assumption of past and future crushing singularities, i.e., under the assumption that space-time comes to an end in the future and past. Although the situation considered in Theorem 1 is just the opposite, we found Bartnik's proof amenable to modification and were able to establish the following.

Theorem 3. Let M be a cosmological space-time which is tgc and has no observer horizons. Then M contains a compact maximal hypersurface.

One interesting feature of the proof is that standard singularity theory is used to obtain an a priori height estimate, which is needed in the proof of the existence of a solution to the mean curvature equation. Once one has established the existence of a compact maximal hypersurface, one can show (see [GC], [B3]) that a cosmological space-time, if tgc, must necessarily split.

We point out that recent work of Bartnik [B3] yields a strengthening of Theorem 3 (and in turn Theorem 1). For space-times M with compact Cauchy surface, the condition that there be no observer horizons has a number of equivalent formulations (see [G1]). For example, M has no observer horizons if and only if for all $p \in M$, $M \setminus (I^-(p) \cup I^+(p))$ is compact. Roughly speaking, this latter condition says that the past and future null cones at each point are able to wrap completely around the universe. The main theorem in [B3] implies that the no observer horizon condition in Theorem 3 can be replaced by: for some point $p \in M$, $M \setminus (I^-(p) \cup I^+(p))$ is compact. This condition is closely related to a condition previously considered by Geroch [GR].

2. The Lorentzian Splitting Theorem.

Actually, Yau had a different approach in mind to the proof of the conjecture stated in the previous section, or to the proof of rigid singularities theorems in general. In his view, such results should follow from a Lorentzian analogue of the Cheeger-Gromoll splitting theorem of Riemannian geometry, the statement of which we now recall.

The Riemannian Splitting Theorem [CG]. Let M be a Riemannian manifold which satisfies the following:

(1) M is geodesically complete;

(2) M has nonnegative Ricci curvature, $\mathrm{Ric}(X,X) \geq 0$ for all

X; and

(3) M contains a complete line.

Then M is isometric to a product $\mathbb{R} \times V$.

A Riemannian line is an inextendible geodesic which minimizes distance between every pair of its points. This theorem is a classic example of a rigidity theorem in Riemannian geometry. Suppose M is as in the theorem, except that it has strictly positive Ricci curvature. Then M cannot contain any lines since, by standard index theory techniques, each complete geodesic contains a pair of conjugate points. The Cheeger-Gromoll splitting theorem shows that if one weakens the curvature assumption to nonnegative Ricci curvature, then M can have lines only under very special circumstances.

The proof involves an analysis of the Busemann functions $b^{\pm}: M \to \mathbb{R}$ associated to the given line γ, defined by

$$b^{\pm}(x) = \lim_{r \to \infty} r - d(x, \gamma(\pm r)) ,$$

where d is the Riemannian distance function. The level sets $b^{\pm} = \text{const.}$, which are called horospheres, are limits of spheres whose centers go to $\pm \infty$ along the line. The Riemannian Busemann functions $b^{\pm}$ are always continuous. The key step in the proof of the Riemannian splitting theorem is to establish the subharmonicity of $b^{\pm}$, in the sense of continuous functions, under the assumption of nonnegative Ricci curvature. The proof of this (which has been greatly simplified in more recent works [W],

[EH]), makes use of the theory of elliptic operators as applied to the Riemannian Laplacian.

In [Y1], Yau posed the problem of establishing a Lorentzian analogue of the Cheeger-Gromoll splitting theorem. His statement of the problem follows.

Problem [Y1]. Let M be a Lorentzian manifold which satisfies the following:

(A) M is tgc;

(B) M obeys the strong energy condition, $Ric(X,X) \geq 0$, for all timelike X; and

(C) M contains a complete timelike line.

Show that M splits isometrically into the product $(\mathbb{R} \times V, -dt^2 \oplus h)$, where (V,h) is a complete Riemannian manifold.

