GRAVITATION WITH GAUSS BONNET TERMS

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Abstract We study general properties of the partial differential equations for generalized gravity arising from the Lovelock Lagrangian.

I - Introduction

The low energy limit of supersymmetric string action leads to an effective Lagrangian which contains, in addition to the usual scalar curvature, polynomial terms in the Riemann curvature tensor; these terms are identically zero if the base manifold is of dimension four. The corresponding field equations are second order partial differential equations for the metric tensor if the Lagrangian was formed with the so called Gauss-Bonnet combination , that is if the polynomial of order p was of the type

$$L(p)= \mathcal{E}^{\lambda_1\lambda_2\ldots\ \lambda_{2p-1}\lambda_{2p}}_{\mu_1\mu_2\ldots\ \mu_{2p-1}\mu_{2p}}\ R_{\lambda_1\lambda_2}{}^{\mu_1\mu_2}\ \ldots\ R_{\lambda_{2p-1}\lambda_{2p}}{}^{\mu_{2p-1}\mu_{2p}}$$

which, if the manifold has dimension 2p, corresponds to a closed form which represents the Euler class of the manifold (up to a constant factor). The equations are non linear even for the second derivatives if the Lagrangian contains a Riemann curvature term of order $p > 1$. Various interesting results are already known for these field equations. First it is obvious that at flat space their first variation is identical whith the variation of Einstein equations. It has also been proved (Boulware and Deser) that they have the same plane wave solutions than Einstein equations. The same authors have constructed spherically symmetric solutions and studied their stability. Kerner has used these equations in a five dimensionnal Kaluza Klein context as a model for non linear electromagnetism. In the case p=2 the characteristics have been studied by Aragone . Other results pertinent to particular cases have also been obtained.

In this article we give some general properties of the generic solutions of the system of non linear partial differential equations, deduced from the Lagrangian with Gauss-Bonnet terms : we show the splitting of the equations like in ordinary gravity , between constraints and evolution, and show that at least in the analytic case the evolution preserves the constraints. We determine the wave fronts ; they are no more tangent to the light cone and not even to a second order cone , neither probably in general to a convex cone even if the polynomial terms in Riemann curvature are coupled with the ordinary Einstein tensor through multiplication by a small constant : the wave cone of the Einstein equations consists in fact of D copies of the light cone of the metric , and by perturbation it becomes a cone of order 2D which may be non convex and non real. Existence of solutions of the Cauchy problem are not known without analyticity hypothesis on the data.

In a second part we give some general results about high frequency waves.

II - Equations

Let V be a d-dimensional C^{∞} manifold, with a metric g of hyperbolic signature (-, +, +, ..+). This metric is said to represent gravitation with Gauss-Bonnet terms if it satisfies the system of partial differential equations

$$A_{\alpha}^{\beta} \equiv K_o \, \delta_{\alpha}^{\beta} + \sum_{p=1}^{\infty} K_p \, \mathcal{E}_{\alpha \, \mu_1 \ldots \mu_{2p}}^{\beta \, \lambda_1 \ldots \lambda_{2p}} R_{\lambda_1 \lambda_2}{}^{\mu_1 \mu_2} \ldots R_{\lambda_{2p-1} \lambda_{2p}}{}^{\mu_{2p-1} \mu_{2p}} = 0 \qquad (2\text{-}1)$$

where the K_p are constants, the $\mathcal{E}_{\alpha \, \mu_1 \ldots \mu_{2p}}^{\beta \, \lambda_1 \ldots \lambda_{2p}}$ the completely antisymmetric Kronecker tensor, and $R_{\alpha\beta}{}^{\lambda\mu}$ the Riemann curvature tensor of g. The sum is indeed finite : all terms with $2p + 1 > d$ are identically zero; for $d = 4$ the only non zero terms are for $p = 1$. For arbitrary d the term in $p = 1$ is proportional to the Einstein tensor :

$$\mathcal{E}_{\alpha\ \mu_1\mu_2}^{\beta\ \lambda_1\lambda_2}\ R_{\lambda_1\lambda_2}{}^{\mu_1\mu_2} \equiv -4\left(R_\alpha^\beta - \frac{1}{2} g_\alpha^\beta R\right) \equiv -4\ S_\alpha^\beta \quad .$$

while the term with p = 0 is the so called cosmological term.

To make appear a coupling constant, eventually small, between the usual Einstein equations in dimension d with possibly a non zero cosmological constant, and the new Gauss-Bonnet terms, we rescale (2-1) to write it under the form

$$A_\alpha^\beta \equiv S_\alpha^\beta + \Lambda\, g_\alpha^\beta + \chi\, B_\alpha^\beta = 0 \qquad (2\text{-}2)$$

with, $P < d/2$ being some positive integer

$$B_\alpha^\beta \equiv \sum_{p=1}^{P} k_p\ \mathcal{E}_{\alpha\ \mu_1..\mu_{2p}}^{\beta\ \lambda_1..\lambda_{2p}}\ R_{\lambda_1\lambda_2}{}^{\mu_1\mu_2} ..\ R_{\lambda_{2p-1}\lambda_{2p}}{}^{\mu_{2p-1}\mu_{2p}}$$

We have, by the symmetries of the Riemann tensor

$$A_{\alpha\beta} \equiv A_{\beta\alpha} \quad ,$$

the equations (2-2) are, in local coordinates, a system of d(d+1)/2 second order partial differential equations for the d(d+1)/2 unknowns $g_{\alpha\beta}$.

