

REGULARITY AND NONREGULARITY FOR
SOLUTIONS OF PRESCRIBED CURVATURE EQUATIONS

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Our aim here is to describe some recent results and conjectures concerning the equations of prescribed m -th mean curvature

$$(1) \quad H_m[u] = g(x)$$

on domains $\Omega \subset \mathbb{R}^n$. Here $H_m[u]$ is defined by

$$(2) \quad H_m[u] = S_m(\kappa_1, \dots, \kappa_n) = \sum_{1 \leq i_1 < \dots < i_m \leq n} \kappa_{i_1} \dots \kappa_{i_m}$$

for any integer m such that $1 \leq m \leq n$, where $\kappa_1, \dots, \kappa_n$ are the principal curvatures, relative to the upward normal, of the graph of the function u . The left hand side of (1) is therefore a function of Du and D^2u , so (1) is a nonlinear second order equation. The cases $m = 1$, $m = 2$ and $m = n$ correspond to the mean, scalar and Gauss curvatures respectively. The mean and Gauss curvature equations have been studied intensively and are quite well understood. The remaining cases are not as well understood, but results for the extreme cases $m = 1$ and $m = n$ give some indication of what one may expect in the general case.

Except for the case $m = 1$, equation (1) is not elliptic on all functions $u \in C^2(\Omega)$. However, Caffarelli, Nirenberg and Spruck [3], [4] and Ivochkina [9] have shown that (1) is elliptic for functions $u \in C^2(\Omega)$ such that at each point of Ω the vector of principal curvatures of the graph of u belongs to the convex cone $\Gamma_m = \Gamma_m^n$ which is the component containing the positive cone $\Gamma_+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \forall i\}$ of the set in \mathbb{R}^n on which S_m is positive. For such solutions to exist we must obviously assume that g is positive in Ω . We shall call such u *admissible*. If the vector of

principal curvatures of the graph of u belongs to $\bar{\Gamma}_m$, then (1) is degenerate elliptic, and for such u to exist we must assume that g is nonnegative. Analogously, we shall say that a C^2 domain $\Omega \subset \mathbb{R}^n$ is m -admissible for some integer m with $0 \leq m \leq n-1$ if at each point of $\partial\Omega$ the vector $(\bar{\kappa}_1, \dots, \bar{\kappa}_{n-1})$ of principal curvatures of $\partial\Omega$ relative to the inner normal belongs to the cone Γ_m^{n-1} . We define $\Gamma_0^{n-1} = \mathbb{R}^{n-1}$.

It is shown in [3], [9] that

$$(3) \quad \Gamma_+ = \Gamma_n \subset \Gamma_{n-1} \subset \dots \subset \Gamma_1 = \{\lambda \in \mathbb{R}^n : \Sigma \lambda_i > 0\} .$$

Alternative characterizations of the cones Γ_m are also known (see [9], [13]), but we shall not be concerned with these. It is also known that $S_m^{1/m}$ is concave on Γ_m (see [3], [9]). Consequently $H_m[u]^{1/m}$ is a concave function of D^2u if u is admissible, and as a result of the well known global second derivative Hölder estimates for fully nonlinear uniformly elliptic equations proved in [6], [14], the solvability of the Dirichlet problem

$$(4) \quad H_m[u] = g(x) \text{ in } \Omega ,$$

$$u = \varphi \text{ on } \partial\Omega$$

is reduced to the derivation of a priori estimates for admissible solutions of (4) and their first and second derivatives on $\bar{\Omega}$. In the mean curvature case the necessary solution and global gradient estimates were found by Serrin [22], Bakel'man [1] and Giaquinta [5]. A number of mathematicians also derived interior gradient bounds, leading to the solvability of the Dirichlet problem with less regular boundary data (see [6]). In the Gauss curvature case one cannot generally solve the Dirichlet problem directly, for reasons which we shall mention later, so one needs to first solve suitable approximating Monge-Ampère equations. Many authors contributed to various aspects of this procedure (see for example [2], [8], [18], [19], [27]), with results specifically for the equation of prescribed Gauss curvature being obtained by Lions [18], Ivochkina [8] and Trudinger and Urbas [27].