The statement of the problem parallels very closely the statement of the Riemannian splitting theorem. However, the status of the concept of geodesic completeness in Lorentzian geometry differs considerably from that of Riemannnian geometry. In the Riemannian case, geodesic completeness insures that any two points can be joined by a minimal geodesic. The naive analogue of this fact does not hold in the Lorentzian case. The standard condition which insures that two timelike related points in a Lorentzian manifold can be joined by a maximal timelike geodesic is not geodesic completeness, but global hyperbolicity. Thus, to the statement of the Lorentzian splitting problem given above one might be inclined to add (or exchange with (A)) the hypothesis:

(D) M is globally hyperbolic.

One readily sees why the proof of the Riemannian splitting theorem fails to carry over to the Lorentzian case. Whereas the Riemannian proof is based on the ellipticity of the Riemannian Laplacian, the space-time Laplacian = the trace of the Hessian = the d'Alembertian is hyperbolic. In particular, the concept of subharmonicity as used in the Riemannian proof is not directly applicable. There are other, more technical difficulties in carrying over the Riemannian proof, as well. Since there exists a Lorentzian distance function [BE], one can formally introduce (using the same defining equation) Lorentzian analogues of the Riemannian Busemann functions. However, as the Lorentzian distance function does not share all the nice properties of the Riemannian distance function, the Lorentzian Busemann functions need not be so well behaved. Indeed, they needn't be continuous or even finite valued. Also, arguments in the Riemannian case frequently involve constructing asymptotes to the given line. In the Lorentzian case one needs to know that the asymptotes to the given timelike line (which arise as the limit of maximal timelike geodesic segments) are themselves timelike. However, in general, these asymptotes can be null, even if M is globally hyperbolic. This problem is a particular instance of a difficulty which frequently arises in doing Lorentzian geometry, and which is a result of the noncompactness of the unit timelike vector bundle.

We now briefly describe the results of four papers written in the last few years which have led to a full resolution of the

Lorentzian splitting problem as considered in this section. Afterwards, we will discuss the bearing of this on the conjecture stated in the first section. All four papers use Busemann function methods.

The first paper to address the Lorentzian splitting problem as presented in this section was the paper of Beem, Ehrlich, Markvorsen and Galloway [B+]. The authors proved that if M is a space-time which (B') has nonpositive timelike sectional curvatures, (C) contains a complete timelike line, and (D) is globally hyperbolic then M splits. The sectional curvature condition is used in several ways. It guarantees that asymptotes (and, more generally co-rays) are timelike and that the Busemann functions $b^{\pm}$ are continuous on the domain of influence of the given timelike line. Also, the sectional curvature condition permits one to study the behavior of $b^{\pm}$ along geodesics, where the relevant differential operator is just ordinary differentiation and a simple one-dimensional maximum principle applies. Although the result is restricted to the sectional curvature case (which, by the way, follows historically the development in the Riemannian case), it is notable in that its proof marks the first use of Busemann functions in Lorenzian geometry. One other interesting feature of the result is that (except for the assumed completeness of the given line) the assumption of timelike geodesic completeness is not needed. Timelike geodesic completeness is derived (via triangle comparison techniques [H1]) from the assumption of global hyperbolicity, the sectional curvature condition, and the completeness of the line.

Subsequent to the work of Beem et al., Eschenburg [E] made a major breakthrough on the Lorentzian splitting problem. He proved that if M is a space-time which (A) is timelike geodesically complete, (B) satisfies the strong energy condition, (C) contains a complete timelike line, and (D) is globally hyperbolic then M splits. Eschenburg manages to overcome the technical difficulties mentioned above by working near the line. For example, he establishes, without the use of curvature conditions, the continuity of $b^{\pm}$ near the line. But the crucial idea of the proof, which enables him to overcome the difficulty in the nonellipticity of the space-time Laplacian, is to restrict the Busemann function b^{+} to smooth spacelike hypersurfaces Σ which approximate the level sets b^{-} = const. Since the induced Laplacians Δ_{Σ} are elliptic, elliptic theory re-enters the picture. In particular, one has at one's disposal the maximum principle, which Eschenburg uses in an intricate manner.