These equations are invariant by diffeomorphisms of V : in fact the lagrangian itself is invariant by these diffeomorphisms, which implies, for any choice of the constants χ and k_p, the d identities

$$\nabla_\alpha A_\beta^\alpha \equiv 0 \quad ; \qquad (2\text{-}3)$$

as can be checked directly.

III - Cauchy problem. Constraints.

The equations (2-2) are, like the equations of ordinary General Relativity both an under determined system and an over determined one : their characteristic determinant is identically zero at most of rank d(d+1)/2 - d = d(d-1)/2 due to (2-3) on the one hand, and on the other hand the unknowns and their first derivatives cannot be given arbitrarily on a d-1 dimensional submanifold of V : the Cauchy data must satisfy constraints.

To make a geometric analysis of the Cauchy problem we use, like in ordinary gravity, a (d-1)+1 decomposition of the metric and, in the case considered here, of the full riemann tensor.

Let $U = S \times I$ be a local slicing of an open set $U \subset V$ by d-1 dimensional space like manifolds $S_t = S \times \{t\}$. The metric reads, in adapted coordinates $x^o = t$, x^i coordinates on S :

$$ds^2 = -\alpha^2 (dx^o)^2 + g_{ij}(dx^i + \beta^i dx^o)(dx^j + \beta^j dx^o) \quad .$$

If the shift β is zero (choice always possible) a simple calculation gives the identities

$$R_{ijh}{}^k \equiv \bar{R}_{ijh}{}^k + K_j{}^k K_{ih} - K_i{}^k K_{hj} \tag{3-1-a}$$

$$R_{ijh}{}^o \equiv -\frac{1}{\alpha}\left(\bar{\nabla}_j K_{ih} - \bar{\nabla}_i K_{jh}\right) \tag{3-1-b}$$

$$R_{ioh}{}^o \equiv -\frac{1}{\alpha}\partial_o K_{ih} + K_{ij} K_h{}^j - \frac{1}{\alpha}\bar{\nabla}_i \partial_h \alpha \tag{3-1-c}$$

where $\bar{\nabla}$ and $\bar{R}_{ijh}{}^k$ are the riemannian covariant derivative and curvature tensor of the metric $\bar{g} = (g_{ij})$ induced on S_t by $g = (g_{\alpha\beta})$, and $K = (K_{ij})$ is the extrinsic curvature of S_t, that is

$$K_{ij} = -\frac{1}{2\alpha}\partial_o g_{ij} \quad . \tag{3-2}$$

We remark on the formulas (3-1) that the derivative $\partial^2_{oo}\alpha$ appears nowhere and that the derivative $\partial_o K_{ij}$, therefore the derivative $\partial^2_{oo} g_{ij}$, appears only in $R_{ioh}{}^o$. As a consequence the quantities A^o_o and A^o_i are determined on a slice S_t by the values on S_t of the first derivatives of the metric, they give constraints on the Cauchy data, namely in the coordinates we have adopted :

$$A^o_o \equiv S^o_o + \Lambda g^o_o + \chi \sum_{p=2}^{P} k_p \, \varepsilon^{i_1 \cdot\cdot i_{2p}}_{j_1 \cdot\cdot j_{2p}} R_{i_1 i_2}{}^{j_1 j_2} \ldots R_{i_{2p-1} i_{2p}}{}^{j_{2p-1} j_{2p}}$$

$$A^o_i \equiv S^o_i - \chi \sum_{p=2}^{P} 2\, p\, k_p \, \varepsilon^{i_1 \cdot\cdot i_{2p}}_{j_2 \cdot\cdot j_{2p}} R_{i_1 i_2}{}^{o j_2} R_{i_3 i_4}{}^{j_3 j_4} \ldots R_{i_{2p-1} i_{2p}}{}^{j_{2p-1} j_{2p}} \quad .$$

We know from Einstein's equations that

$$S^o_o \equiv -\frac{1}{2}\left(\overline{R} - K_i{}^j \, K_j{}^i + \left(K^h_h\right)^2\right)$$

$$\alpha \, S^o_i \equiv \overline{\nabla}_h \, K_i{}^h - \partial_i \, K^h_h$$

Using (3-1) the other terms in A^o_o and $\alpha \, A^o_i$ can also be expressed in terms only of the geometric elements $\overline{g}_t$ and K_t on S_t.