In the last few years progress has been made on the Dirichlet problem (4) for the intermediate cases as well. Caffarelli, Nirenberg and Spruck [4] and Ivochkina [10], [11] independently solved the Dirichlet problem (4) on sufficiently smooth, uniformly convex domains with constant boundary data. More recently there have been two important developments pertaining to the Dirichlet problem (4). The first of these is the derivation by Ivochkina [12] of global second derivative bounds for admissible solutions of (4) in the case $1 < m < n$, $g \in C^{1,1}(\bar{\Omega})$ is positive, $\varphi \in C^{3,1}(\bar{\Omega})$ and $\partial\Omega \in C^{3,1}$ is m -admissible. Some curvature condition is of course necessary for the solvability of the Dirichlet problem (4) for arbitrary smooth boundary data.

The second important contribution was made by Trudinger [23], [24], [25], [26] who established precise necessary conditions on the function g for the existence of an admissible solution of (1), and in addition, obtained a priori estimates for u and Du on $\bar{\Omega}$ for admissible solutions of the Dirichlet problem (4), as well as for solutions of more general curvature equations. Coupled with a suitable regularization procedure this led to a proof of the existence and uniqueness of Lipschitz continuous generalized solutions (in a sense to be explained below) of the Dirichlet problem (4) under natural geometrical restrictions on the domain and under relatively weak regularity hypotheses on the data. Ivochkina [11] also proved solution and gradient estimates for admissible solutions of (4), but her hypotheses are stronger than those of Trudinger [24], [26]. The combination of the results of Trudinger and Ivochkina in the case $m < n$, and those of Trudinger and Urbas [27] in the case $m = n$, leads to the following existence theorem for classical solutions of the Dirichlet problem (4).

THEOREM 1. *a) Assume $1 \leq m < n$ and let Ω be a $C^{3,1}$ bounded domain in \mathbb{R}^n with m -admissible boundary. Let $g \in C^{1,1}(\bar{\Omega})$ be positive on $\bar{\Omega}$ and assume that the following conditions are satisfied:*

- (i) *There is a constant $\chi > 0$ such that for every set $E \subset \Omega$ with C^2 $(m-1)$ -admissible boundary ∂E we have*

$$(5) \quad \int_E g \leq \frac{1-\chi}{m} \int_{\partial E} H_{m-1}[\partial E]$$

where $H_{m-1}[\partial E]$ denotes the $m-1$ curvature of ∂E (we take $H_0 \equiv 1$).

(ii) At any point of $\partial\Omega$ we have

$$(6) \quad S_m(\tilde{\kappa}_1, \dots, \tilde{\kappa}_{n-1}, 0) \geq g$$

where $\tilde{\kappa}_1, \dots, \tilde{\kappa}_{n-1}$ are the principal curvatures of $\partial\Omega$ relative to the inner normal. Then for any $\varphi \in C^{3,1}(\bar{\Omega})$ the Dirichlet problem (4) has a unique admissible solution $u \in C^2(\bar{\Omega})$.

b) If $m = n$, Ω is a $C^{1,1}$ uniformly convex domain in \mathbb{R}^n , $g \in C^{1,1}(\Omega) \cap C^{0,1}(\bar{\Omega})$ is positive in Ω and conditions i) and ii) are satisfied, then for any $\varphi \in C^{1,1}(\bar{\Omega})$ the Dirichlet problem (4) has a unique admissible solution $u \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$.

In the mean curvature case it suffices to assume the weaker regularity hypotheses $\partial\Omega, \varphi \in C^{2,\alpha}$ for some $\alpha > 0$ and $g \in C^{0,1}(\bar{\Omega})$ (see [6], Theorem 16.10). The Gauss curvature case is somewhat anomalous, and we shall explain the reasons for this shortly.