Eschenburg's splitting result is very satisfying. However, the fact that the Beem et al. result does not require the assumption of timelike geodesic completeness suggests that there may be some redundancy in his hypotheses. Subsequent to Eschenburg's work, we [G3] were able to give a proof of the Lorentzian splitting theorem without the assumption of timelike geodesic completeness, thus simultaneously generalizing the splitting results of Beem et al. and Eschenburg. One interesting aspect of the proof is that it combines the Busemann function approach with maximal surface theory. It also emphasizes the geometry of the Busemann functions, and hence permits one to utilize them in a somewhat more natural way. We consider $b^{\pm}$ restricted to a maximal hypersurface Σ whose edge is contained in the level set $b^{+} = 0$. Using the ellipticity of the induced Laplacian, we show that $b^{\pm}$ obeys a maximum principle: if $b^{\pm}|_{\Sigma}$ attains an interior minimum then $b^{\pm}|_{\Sigma}$ is constant. Although the level sets of $b^{\pm}$ are not in general smooth, one can interpret this result as saying that the level sets are mean convex. (In the Riemannian case, this follows from the subharmonicity.) This result is used to show that, near the line, the level sets $b^{\pm} = 0$ agree and are smooth. At this point one is well on the way to establishing the desired splitting. The existence of the maximal hypersurface Σ is guaranteed by recent results of Bartnik [B2] concerning the existence and regularity of solutions to the Dirichlet problem for the prescribed mean curvature equation with rough boundary data.

Thus, in [G3] we have obtained the strongest form of the splitting theorem for globally hyperbolic space-times. But this still leaves open Yau's version of the splitting problem, which assumes timelike geodesic completeness but not global hyperbolicity. Shortly after the completion of our work, Newman [N] made a very nice observation which enabled him to eliminate the global hyperbolicity assumption from Eschenburg's theorem and, hence, to establish Yau's version of the splitting theorem. The global hyperbolicity assumption is used primarily to guarantee the existence of maximal geodesic segments joining timelike related points. Using the timelike geodesic completeness assumption, Newman shows that each point p sufficiently close to the given line

γ and each point q sufficiently far in the future of p on γ can be joined by a maximal timelike geodesic segment. Newman uses a limit curve argument, which we briefly describe, to accomplish this. Since γ is a line we are assured that $d(p,q) < \infty$. Let $\{\sigma_n\}$ be a sequence of causal curves from p to q, the lengths of which converge to $d(p,q)$, and let σ be a limit curve of this sequence. Newman shows, by using timelike geodesic completeness, that σ either is the desired maximal segment or a null ray (half-line) from p. Using Busemann function estimates of Eschenburg, which Newman shows are valid without global hyperbolicity, he is able to eliminate the latter possibility.

Thus, having established the Lorentzian splitting theorem for space-times with timelike lines, one would now like to use it to prove the conjecture considered in Section 1. For this, one needs to establish the existence of a timelike line in a cosmological space-time, which, for the purpose of proving the conjecture, one may even assume is tgc. Recall from our discussion of the proof of the singularity theorem presented in Section 1 that there are standard arguments for producing a timelike or null line in a space-time with compact Cauchy surface. What one needs to do is to come up with a construction which insures that the line obtained is actually timelike. We remark that the no observer horizon condition, as well as the weaker criterion considered by Bartnik, is sufficient to guarantee that a space-time with compact Cauchy surface has a timelike line. Thus, Theorem 1, and its improvement by Bartnik, are consequences of the Lorentzian splitting theorem. Moreover, it can be shown [B+] that the nonpositive timelike sectional curvature assumption, together with the timelike geodesic completeness assumption, are sufficient to establish the existence of a timelike line in a space-time with compact Cauchy surface. Hence, Corollary 2 is also a consequence of the Lorentzian splitting theorem.