The intrinsic Cauchy data on a slice S_o for gravitation with Gauss-Bonnet terms are like for usual gravity a metric and a symmetric 2-tensor on S_o, satisfying the constraints,

$$A_{\perp\perp} \equiv A_{\alpha\beta} \, n^\alpha \, n^\beta = 0 \qquad \text{"hamiltonian" constraint}$$

$$A^{\parallel}_{\perp} = (A_{\perp i}) = \pi^\beta_i \, n_\alpha \, A^\alpha{}_\beta = -\alpha \, A^o_i = 0 \qquad \text{"momentum" constraint.}$$

An analytic solution of the equations $A_{ij} = 0$ on $U = S \times I$ which satisfies the constraints on S_o satisfies the equations $A_{\alpha\beta} = 0$ in a neighborhood of S_o, for any analytic choice of lapse (and shift if we introduce it) due to the identities (1-3) which are then a first order homogeneous system of the Cauchy Kovalevski type for the d quantities A^o_α.

The same is true if, instead of $A_{ij} = 0$, we consider the equations

$$\tilde{A}_{ij} \equiv A_{ij} - \frac{1}{d-2} g_{ij} A^{\alpha}_{\alpha} \equiv R_{ij} - \frac{2}{d-2} \Lambda g_{ij} + B_{ij} - \frac{1}{d-2} g_{ij} B^{\alpha}_{\alpha} = 0 \quad .$$

IV - Evolution. Analytic case.

The equations $A_{ij} = 0$, or $\tilde{A}_{ij} = 0$ are, when the lapse is given as well as the shift (here taken to be zero) a system of non linear $d(d-1)/2$ partial differential equations for the unknown g_{hk}. It results from §3, as already remarked by Aragone that they are linear in the second derivatives $\partial^2_{oo} g_{hk}$. They are of the Cauchy Kovalevski type in a neighborhood in $S \times \mathbb{R}$ of the manifold $S_o = S \times \{0\}$, for the Cauchy data $\bar{g}$ and K, if they can be solved with respect to the $\partial^2_{oo} g_{hk}$, that is if the determinant of the coefficients of these derivatives is non zero.

We denote by $\simeq$ equality modulo the addition of terms which contain no second derivatives $\partial^2_{oo} g_{hk}$. We have

$$R_{ij} \simeq \frac{1}{2\alpha^2} \partial^2_{oo} g_{ij}$$

$$B_{ij} \simeq \sum_{p=2}^{P} g_{j\ell} k_p (2p)^2 \varepsilon^{\ell \; \ell_2 \ell_3 .. \ell_{2p}}_{i \; m_2 m_3 .. m_{2p}} R_{\ell_2 o}{}^{m_2 o} R_{\ell_3 \ell_4}{}^{m_3 m_4} .. R_{\ell_{2p-1} \ell_{2p}}{}^{m_{2p-1} m_{2p}}$$

$$\simeq \frac{1}{2\alpha^2} X^{hk}_{ij} \partial^2_{oo} g_{hk}$$

with

$$X^{hk}_{ij} = g_{j\ell} g^{mk} \sum_{p=2}^{P} k_p (2p)^2 \varepsilon^{\ell \; h \; \ell_3 .. \ell_{2p}}_{i \; m \; m_3 .. m_{2p}} R_{\ell_3 \ell_4}{}^{m_3 m_4} .. R_{\ell_{2p-1} \ell_{2p}}{}^{m_{2p-1} m_{2p}} \quad .$$

The system $\tilde{A}_{ij} = 0$ is of the Cauchy-Kovalevski type, for the unknown g_{ij} and an arbitrarily given α if the determinant of the matrix M with elements (a capital index is a pair of ordered indices)

$$M_I^J \equiv M_{ij}^{hk} = \frac{1}{2\alpha^2}\left(\delta_I^J + \chi\, Y_{ij}^{hk}\right) \quad ,$$

$$Y_{ij}^{hk} \equiv X_{ij}^{hk} - \chi \frac{1}{d-2} g_{ij}\, g^{\ell m}\, X_{\ell m}^{hk} \quad ,$$

$$g^{\ell m}\, X_{\ell m}^{hk} \equiv g^{mk} \sum_{p=2}^{P} k_p (2p)^2 (d-1) \, \varepsilon_m^{h}{}_{m_3..m_{2p}}^{\ell_3..\ell_{2p}}\, R_{\ell_3\ell_4}{}^{m_3m_4} .. R_{\ell_{2p-1}\ell_{2p}}{}^{m_{2p-1}m_{2p}}$$

is non identically zero.

We have, $\mathbb{1}$ denoting the unit matrix and $D = d(d-1)/2$,

$$\det M = \left(\frac{1}{2\alpha^2}\right)^D \det(\mathbb{1} + \chi Y)$$

and

$$\det(\mathbb{1} + \chi Y) = 1 + \chi\, a_1 + \chi^2 a_2 + \dots + \chi^D a_D$$

with

$$a_1 = \delta_J^I\, Y_I^J = \text{tr}\, Y = \delta_h^i\, \delta_k^j\, Y_{ij}^{hk} = X_{ij}^{ij} - \frac{1}{d-2} g_{ij}\, g^{\ell m}\, X_{\ell m}^{ij} =$$

$$= \sum_{p=2}^{P} k_p\, (2p)^2\, \varepsilon_{m_3..m_{2p}}^{\ell_3..\ell_{2p}}\, R_{\ell_3\ell_4}{}^{m_3m_4} .. R_{\ell_{2p-1}\ell_{2p}}{}^{m_{2p-1}m_{2p}}$$