Condition (i) is used to prove an a priori bound for the solution u . In fact, the results of Trudinger [24], [26] show that the condition

$$(7) \quad \int_E g \leq \frac{1}{m} \int_{\partial E} H_{m-1}[\partial E]$$

for any set $E \subset \Omega$ with C^2 $(m-1)$ -admissible boundary ∂E , with strict inequality unless $E = \Omega$, is necessary for the existence of an admissible solution $u \in C^2(\Omega)$ of (1) on a bounded domain $\Omega \subset \mathbb{R}^n$, while the stronger condition (i) is necessary for the existence of an admissible solution $u \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$. This is proved explicitly in the case $u = 0$ on $\partial\Omega$ in [24], while the general case follows from this and the

existence results of [25]. In the mean curvature case condition (i) reduces to the well known condition (see [5])

$$(8) \quad \int_E g \leq (1-\chi)\mathcal{H}^{n-1}(\partial E)$$

for any set $E \subset \Omega$ with $\partial E \in C^2$, where \mathcal{H}^{n-1} is the $n-1$ dimensional Hausdorff measure, while in the Gauss curvature case it reduces to the simple inequality

$$(9) \quad \int_{\Omega} g \leq (1-\chi)\omega_n,$$

where ω_n is the measure of the unit ball in \mathbb{R}^n .

Condition (ii) is used to prove a boundary gradient estimate. In the case $m=1$ it reduces to the well known condition of Serrin [22] and Bakel'man [1]. In the Gauss curvature case it reduces to the condition

$$(10) \quad g = 0 \text{ on } \partial\Omega$$

which makes the equation degenerate on $\partial\Omega$. This condition is necessary if we wish to solve the Dirichlet problem for arbitrary smooth boundary data (see [27], [30]). Ivochkina [8] has shown that if $\|\varphi\|_{C^2(\bar{\Omega})}$ and $\sup_{\Omega} g$ are small enough, barriers may be constructed in the Gauss curvature case without assuming (10). If g is positive on $\bar{\Omega}$ and $\partial\Omega$, $\varphi \in C^{3,1}$, we then obtain a solution $u \in C^2(\bar{\Omega})$. In general, however, the degeneracy condition (10) precludes us from obtaining global $C^{1,1}$ or higher estimates in the Gauss curvature case, and for this reason the regularity hypotheses on the boundary data may be weakened to $\partial\Omega$, $\varphi \in C^{1,1}$ in this case. If $\partial\Omega \in C^{2,1}$, $g^{1/n} \in C^{1,1}(\bar{\Omega})$ and $\varphi \equiv 0$, we have $u \in C^{1,1}(\bar{\Omega})$ (see [27]), but in general this degree of global regularity is not known for nonconstant boundary data, even under stronger regularity hypotheses. However, some recent work of Krylov [15], where he establishes the analogous assertions for equations involving symmetric functions of the eigenvalues of the Hessian of u , suggests that this will be the case.

As mentioned above, Trudinger [25] has proved the existence of Lipschitz continuous generalized solutions of (4) under the conditions of Theorem 1, but with the weaker regularity hypotheses $\partial\Omega \in C^2$, $g \in C^{0,1}(\bar{\Omega})$ and $\varphi \in C^{1,1}$ or even $\varphi \in C^0$, in which case the solutions are only locally Lipschitz continuous. Nonnegative g with $g^{1/m} \in C^{0,1}(\bar{\Omega})$ are also permitted. It is of some interest therefore to know whether these solutions are smooth in the interior of Ω if $g \in C^{1,1}(\Omega)$ is positive. Before addressing this question, let us recall the definition of a generalized solution of (1) in the sense of [25].

Definition. A function $u \in C^0(\Omega)$ is said to be a *generalized* or *viscosity subsolution* (*supersolution*) of (1) if for any admissible function $\varphi \in C^2(\Omega)$ and any local maximum (minimum) $x_0 \in \Omega$ of $u - \varphi$ we have

$$H_m[\varphi](x_0) \geq g(x_0) \quad (\leq g(x_0)) .$$

u is said to be a *generalized* or *viscosity solution* if it is both a generalized or viscosity subsolution and supersolution.

This definition is the one used in [32] and it differs slightly from the one of [25] in that in the subsolution case, all $\varphi \in C^2(\Omega)$ are allowed as comparison functions in [25]. However the two notions are equivalent, at least for positive g . Perhaps the most important point to observe is that any C^2 admissible solution is a generalized solution, and conversely, any generalized solution of (1) which is of class C^2 is admissible (see [25], [32]). Thus the notion of ellipticity is implicit in the definition. It is also not difficult to verify that this notion of generalized solution is stable with respect to uniform convergence. Furthermore, Trudinger [25] has proved comparison principles which imply the uniqueness of generalized solutions of (4) if g is positive.

