

OPERATORS OF TYPE ω WITHOUT A BOUNDED H_∞ FUNCTIONAL CALCULUS

*Alan McIntosh*¹ and *Atsushi Yagi*

1. Introduction

At the previous C.M.A. Miniconference which was held at Macquarie University, one of us considered operators T of type ω acting in a Hilbert space \mathcal{H} , and listed several conditions equivalent to such an operator T having a bounded H_∞ functional calculus [M^cI]. We shall list these again shortly. It is sometimes asked whether every operator of type ω satisfies one or other of these conditions, so we would like to take this opportunity to show that they do not. On other occasions we have considered operators T of type ω with respect to a double sector, and discussed the conditions under which such a T has a bounded H_∞ functional calculus, which implies in particular that T has bounded spectral projections associated with each sector. An example of an operator with no such bounded projections will also be presented.

In the next section we shall recall some results from [M^cI] and other papers. We shall then define and study some operators which will be used in the following two sections to construct the counter-examples. These examples are really modifications of those presented in earlier papers [M^c,1,2,3,4], and their existence comes as no surprise to those who are familiar with this material.

In the final section we shall show that every operator T of type ω does have a bounded H_∞ functional calculus if it is considered as acting in a Hilbert space \mathcal{H}_T in which the norm of \mathcal{H} is replaced by a square function norm.

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2. Operators of type ω

We shall consider operators T in a Hilbert space \mathcal{H} over the complex field \mathbb{C} . By this we mean that T is a linear mapping from its domain $\mathcal{D}(T) \subset \mathcal{H}$ to \mathcal{H} , where $\mathcal{D}(T)$ is a linear subspace of \mathcal{H} .

We shall need the sets, defined for $0 \leq \mu < \pi$, by

$$S_{\mu+} = \{ z \in \mathbb{C} : \text{larg}(z) \leq \mu \text{ or } z = 0 \}, \quad S_{\mu-} = -S_{\mu+}, \quad S_{\mu} = S_{\mu+} \cup S_{\mu-} \quad (\mu < \frac{1}{2}\pi)$$

which are closed sets with interiors $S_{\mu+}^{\circ}$, $S_{\mu-}^{\circ}$ and S_{μ}° . We shall employ the spaces of functions defined on these open sets by

$$H_{\infty}(S_{\mu+}^{\circ}) = \{ f : S_{\mu+}^{\circ} \rightarrow \mathbb{C} \mid f \text{ is holomorphic and } \|f\|_{\infty} < \infty \}$$

which are Banach spaces with norms $\|f\|_{\infty} = \sup\{|f(z)| : f \in S_{\mu+}^{\circ}\}$, and $H_{\infty}(S_{\mu}^{\circ})$, which are defined likewise and are also Banach spaces, as well as

$$\Psi(S_{\mu+}^{\circ}) = \{ \psi \in H_{\infty}(S_{\mu+}^{\circ}) : \exists s > 0, c \geq 0 \text{ such that } |\psi(z)| \leq \frac{c|z|^s}{1+|z|^{2s}} \text{ for all } z \in S_{\mu+}^{\circ} \}.$$

Definition. An operator T in \mathcal{H} is said to be of type ω (or type $\omega+$) provided that T is a closed operator in \mathcal{H} , its spectrum $\sigma(T)$ is a subset of S_{ω} (or $S_{\omega+}$), and for all $\mu > \omega$ there is a number c_{μ} such that $\|(T-\lambda)^{-1}\| \leq c_{\mu}|\lambda|^{-1}$ whenever $\lambda \notin S_{\mu}^{\circ}$ (or $\lambda \notin S_{\mu+}^{\circ}$).

Every such operator has domain $\mathcal{D}(T)$ dense in \mathcal{H} . In this talk we shall assume that T is one-one, which (for operators of type ω or $\omega+$) is equivalent to T having range $\mathcal{R}(T)$ dense in \mathcal{H} . (There is a slight change of terminology from [M^cI] where type ω was defined with respect to $S_{\omega+}$.) Actually, if T is not one-one, then $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{R}$ where $\mathcal{N}(T)$ is the kernel of T and \mathcal{R} is the closure of $\mathcal{R}(T)$, and we can focus our attention on $T|_{\mathcal{R}}$, which is a one-one operator of type ω or $\omega+$ in the Hilbert space \mathcal{R} . (The direct sum \oplus is typically not -orthogonal.)