$$a_2 = \varepsilon_{J_1J_2}^{I_1I_2}\, Y_{I_1}^{J_1}\, Y_{I_2}^{J_2} = (\text{tr}\, Y)^2 - Y_{ij}^{hk}\, Y_{hk}^{ij}$$

$$\vdots$$

$$a_D = \varepsilon_{J_1..J_D}^{I_1..I_D}\, Y_{J_1}^{I_1} \dots Y_{J_D}^{I_D} = \det Y$$

a_q is a polynomial in the components $R_{ij}{}^{k\ell}$ of the Riemann tensor of g which can be expressed on S_o, using (3-1), in terms of the Cauchy data $\bar{g}$ and K.

<u>Theorem</u> Let $(S,\bar{g},K)$ be an analytic initial data set, satisfying the constraints and such that $\det(\mathbb{1} + \chi\, Y)_S \neq 0$. There exists an analytic space

time (V,g) taking these initial data and solution of the equations of stringy gravitation with Gauss-Bonnet terms.

The (analytic) lapse is arbitrary : to different choices of lapse correspond locally isometric space times.

Proof : The Cauchy-Kovalevski theorem.

Remark if χ, or if K and the Riemann tensor of $\bar{g}$, are small enough then $\det(\mathbb{1} + \chi Y)_S \neq 0$.

V - Characteristics as possible wave fronts.

In order to study possible propagation of Gauss-Bonnet gravity, and eventually get rid of the analyticity hypothesis in the solution of the Cauchy problem, we now look for the possible significant discontinuities of the second derivatives of the metric across a d-1 dimensional submanifold S of a given space time (V,g) solution of the equations of Gauss-Bonnet gravity. Such hypersurfaces are called wave fronts.

We know (cf Lichnerowicz[11]) that the significant discontinuities of the second derivatives of the metric - that is those which cannot be removed by a C^2 by pieces change of coordinates - are the discontinuities in $\partial^2_{oo}\, g_{ij}$ if S has local equation x^o = constant. The calculations made in the previous paragraph show that these discontinuities can occur across S if and only if

$$\det M|_S = 0 \quad . \qquad (5\text{-}1)$$

In arbitrary coordinates, where the equation of S is $f(x^\alpha) = 0$, the condition for S to be a wave front reads

$$\left(-\, g^{\lambda\mu}\, \partial_\lambda f\, \partial_\mu f\right)^D \left(1 + \chi\, a_1 + \ldots + \chi^D\, a_D\right) \neq 0$$

where a_q is an invariant polynomial of order (P-1)q in the tensor ρ, projection on S of the Riemann tensor. It can be seen using the expression of

ρ and the antisymmetry of the $\mathcal{E}$ tensor that a_q is only a polynomial of order 2q in n. We find for instance when P = 2

$$a_1 = C\, S^{\alpha\beta}\, n_\alpha\, n_\beta \qquad , \qquad (\text{using } n^\alpha\, n_\alpha = -1)$$

(result found by Aragone[10])

and a_2 is of the form, with C, C_o, C_1, C_2 numbers depending on d

$$a_2 = C_o \left(\rho^\alpha_\alpha\right)^2 + C_1\, \rho^\lambda_\alpha\, \rho^\alpha_\lambda + C_2\, \rho_{\alpha\beta}{}^{\lambda\mu}\, \rho^{\alpha\beta}{}_{\lambda\mu}$$

which, using antisymmetries and $n^\alpha\, n_\alpha = -1$, reduces to a polynomial of degree 4 in n.

We obtain the equation for the wave fronts by replacing $n_\alpha\, n_\beta$ in a_q by

$$n_\alpha\, n_\beta = - \frac{\partial_\alpha f\, \partial_\beta f}{g^{\lambda\mu}\, \partial_\lambda f\, \partial_\mu f}$$

and we see that the hypersurface S, f = constant can be a wave front if

$$\Delta \equiv \left(- g^{\lambda\mu}\, \partial_\lambda f\, \partial_\mu f\right)^D + \chi \left(- g^{\lambda\mu}\, \partial_\lambda f\, \partial_\mu f\right)^{D-1} b_1(\nabla f)$$

$$+ \chi^2 \left(- g^{\lambda\mu}\, \partial_\lambda f\, \partial_\mu f\right)^{D-2} b_2(\nabla f) + \ldots + \chi^D\, b_D(\nabla f) = 0$$

where $b_q(\nabla f)$ is an homogeneous polynomial of degree 2q in ∇f, whose coefficients vanish when the curvature tensor vanishes.

The wave front cone at a point of V, is a cone in the cotangent space, obtained by replacing $\partial_\lambda f$ by a covariant vector ξ_λ, of degree 2D. By taking the parameter χ small - or the curvature small - it is possible to insure that this cone remains in a region close to the null cone of the metric, using the property that this null cone is convex (cf 12), however there is no reason to consider that the product of the null cone and the cone $g^{\lambda\mu}\, \xi_\lambda\, \xi_\mu - \chi\, b_1(\xi) = 0$ approximates the full cone C.