Let us now return to the question of regularity for generalized solutions. We have recently proved the following negative result (see [32]).

THEOREM 2. *For any integers m, n such that $3 \leq m \leq n$ and any positive*

function $g \in C^\infty(\bar{B}_1)$, where B_1 is the open unit ball in \mathbb{R}^n , there exist an $\epsilon > 0$ and a generalized solution $u \in C^{0,1}(\bar{B}_\epsilon)$ of (1) such that u does not belong to $C^{1,\alpha}(B_\epsilon)$ for any $\alpha > 1 - 2/m$.

The proof of Theorem 2 uses a result of Pogorelov [19], which asserts that for $m \geq 3$ the function

$$(11) \quad w(x_1, \dots, x_m) = (1+x_1^2) \left[\sum_{k=2}^m x_k^2 \right]^{1-1/m}$$

is a convex generalized solution of the equation

$$(12) \quad \det D^2 w = f(x_1, \dots, x_m) = (2\beta)^{m-1} (1+x_1^2)^{m-2} (\beta-1 - (\beta+1)x_1^2)$$

in a small ball $B_{\epsilon_0}^m \subset \mathbb{R}^m$, where $\beta = 2 - 2/m$. Evidently $w \in C^{1,1-2/m}(\overline{B_{\epsilon_0}^m})$. If we now set $v = Aw$ for a large positive constant A , and extend v to be constant in the x_{m+1}, \dots, x_n coordinate directions, we see that

$$H_m[v] \geq g \text{ in } B_\epsilon$$

in the generalized sense for sufficiently small $\epsilon > 0$. Furthermore, the function

$$v_0(x) = 2A \left[\sum_{k=2}^m x_k^2 \right]^{1-1/m}$$

solves

$$H_m[v_0] = 0 \text{ in } B_\epsilon$$

in the generalized sense, and $v_0 \geq v$ in B_ϵ with equality on $\{x \in B_\epsilon : x_2 = \dots = x_m = 0\}$. By a suitable approximation argument we may now find a generalized solution $u \in C^{0,1}(\bar{B}_\epsilon)$ of

$$H_m[u] = g \text{ in } B_\epsilon,$$

$$u = v_0 \text{ on } \partial B_\epsilon,$$

and by the comparison principle for generalized solutions we have

$$v_0 \geq u \geq v \text{ in } B_\epsilon .$$

Since equality holds on $\{x \in B_\epsilon : x_2 = \dots = x_m = 0\}$, it follows that u cannot be of class $C^{1,\alpha}(B_\epsilon)$ for any $\alpha > 1-2/m$.

In [32] we have also proved similar assertions for equations of the form

$$(13) \quad F_m(D^2u) = g(x,u,Du) ,$$

where $F_m(D^2u)$ is the m -th elementary symmetric function of the eigenvalues of D^2u .

It should be noted that the technique of Theorem 2 can be used to construct singular solutions of (1) whenever we can find convex functions u_1 and u_2 on some ball $B_\epsilon^m \subset \mathbb{R}^m$ such that

$$\det D^2u_1 \leq \inf_{B_1} g , \quad \sup_{B_1} g \leq \det D^2u_2$$

in B_ϵ^m in the generalized sense, and $u_1 \geq u_2$ in B_ϵ^m with equality holding on a nonempty set $E \subset B_\epsilon^m$ on which u_1 and u_2 fail to be of class $C^{1,1}$. Thus, for example, it is clear that we may construct generalized solutions of (1) for $m \geq 3$ which are at best of class $C^{1,1/3}$. We simply observe, by setting $m = 3$ in (11), that

$$w(x_1, x_2, x_3) = (1+x_1^2)(x_2^2 + x_3^2)^{2/3}$$

is a convex generalized solution of an equation

$$\det D^2u = f(x_1, x_2, x_3) > 0$$

in $B_{\epsilon_0}^3 \subset \mathbb{R}^3$ for $\epsilon_0 > 0$ small enough. Letting