Every operator T of type ω (or type $\omega+$) has a natural functional calculus defined over $H_{\infty}(S_{\mu}^{\circ})$ (or $H_{\infty}(S_{\mu+}^{\circ})$) where $0 \leq \omega < \mu < \frac{1}{2}\pi$ (or $0 \leq \omega < \mu < \pi$) provided that we admit the possibility of the functional calculus containing unbounded operators. In particular, if $f, f_1 \in H_{\infty}(S_{\mu}^{\circ})$ (or $H_{\infty}(S_{\mu+}^{\circ})$) and $\alpha \in \mathbb{C}$,

then $f(T)$ and $f_1(T)$ are closed operators with domains dense in \mathcal{H} which satisfy

$$(i) \quad \alpha(f(T)) + f_1(T) = (\alpha f + f_1)(T)|_{\mathcal{D}(f(T))} \quad \text{and}$$

$$(ii) \quad f_1(T)f(T) = (f_1 f)(T)|_{\mathcal{D}(f(T))}$$

(where the operator on the right of (ii) has domain $\mathcal{D}(f_1 f)(T) \cap \mathcal{D}(f(T))$). The interesting problem is to show that for certain such T , all these operators $f(T)$ are bounded. Nevertheless, regardless of such boundedness, the functional calculus is of interest. For example, if T is a one-one operator of type $\omega+$ and s is a real number, define T^{is} by $T^{is} = f_s(T)$ where $f_s(\zeta) = \zeta^{is}$. Then, for all $s, t \in \mathbb{R}$, $T^{is}T^{it}u = T^{i(s+t)}u$ whenever $u \in \mathcal{D}(T^{is}T^{it}) = \mathcal{D}(T^{i(s+t)}) \cap \mathcal{D}(T^{it})$. As a second example, take T to be a one-one operator of type ω , and define the spectral projections E_+ and E_- associated with the right and left sectors by $E_+ = \chi_+(T)$ and $E_- = \chi_-(T)$ where

$$\chi_+(\zeta) = \begin{cases} 1 & \text{if } \operatorname{Re}(\zeta) > 0 \\ 0 & \text{if } \operatorname{Re}(\zeta) < 0 \end{cases} \quad \text{and} \quad \chi_-(\zeta) = \begin{cases} 1 & \text{if } \operatorname{Re}(\zeta) < 0 \\ 0 & \text{if } \operatorname{Re}(\zeta) > 0 \end{cases}.$$

Then, by (i) and (ii), $\mathcal{D}(E_+) = \mathcal{D}(E_-) =: \mathcal{D}$, $E_+ + E_- = I|_{\mathcal{D}}$, $E_+^2 = E_+$, $E_-^2 = E_-$ and $E_+E_- = E_-E_+ = 0|_{\mathcal{D}}$.

The significance of this approach is that the above operators, along with fractional powers T^α and semigroups, all fit into the same framework. Incidentally, it relies on repeated use of the following result.

Convergence Lemma. *Let T be a one-one operator of type $\omega+$ in \mathcal{H} . Let $\mu > \omega$. Let f_α be a uniformly bounded net of functions in $\Psi(S_{\mu+}^\circ)$ which converges to a function $f \in H_\infty(S_{\mu+}^\circ)$ uniformly on every set of the form $\{z \in S_{\mu+}^\circ : 0 < \delta \leq |z| \leq \Delta < \infty\}$. Suppose that the operators $f_\alpha(T)$ are uniformly bounded operators on \mathcal{H} . Then $f_\alpha(T)u$ converges to $f(T)u$ for all $u \in \mathcal{H}$, and consequently $f(T)$ is a bounded linear operator on \mathcal{H} with $\|f(T)\| \leq \sup_\alpha \|f_\alpha(T)\|$.*

If T is a one-one operator of type ω in \mathcal{H} , then the analogous result holds with $S_{\mu+}^\circ$ replaced by S_μ° .

The important thing to know about an operator T of type ω (or $\omega+$) is whether or not it has a bounded H_∞ functional calculus. If it does, then, in particular, the operators T^{is} and E_\pm are bounded.

Let us first summarize some positive results. The first, which builds upon the work of Yagi [Y] and many others before him, is taken (slightly modified) from [M^cI]. (Let us note that, if $\psi \in \Psi(S_{\mu+}^\circ)$, then $\psi(T)$ is a bounded

linear operator on \mathcal{H} which can be represented as $\psi(T) = \frac{1}{2\pi i} \int_{\delta} (T - \lambda I)^{-1} \psi(\lambda) d\lambda$

where δ is an unbounded contour, $\delta = \{\lambda = re^{\pm iv} : r \geq 0\}$, $\omega < v < \mu$, parametrized clockwise around $S_{\omega+}$.)