VI - Harmonic coordinates.

Coordinates are harmonic if the metric satisfies the conditions

$$F^{\lambda} \equiv g^{\alpha\beta} \Gamma_{\alpha}{}^{\lambda}{}_{\beta} = 0 \qquad . \qquad (6\text{-}1)$$

It is well known that

$$R_{\alpha\beta} \equiv -\frac{1}{2} g^{\lambda\mu} \partial_{\lambda}{}^{2}{}_{\mu} g_{\alpha\beta} + g_{\alpha\lambda} \partial_{\beta} F^{\lambda} + g_{\beta\lambda} \partial_{\alpha} F^{\lambda} + h_{\alpha\beta}(g,\partial g) \qquad (6\text{-}2)$$

where $H_{\alpha\beta}$ depends only on the metric and its first derivative. We set

$$R^{(h)}_{\alpha\beta} \equiv -\frac{1}{2} g^{\lambda\mu} \partial^{2}_{\lambda\mu} g_{\alpha\beta} + h_{\alpha\beta} \qquad (6\text{-}3)$$

and

$$\tilde{A}^{(h)}_{\alpha\beta} \equiv R^{(h)}_{\alpha\beta} + \chi \tilde{B}_{\alpha\beta} \qquad (6\text{-}4)$$

where we do not truncate $B_{\alpha\beta}$ by the use of (6-1).

We deduce from the conservation identities that a solution of $\tilde{A}^{(h)}_{\alpha\beta} = 0$ satisfies the homogeneous wave equations in F^{λ} :

$$\nabla^{\alpha} \partial_{\alpha} F^{\lambda} = 0 \qquad . \qquad (6\text{-}5)$$

On the other hand a hyperbolic metric $g_{\alpha\beta}$ solution of $\tilde{A}^{(h)}_{\alpha\beta} = 0$ which satisfies the constraints, on S_o ($x^o = 0$) satisfies also

$$\partial_o F^{\lambda}|_{x^o=0} = 0 \qquad . \qquad (6\text{-}6)$$

We deduce from this remark :

<u>Proposition</u> A solution of $A^{(h)}_{\alpha\beta} = 0$ which satisfies the constraints on S_o and

$$F^{\lambda}|_{S_o} = 0 \tag{6-7}$$

satisfies $F^{\lambda} = 0$ in all the future of S_o, determined by the isotropic cone of the metric, under only mild regularity hypothesis (as necessary for the uniqueness theorem for the wave equations), and hence satisfies $A_{\alpha\beta} = 0$.

In contradiction with $A_{\alpha\beta} = 0$, the system $\tilde{A}^{(h)}_{\alpha\ \beta} = 0$ is of the Cauchy-Kovalevski type. Its characteristic determinant is non zero except on a cone, the characteristic cone.

However this cone is not the light cone : the elements of the characteristic determinant are

$$P^{\rho\sigma}_{\alpha\beta} = \frac{\partial\, \tilde{A}^{(h)}_{\alpha\ \beta}}{\partial\left(\partial^2_{\lambda\mu}\, g_{\rho\sigma}\right)}\, \xi_\lambda\, \xi_\lambda = -\frac{1}{2}\, g^{\lambda\mu}\, \xi_\lambda\, \xi_\mu\, \delta^\rho_\alpha\, \delta^\sigma_\beta \tag{6-8}$$

$$+ \chi\, \frac{\partial\, B_{\alpha\beta}}{\partial\left(\partial^2_{\lambda\mu}\, g_{\rho\sigma}\right)}\, \xi_\lambda\, \xi_\mu$$

where rows and columns are numbered by ordered pairs of indices $(\alpha\beta)$, $(\rho\sigma)$.

<u>Proposition</u> (cf 12) The characteristic determinant of the system

$$\tilde{A}^{(h)}_{\alpha\ \beta} = 0 \tag{6-9}$$

is, with $C = d^{-d(d+1)/2}$

$$\det P^{(h)} = C\, \left(g^{\lambda\mu}\, \xi_\lambda\, \xi_\mu\right)^d\, \Delta$$

where Δ is the polynomial giving the wave front cone determined in the previous paragraph.

Writing the Gauss-Bonnet gravity in harmonic coordinates as $A^{(h)}_{\alpha\ \beta} = 0$ introduces the isotropic cone as a spurious wave front cone, but preserves the true one.

Without further information on this cone : reality, simplicity, convexity, it is not easy to give more results on the general Cauchy problem for the classical system of partial differential equations of Gauss-Bonnet gravity.

VII - Shocks and High frequency waves.

The equations of gravitation with Gauss-Bonnet terms are fully non linear ; as such they offer a new challenge to the specialists in partial differential equations, in particular the study of shock waves as well as the determination of high frequency waves by asymptotic expansion could be untractable . However the non linear terms have a remarquable property , a consequence of which is the result obtained by Boulware and Deser that gravitational plane waves are also solution of the equations with Gauss-Bonnet terms . We now give this remarkable property in its full generality.