$$\tilde{w}(x_1, \dots, x_m) = w(x_1, x_2, x_3) + \frac{1}{2} \sum_{k=4}^m x_k^2 ,$$

we see that \tilde{w} solves

$$\det D^2 \tilde{w} = f(x_1, x_2, x_3)$$

in $B_{\epsilon_0}^m$ in the generalized sense. We now proceed as before, with w replaced by \tilde{w} and v_0 by $\tilde{v}_0(x) = 2A(x_2^2 + x_3^2)^{2/3} + \frac{1}{2} \sum_{k=4}^m x_k^2$.

We also note that we cannot obtain singular solutions of (1) in the case $m = 2$ using the technique of Theorem 2, because any generalized solution of

$$(14) \quad \det D^2 u = g(x, u, Du)$$

in a domain $\Omega \subset \mathbb{R}^2$ is of class $C^{2,\alpha}$ if g is positive and of class $C^{0,\alpha}$ for some $\alpha \in (0,1)$ (see Sabitov [20] and Schulz [21]).

Theorems 1 and 2 naturally lead us to ask a number of questions. First, how regular must a generalized solution of (1) be on $\partial\Omega$ in order that u have classical regularity in the interior of Ω , assuming g is positive and sufficiently smooth, say $g \in C^{1,1}(\Omega)$? Second, what kind of interior regularity results can one obtain without assuming any regularity of the solution on $\partial\Omega$? Third, do there exist singular solutions in the scalar curvature case? These questions are primarily of interest for the cases $m \geq 2$, because it is known that in the mean curvature case we have classical interior regularity if $g \in C^{0,1}(\Omega)$, regardless of the behaviour of u on $\partial\Omega$. At present we do not have complete answers to these questions, but the Gauss curvature case is reasonably well understood. We have the following interior regularity theorem in the case $m = n$ (see [31]).

THEOREM 3. *Suppose Ω is a bounded convex domain in \mathbb{R}^n and $u \in C^0(\bar{\Omega})$ is a generalized solution of*

$$(15) \quad H_n[u] = g(x) \text{ in } \Omega ,$$

where $g \in C^{1,1}(\Omega)$ is a positive function. If $\partial\Omega$ and $u|_{\partial\Omega}$ are of class $C^{1,\alpha}$ for some $\alpha > 1-2/n$, then $u \in C^2(\Omega)$, and for any $\Omega' \subset\subset \Omega$ we have an estimate

$$(16) \quad \sup_{\Omega'} |D^2u| \leq C$$

where C depends only on $n, \alpha, \Omega, \Omega', \sup_{\Omega} |u|, \|u\|_{C^{1,\alpha}(\partial\Omega)}, g$ and its first and second derivatives, and the modulus of continuity of u on $\partial\Omega$.

Theorem 3 is in fact valid for general Monge–Ampère equations of the form (14) with positive $g \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Furthermore, we do not need to assume $\partial\Omega, u|_{\partial\Omega} \in C^{1,\alpha}$ for some $\alpha > 1-2/n$; any modulus of continuity better than $C^{1,1-2/n}$ suffices.

Comparing Theorems 1 and 2 it is tempting to make the following conjecture.

CONJECTURE 1. *Let $u \in C^0(\bar{\Omega})$ be a generalized solution of (1) on a bounded domain $\Omega \subset \mathbb{R}^n$, possibly with $\partial\Omega$ $(m-1)$ -admissible, and assume that $g \in C^{1,1}(\Omega)$ is positive. If $\partial\Omega$ and $u|_{\partial\Omega}$ are of class $C^{1,\alpha}$ for some $\alpha > 1-2/m$, then $u \in C^2(\Omega)$ and for any $\Omega' \subset\subset \Omega$ there is an estimate of the form (16).*

Theorem 3 evidently implies that we cannot construct singular solutions u of (1) with $u|_{\partial\Omega}$ of class $C^{1,\alpha}$ for some $\alpha > 1-2/m$ by the method used to prove Theorem 2. Of course, this does not exclude the possibility that singularities may arise in some other way, but we do not believe that this will be the case.