As it stands, the conjecture of Section 1 remains open. Even so, one is naturally led to consider more general versions of this conjecture. Dropping the global hyperbolicity assumption leads to the following conjecture.

Conjecture. Let M be a noncompact space-time which satisfies the following:

(1) M contains a compact spacelike hypersurface; and

(2) M obeys the strong energy condition.

Then either M is timelike geodesically incomplete, or else M splits.

Note the assumption that M is noncompact. Strictly speaking, the conjecture would be false without it (consider the flat space-time torus). However, this assumption is not essential. If we omit it, then the conclusion should be modified to state that either M is timelike geodesically incomplete or it is covered by a space-time which splits. Some evidence supporting the more general conjecture stated above is provided in [G3], where it is proved in two special cases: case 1, where M has nonpositive timelike sectional curvatures, and case 2, where M has a compact maximal hypersurface. In both cases, we prove that M is globally hyperbolic, thereby reducing the setting to situations covered by previous results. Another (in some sense dual) approach to proving the conjectures discussed here would be to establish the existence of a compact maximal hypersurface under sufficiently general circumstances. We refer the reader to [B3] for a current discussion of this approach.

REFERENCES

[A] A. Avez, Essais de geometrie riemanniene hyperbolique globale. Applications a la relativite generale, Ann. Inst. Fourier 13 (1963) 105-190.

[B1] R. Bartnik, Existence of maximal surfaces in asymptotically flat space-times, Commun. Math. Phys. 94 (1984) 155-175.

[B2] ———, Regularity of variational maximal surfaces, to appear in Acta Mathematica.

[B3] ———, Remarks on cosmological spacetimes and constant mean curvature surfaces, to appear in Commun. Math. Phys.

[BE] J.K. Beem, P.E. Ehrlich, Global Lorentzian geometry, Pure and Applied Math., Vol. 67, Dekker, New York, 1981.

[B+] J.K. Beem, P.E. Ehrlich, S. Markvorsen, G.J. Galloway, Decomposition theorems for Lorentzian manifolds with nonpositive curvature, J. Diff. Geom. 22 (1985) 29-42.

[CE] J. Cheeger and D. Ebin, Comparison theorems in Riemannian geometry, Amsterdam, North-Holland, 1975.

[CG] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Diff. Geom. 6 (1971) 119-128.

[E] J.-H. Eschenburg, The splitting theorem for space-times with strong energy condition, J. Diff. Geom. 27 (1988) 477-491.

[EH] J.-H. Eschenburg and E. Heintze, An elementary proof of the Cheeger-Gromoll splitting theorem, Ann. Glob. Anal. and Geom. 2 (1984) 141-151.

[G1] G.J. Galloway, Splitting theorems for spatially closed space-times, Commun. Math. Phys. 96 (1984) 423-429.

[G2] ———, The Lorentzian splitting theorem without completeness assumption, to appear in J. Diff. Geom.

[G3] ———, Some connections between global hyperbolicity and geodesic completeness, to appear in J. Geom. and Phys.

[GC] C. Gerhardt, H-surfaces in Lorentzian manifolds, Commun. Math. Phys. 89 (1983) 523-553.

[GR] R. Geroch, Singularities in closed universes, Phys. Rev. Lett. 17 (1966) 445-447.

[H1] S.G. Harris, A triangle comparison theorem for Lorentz manifolds, Indiana Math. J. 31 (1982) 289-308.

[H2] ———, On maximal geodesic diameter and causality in Lorentzian manifolds, Math. Ann. 261 (1982) 307-313.

[MT] J.E. Marsden, F.J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in general relativity, Physics Reports 66 (1980) 109-139.

[N] R.P.A.C. Newman, A proof of the splitting conjecture of S.T. Yau, to appear in J. Diff. Geom.

[W] H. Wu, An elementary method in the study of nonnegative curvature, Acta Math. 142 (1979) 57-78.

[Y1] S.-T. Yau, Problem section, Ann. of Math. Studies, 102 (1982) 669-706.

[Y2] ———, private communication.

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