Theorem 1. If the Riemann tensor is of the pure radiative form , then the Gauss Bonnet correction in the equations $A_{\alpha\beta} = 0$ is identically zero .

Proof : The Riemann tensor is called purely radiative if it reads (Lichnerowicz 1961)

$$R_{\alpha\beta,\lambda\mu} = h_{\alpha\lambda} n_\beta n_\mu - h_{\beta\lambda} n_\alpha n_\mu + h_{\beta\mu} n_\alpha n_\mu - h_{\alpha\mu} n_\beta n_\lambda$$

The result follows from the antisymmetries of the Kronecker tensor .

The theorem 1 shows that if we try to determine shocks as solutions which admit discontinuities in the first derivatives of the metric across a hypersurface S we shall indeed find equations which do not contain the meaningless square of the measure $\delta(S)$. Unfortunately these equations still appear to contain in general the product of $\delta(S)$ by a function which is discontinuous across S . This is not defined , but it has been shown in special cases that such products in fact do not occur[21] and the study of the generic case remains to be done . We shall not pursue this way here , but instead we shall give some results about high frequency waves : it is a more flexible subject and perhaps more interesting physically in this context .

A metric g is said to represent an <u>high frequency wave</u> if it depends on the point x of the manifold V with two different scales : it is defined through a mapping from V x R into the space of metrics on V ,

$$(x , \xi) \mapsto g(x , \xi) , x \in V , \xi \in R$$

and by replacing ξ by the product $\omega \varphi(x)$, where ω is a large parameter and φ is a scalar function on V . With this hypothesis the partial derivatives of the components of g in local coordinates are given by , with obvious notations ,

$$\partial_\lambda g_{\alpha\beta} = \underline{\partial}_\lambda g_{\alpha\beta} + \omega\, g'_{\alpha\beta} n_\lambda \quad , \quad n_\lambda = \partial_\lambda \varphi \quad , \quad g'_{\alpha\beta} = \partial g_{\alpha\beta} / \partial\xi$$

and make appear , by choice of a large enough ω , a rapid variation of g in the direction n transversal to the submanifolds φ = constant , called submanifolds of constant phase , or sometimes wave fronts .

We then look for a solution of the system of partial differential equations under the form of a formal series in inverse powers of ω . When we report formally this series in the equations we obtain another formal series in these inverse powers and we say that we have an asymptotic solution of order p if all coefficients of powers of ω vanish up to the coefficient of ω^{-p} . An asymptotic solution is also an approximate solution if the remainder satisfies appropriate boundedness conditions .

There are several possible choices , already in general relativity, to look for these asymptotic expansions starting from a metric independant of ω called a background , and perturbating it either with terms of order ω^{-2} (cf 13, 16) or by terms of order ω^{-1} which induce a "back reaction" (cf 14, 15) : in this case the background metric cannot be a solution of the vacuum Einstein equations. In the case of the equations with Gauss Bonnet terms the situation is enriched by the presence of the coupling constant . We shall present below some results which take advantage of this possibility .

We consider the case where the coupling constant between the usual Einstein tensor and the Gauss Bonnet terms is of order ω^{-1} , for simplicity in writing we take only the quadratic polynomial in the Riemann curvature : the results are essentially the same if higher order terms are also considered . The equations read :

$$A_{\alpha\beta} \equiv S_{\alpha\beta} + \omega^{-1}B_{\alpha\beta} \quad , \quad B_{\alpha\beta} \equiv \epsilon^{\gamma\lambda_1\lambda_2\lambda_3\lambda_4}_{\alpha\mu_1\mu_2\mu_3\mu_4} R_{\lambda_1\lambda_2}{}^{\mu_1\mu_2} R_{\lambda_3\lambda_4}{}^{\mu_3\mu_4} \quad .$$

We look for a solution which is a perturbation of order ω^{-2} of some given metric $\underline{g}$ called background , independant of ω

$$g_{\alpha\beta}(x \; , \; \omega\, \varphi(x)) \equiv \underline{g}_{\alpha\beta}(x) + \omega^{-2}h_{\alpha\beta}(x \; , \; \omega\, \varphi(x)) + \omega^{-3}\kappa_{\alpha\beta}(x \; , \; \omega\, \varphi(x))$$

We suppose that h and k are uniformly bounded independently of ω as well as their first and second derivatives with respect to the variables x and $\xi = \omega\varphi$, so that this expansion is meaningfull, as well as the ones we shall now compute.