The important point in proving the above conjecture is the derivation of an estimate such as (16) for smooth solutions; higher estimates then follow from known results for uniformly elliptic equations (see [6], [14]), and the result for generalized solutions follows by an approximation argument. The interior gradient bounds of Korevaar [13], or more precisely, their generalizations proved in [25], imply that

generalized solutions of (1), with g nonnegative and $g^{1/m} \in C^{0,1}(\Omega)$, are locally Lipschitz continuous. It is not known in general whether any higher regularity holds for example if $g \in C^{1,1}(\Omega)$ is positive.

We now move on to the question of interior regularity without any hypotheses on $u|_{\partial\Omega}$. We have recently proved the following interior regularity theorem for the equation of prescribed Gauss curvature (see [33]). Recall that the necessary condition (7) in this case takes the form

$$(17) \quad \int_{\Omega} g \leq \omega_n .$$

THEOREM 4. *Let u be a generalized solution of (15) in a uniformly convex domain $\Omega \subset \mathbb{R}^n$, where $g \in C^2(\Omega) \cap L^n(\Omega)$ is a positive function. Then for any $\theta > 0$ such that $\Omega_{\theta} = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \theta\} \neq \emptyset$, there is a number $\delta = \delta(n, \Omega, g, \theta) > 0$ such that if*

$$(18) \quad \int_{\Omega} g \geq \omega_n - \delta ,$$

then $u \in C^2(\bar{\Omega}_{\theta})$ and we have

$$(19) \quad \sup_{\Omega_{\theta}} |D^2u| \leq C$$

where C depends only on n, Ω, g and θ . Furthermore

$$(20) \quad \lim_{\theta \rightarrow 0^+} \delta = 0 .$$

Alternatively, for any $\delta > 0$ there is a number $\theta = \theta(n, \Omega, g, \delta) > 0$ such that if (18) holds, then $u \in C^2(\bar{\Omega}_{\theta})$ if $\Omega_{\theta} \neq \emptyset$, and (19) holds with C depending only on n, Ω, g and δ . Furthermore

$$(21) \quad \lim_{\delta \rightarrow 0^+} \theta = 0 .$$

In [33] we also showed that Theorem 4 cannot be qualitatively improved in the sense that if (9) holds for some $\chi > 0$, then there may be singularities near $\partial\Omega$.

In view of Theorem 4 we make the following conjecture.

CONJECTURE 2. *Let Ω be a C^2 bounded domain in \mathbb{R}^n , possibly with $\partial\Omega$ $(m-1)$ -admissible, and let $g \in C^{1,1}(\bar{\Omega})$ be a positive function such that (7) holds for any set $E \subset \Omega$ with C^2 $(m-1)$ -admissible boundary, with strict inequality if $E \neq \Omega$ and equality if $E = \Omega$. Then for any $\Omega' \subset \subset \Omega$ there exist two positive numbers ϵ and M such that if u is a generalized solution of (1) on Ω_ϵ , we have*

$$(22) \quad \sup_{\Omega'} |D^2 u| \leq M .$$

The estimate (22) has been proved by Giusti [7] in the case $m = 1$. In fact, Giusti proves an estimate for $|Du|$ rather than $|D^2 u|$, but (22) then follows by virtue of elliptic regularity theory [6], Lemma 17.16, since in this case the equation is uniformly elliptic once the gradient is bounded. Of course, for the case $m = 1$ this result is not so interesting as far as regularity theory is concerned. Nevertheless, it provides a reason for believing that analogous results are true for the other cases.

Let us now make some remarks about the case $m = 2$. If $m = n = 2$, we are dealing with the two dimensional prescribed Gauss curvature equation

$$\det D^2 u = g(x)(1 + |Du|^2)^2$$

in $\Omega \subset \mathbb{R}^2$ which, as we have already observed, has no singular solutions if g is positive and of class $C^{0,\alpha}$ for some $\alpha > 0$. We have also observed that the technique of Theorem 2 cannot be used to construct singular solutions of (1) if $2 = m < n$. This leads us to make the following conjecture.

CONJECTURE 3. *If u is a bounded generalized solution of (1) in the case $2 = m < n$ and $g \in C^{1,1}(\Omega)$ is positive, then $u \in C^2(\Omega)$, and for any $\Omega' \subset \subset \Omega$ there is an estimate of the form (16).*

Most of the previous results have been concerned with the interior regularity of generalized solutions of (1). We would like to conclude by mentioning a recent result concerning the boundary regularity of solutions of the equation of prescribed Gauss curvature (see [34]).

THEOREM 5. *Let Ω be a bounded domain in \mathbb{R}^n with a relatively open, connected, uniformly convex, $C^{2,1}$ portion $\Gamma = \partial\Omega \cap B_{R_0}(0)$, where $0 \in \partial\Omega$. Suppose that $u \in C^2(\Omega)$ is a convex solution of the equation of prescribed Gauss curvature (15) where $g \in C^{1,1}(\bar{\Omega})$ satisfies*

$$(23) \quad \mu_0 \leq g \leq \mu_1$$

and

$$(24) \quad |Dg| + |D^2g| \leq \mu_2$$

for some positive constants μ_0, μ_1 and μ_2 , and assume that

$$(25) \quad |Du| = \infty \text{ on } \Gamma.$$

Then there is a number $\rho > 0$, depending only on $n, R_0, \Gamma, \mu_0, \mu_1$, and μ_2 , such that the following are true:

(i) $u \in C^{0,1/2}(\overline{\Omega \cap B_\rho})$ and for any $x, y \in \overline{\Omega \cap B_\rho}$ we have

$$(26) \quad |u(x) - u(y)| \leq C_1 |x-y|^{1/2}$$

where C_1 depends only on $n, R_0, \Gamma, \mu_0, \mu_1$, and μ_2 .

(ii) $\text{graph } u \Big|_{\overline{\Omega \cap B_\rho}}$ is a $C^{2,\alpha}$ hypersurface for any $\alpha \in (0,1)$ and the unit normal vector field to $\text{graph } u$, $\nu = \frac{(Du, -1)}{\sqrt{1+|Du|^2}}$, satisfies

$$(27) \quad \|\nu\|_{C^{1,\alpha}(\overline{(\Omega \cap B_\rho) \times \mathbb{R}}) \cap \text{graph } u}} \leq C_2$$

where C_2 depends only on $n, R_0, \Gamma, \mu_0, \mu_1, \mu_2$ and α .

(iii) $u|_{\overline{\Gamma \cap B_\rho}}$ is of class $C^{1,\alpha}$ for any $\alpha \in (0,1)$ and we have

$$(28) \quad \|u\|_{C^{1,\alpha}(\overline{\Gamma \cap B_\rho})} \leq C_3$$

where C_3 depends only on $n, R_0, \Gamma, \mu_0, \mu_1, \mu_2, \alpha$ and $\inf_\Omega u$.

If Γ and g are more regular, then we get correspondingly better regularity assertions in (ii) and (iii).

The boundary condition (25) may seem unusual, but in fact it arises quite naturally if we attempt to solve the Dirichlet problem for (15) assuming condition (9) but not condition (10). In this case it is not generally possible to satisfy the Dirichlet boundary condition everywhere on $\partial\Omega$ (see [27], [30]), but as shown in [29], there is a convex solution u which satisfies it in a certain optimal sense. At points at which it is not satisfied in the classical sense, we have $|Du| = \infty$, provided $g \in L^n(\Omega)$.

The boundary condition (25) also arises in the extremal case

$$(29) \quad \int_\Omega g = \omega_n.$$

In this case it is shown in [28], [30] that if Ω is a uniformly convex domain, then there is a generalized solution u of (15) which is unique up to additive constants. Furthermore, if $g \in C^2(\Omega) \cap L^n(\Omega)$ is positive, then $|Du| = \infty$ everywhere on $\partial\Omega$ and $u \in C^2(\Omega)$.

Similar phenomena occur for the prescribed mean curvature equation when either of conditions i) or ii) of Theorem 1 is not satisfied (see [5], [7]), and in fact, boundary regularity results for the mean curvature equation which are completely analogous to Theorem 5 have been proved by Lin [16], [17]. We expect that similar assertions will prove to be true for the other curvature equations.

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