We first deduce from the above formula

$$g^{\alpha\beta} \equiv \underline{g}^{\alpha\beta} - \omega^{-2}h^{\alpha\beta} + \omega^{-3}K^{\alpha\beta} \quad , \quad h^{\alpha\beta} \equiv \underline{g}^{\alpha\rho}\, \underline{g}^{\beta\sigma}h_{\rho\sigma} \; , \; \kappa^{\alpha\beta} \text{ bounded} \quad .$$

We find for the Christoffel symbols

$$\Gamma^{\lambda}_{\alpha\beta} \equiv \underline{\Gamma}^{\lambda}_{\alpha\beta} + \omega^{-1}\, \gamma^{\lambda}_{\alpha\beta} + \omega^{-2}\, N^{\lambda}_{\alpha\beta}$$

where

$$\gamma^{\lambda}_{\alpha\beta} \equiv \tfrac{1}{2}\, \underline{g}^{\lambda\mu}\, (\, h'_{\alpha\mu}n_{\beta} + h'_{\beta\mu}n_{\alpha} - h'_{\alpha\beta}n_{\mu} \,)$$

which gives for the Riemann tensor

$$R_{\alpha\beta}{}^{\lambda}{}_{\mu} \equiv \underline{R}_{\alpha\beta}{}^{\lambda}{}_{\mu} + r_{\alpha\beta}{}^{\lambda}{}_{\mu} + \omega^{-1}\, T_{\alpha\beta}{}^{\lambda}{}_{\mu} + \omega^{-2}\, M_{\alpha\beta}{}^{\lambda}{}_{\mu} \quad ,$$

where r is of the purely radiative form defined previously : its components beeing raised or lowered with the background metric we have

$$r_{\alpha\beta,\lambda\mu} = \tfrac{1}{2}\ (\ h''_{\beta\lambda}n_\mu - h''_{\beta\mu}n_\lambda)\ n_\alpha\ -\ (h''_{\lambda\alpha}n_\mu - h''_{\alpha\mu}n_\lambda)\ n_\beta \quad .$$

The Ricci tensor is then given by

$$R_{\alpha\beta} \equiv \underline{R}_{\alpha\beta} + r_{\alpha\beta} + \omega^{-1}\ T_{\alpha\beta} + \omega^{-2}\ N_{\alpha\beta}$$

where underlines quantities are relative to the background metric $\underline{g}$ and $r_{\alpha\beta}$ is given by

$$r_{\alpha\beta} \equiv \tfrac{1}{2}\ \underline{g}^{\lambda\mu}\ (\ h''_{\lambda\beta}\ n_\alpha\ n_\mu + h''_{\lambda\alpha}n_\beta\ n_\mu - h''_{\lambda\mu}\ n_\alpha\ n_\beta - h''_{\alpha\beta}\ n_\lambda\ n_\mu\ )$$

We deduce from these expressions that the Gauss Bonnet equations admit the expansion , whose coefficients of powers of ω are uniformly bounded,

$$A_{\alpha\beta} \equiv \underline{S}_{\alpha\beta} + r_{\alpha\beta} - \tfrac{1}{2}\underline{g}_{\alpha\beta}r + \omega^{-1}\left(T_{\alpha\beta} - \tfrac{1}{2}\underline{g}_{\alpha\beta}T + 2\underline{g}_{\beta\gamma}\epsilon^{\gamma\lambda_1\lambda_2\lambda_3\lambda_4}_{\alpha\mu_1\mu_2\mu_3\mu_4}\underline{R}_{\lambda_1\lambda_2}{}^{\mu_1\mu_2}\ r_{\lambda_1\lambda_2}{}^{\mu_1\mu_2}\right) + \omega^{-2}M_{\alpha\beta}$$

To obtain this formula we have used the theorem 1 which shows that the quadratic term in the radiative part r vanishes identically. We see that if the high frequency wave is to satisfy the equations $A_{a\beta} = 0$ at the first order of approximation in ω^{-1} then we must have

$$\underline{S}_{\alpha\beta} + r_{\alpha\beta} - \tfrac{1}{2}\ \underline{g}_{\alpha\beta}r = 0 \quad , \quad \text{or equivalently,} \quad \underline{R}_{\alpha\beta} + r_{\alpha\beta} = 0$$

These equations are identical with the equations obtained in ordinary Einstein gravity when looking for high frequency waves at zero order in ω from known results in this case (cf 18) we deduce the following

Theorem 2. The high frequency wave is a significant perturbation of the background metric, solution of order zero in ω of the Einstein equations with a Gauss-Bonnet perturbation of order -1 in ω if and only if

1. The background metric is a solution of the vacuum Einstein equations $\underline{R}^{\alpha\beta} = 0$, and the wave fronts are null hypersurfaces of this metric,
2. The perturbation $h_{\alpha\beta}$ satisfies the relations

$$n_\alpha h - 2 n^\lambda h_{\alpha\lambda} = 0 \quad , \quad \text{with } h = \underline{g}^{\lambda\mu} h_{\lambda\mu} \quad , \quad n^\lambda = \underline{g}^{\lambda\mu} n_\mu \quad .$$

Note : a perturbation is called significant if it cannot be made to vanish by a high frequency change of coordinates of the same type as the wave itself.

Radiative coordinates. We have denoted by "radiative" coordinates in which the wave fronts $\varphi(x)$ = constant are the hypersurfaces x^0 = constant. In such coordinates the above results read:

$$\underline{g}^{00} = 0 \ , \ n_0 = 1 \ , \ n_i = 0 \ , \ n^0 = 0 \ , \ n^i = \underline{g}^{0i} \text{ and}$$

$$n^j h_{ij} = 0 \quad , \quad \underline{g}^{ij} h_{ij} = 0$$

We now look for the conditions which insure that our high frequency wave is a solution of the equations up to the order ω^{-2}. They will give us as usual in these problems propagation equations for the first order perturbation h.

The coefficients of the term in ω^{-1} in each equation is the sum of a term coming from the pure Einstein tensor and a term coming from the Gauss Bonnet correction. The first term has already been determined in previous works (cf 17, 18), in particular in radiative coordinates, it has been found that for the Ricci tensor these coefficients split between linear differential operators in the significant part of the perturbation h_{ij}

$$R^1_{ij} \equiv - n^\lambda \underline{\nabla}_\lambda h'_{ij} - \tfrac{1}{2} h'_{ij} \underline{\nabla}_\lambda n^\lambda$$

and terms $\overset{1}{R}_{0\alpha}$ which contain the non significant part $h_{0\alpha}$.

We shall now compute the Gauss Bonnet term. Due to the theorem 1 it will be of the form

$$\epsilon^{\cdot\cdot\cdot\cdot\cdot}_{\cdot\cdot\cdot\cdot\cdot}\ \underline{R}_{\cdot\cdot}{}^{\cdot\cdot}\ \underline{R}_{\cdot\cdot}{}^{\cdot\cdot} + 2\epsilon^{\cdot\cdot\cdot\cdot\cdot}_{\cdot\cdot\cdot\cdot\cdot}\ \underline{R}_{\cdot\cdot}{}^{\cdot\cdot}\ r_{\cdot\cdot}{}^{\cdot\cdot}$$

The first term of the sum is independent of $\omega\varphi$, it must vanish if we want the perturbation to be uniformly bounded in ω. We compute the second term in radiative coordinates.

We deduce from the values of $\underline{g}$ and n in radiative coordinates that all components of $r_{\alpha\beta\lambda\mu}$ are zero, except those which contain two zero indices which are given by:

$$r_{i0j0} = \tfrac{1}{2}\, h''_{ij}$$

from this result we deduce that, for a perturbation satisfying the theorem 2, all the components of $r_{\alpha\beta}{}^{\lambda\mu}$ are zero except those which contain only one zero index, situated moreover in a bottom position, and then given by :

$$r_{0i}{}^{hk} = \tfrac{1}{2}\ (\ n^h\ \underline{g}^{kj} - n^k\ \underline{g}^{hj}\ )\ h''_{ij}$$

We deduce from these facts that if we set

$$\Theta^{\alpha}_{\gamma} \equiv 2\ \epsilon^{\gamma\lambda_1\ldots\lambda_4}_{\alpha\mu_1\ldots\mu_4}\ \underline{R}_{\lambda_1\lambda_2}{}^{\mu_1\mu_2}\ \ r_{\lambda_3\lambda_4}{}^{\mu_3\mu_4}$$

we have in radiative coordinates

$$\Theta^{\alpha}_{\gamma} = 4\ \epsilon^{\alpha l_1 l_2 0\ i}_{\gamma\mu_1\mu_2 h\ k}\ \ r_{0i}{}^{hk}\ \underline{R}_{l_1\ l_2}{}^{\mu_1\mu_2}$$

hence

$$\Theta^{0}_{\alpha} = 0\quad ,\quad \Theta^{j}_{0} = -\ 8\ \epsilon^{jlpi}_{mqhk}\ n^h\ \underline{g}^{kn}\ h''_{in}\ \underline{R}_{lp}{}^{qh}\quad ,$$

$$\Theta^{c}_{j} = 16\ \epsilon^{clpi}_{jmhk}\ n^h\ \underline{g}^{kn}\ h''_{in}\ \underline{R}_{lp}{}^{0m}\quad ,\quad \Theta \equiv \Theta^{\alpha}_{\alpha} = \Theta^{j}_{j}$$

The equations satisfied by the significant part h_{ij} of the perturbation read

$$\overset{1}{R}_{ij} + g_{ic}\,\Theta^{c}_{j} - g_{ij}\,\frac{\Theta}{d-2} = 0 \qquad \text{(P)}$$

We shall enunciate as a theorem the results we have just obtained.

Theorem 3. For a general background metric g, a high frequency wave solution to order zero in ω of the Einstein equations with Gauss Bonnet terms (coupling constant $\chi = \omega^{-1}$) is also solution to order ω^{-1} if

1. The background metric annuls the Gauss Bonnet term.

2. The significant part of the perturbation h_{ij} satisfies the system of differential equations (P), called propagation equations. these are linear differential equations of the first order in h'_{ij} which contain both a derivation in the direction of the "light ray" n and a derivation with respect to the supplementary variable $\xi = \omega\varphi$.

The fact that the derivative h''_{ij} does appear in the propagation equation shows that signals will be, in the case of a generic background, distorted along their propagation (cf 19); the wave fronts $\varphi(x)$ = constant will not be exceptionnal in the sense of Lax and Boillat (cf 20).

Many developments can originate from this general study. But some idea of physical applications would be usefull both to motivate the effort and to guide the researcher. Physical situations where these partial differential equations could be relevant are considered by R. Kerner and collaborators (cf 8, 9, 22).

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