Theorem 1. *Let T be a one-one operator of type $\omega+$ in \mathcal{H} . Then the following statements are equivalent.*

(a) *for each $\mu > \omega$, T has a bounded $H_{\infty}(S_{\mu+}^{\circ})$ functional calculus (i.e. there exists c_{μ} such that $\|f(T)\| \leq c_{\mu} \|f\|_{\infty}$ for all $f \in H_{\infty}(S_{\mu+}^{\circ})$);*

(b) *there exists $\mu > \omega$ such that T has a bounded $H_{\infty}(S_{\mu+}^{\circ})$ functional calculus;*

(c) *{ $T^{is} \mid s \in \mathbb{R}$ } is a C^0 group, and, for all $\mu > \omega$, there exist c_{μ} such that $\|T^{is}\| \leq c_{\mu} e^{\mu|s|}$ when $s \in \mathbb{R}$;*

(d) *if A is a non-negative self-adjoint operator and U is an isomorphism satisfying $T = UA$, and $0 < \alpha < 1$, then $\mathcal{D}(T^{\alpha}) = \mathcal{D}(A^{\alpha})$ and $c^{-1} \|A^{\alpha} u\| \leq \|T^{\alpha} u\| \leq c \|A^{\alpha} u\|$ for some $c > 0$ and all $u \in \mathcal{D}(T^{\alpha})$;*

(e) *for all $\mu > \omega$ and all $\psi \in \Psi(S_{\mu+}^{\circ})$ there exist $q > 0$ such that, when $u \in \mathcal{H}$,*

$$q^{-1} \|u\| \leq \left\{ \int_0^{\infty} \|\psi(tT)u\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \leq q \|u\|;$$

(f) *there exists $\mu > \omega$, $\psi \in \Psi(S_{\mu+}^{\circ})$ and $q > 0$ such that $\psi(x) > 0$ when $x > 0$, and*

$$q^{-1} \|u\| \leq \left\{ \int_0^{\infty} \|\psi(tT)u\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \leq q \|u\|$$

for all $u \in \mathcal{H}$.

The second result, which was proved in [M^cQ], is based upon the proof by Jones and Semmes of the L_2 boundedness of the Cauchy integral operator C_{γ} on a Lipschitz curve γ in the complex plane [CJS]. Note that if $\gamma = \{x+ig(x) : x \in \mathbb{R}\}$, where g is a Lipschitz function, then $C_{\gamma} = i(\chi_{+}(D_{\gamma}) - \chi_{-}(D_{\gamma}))$ where D_{γ} denotes differentiation on γ with respect to the complex variable. See [M^cQ] for details. The boundedness of C_{γ} on $L_2(\gamma)$ was first proved in [CM^cM].

We call $\langle T, T' \rangle$ a dual pair of operators with respect to a dual pairing $\langle \dots \rangle$ between \mathcal{H} and itself if $\langle Tu, v \rangle = \langle u, T'v \rangle$ for all $u \in \mathcal{D}(T)$ and all $v \in \mathcal{D}(T')$.

Theorem 2. Suppose $0 \leq \omega < \mu < \frac{1}{2}\pi$. Let $\langle T, T' \rangle$ be a dual pair of one-one operators of type ω in \mathcal{H} . Let \mathcal{H}_\pm be the closure of the range $\mathcal{R}(E_\pm)$, let \mathcal{K}_\pm be the closure of $\mathcal{R}(\chi_\pm(T'))$ and let $T_\pm = T|_{\mathcal{H}_\pm}$ and $T'_\pm = T'|_{\mathcal{K}_\pm}$. Then the following statements are equivalent.

(a) The operator T has a bounded $H_\infty(S_\mu^\circ)$ functional calculus (i.e. there exists c_μ such that $\|f(T)\| \leq c_\mu \|f\|_\infty$ for all $f \in H_\infty(S_\mu^\circ)$);

(b) the operators T_\pm and T'_\pm have bounded $H_\infty(S_{\mu\pm}^\circ)$ functional calculi in the Hilbert spaces \mathcal{H}_\pm and \mathcal{K}_\pm .

We find (b) an intriguing result, because it has the consequence that estimates within \mathcal{H}_\pm and \mathcal{K}_\pm imply that the operators $E_\pm = \chi_\pm(T)$ are bounded, and hence that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = \mathcal{K}_+ \oplus \mathcal{K}_-$. In the case of the operator D_γ , this means that the quadratic estimates proved by Kenig *within* the Hardy spaces associated with the regions above and below γ imply that $L_2(\gamma)$ is the direct sum of these Hardy spaces, or equivalently, that C_γ is a bounded operator on $L_2(\gamma)$ [CJS].

3. An Operator Equation

We shall now define some matrices which will be useful in constructing the counter-examples. Similar examples were used in [Mc,1,2,3,4]. See also [K,1,2] where Kahan obtained estimates which depend on the size of the matrices.

For $N \geq 1$, consider \mathbb{C}^{N+1} as a Hilbert space as usual. For $\beta > 0$, let A, B and Z be the operators in \mathbb{C}^{N+1} given by the matrices $A = \text{diag}(2^j)$, $B = (B_{j,k})$ and $Z = (Z_{j,k})$, where

$$B_{j,k} = \begin{cases} \frac{\beta}{\pi(k-j)} & \text{if } j \neq k \\ 0 & \text{if } j = k \end{cases} \quad Z_{j,k} = \begin{cases} \frac{2^k \beta}{(2^{k+2j}\pi(k-j))} & \text{if } j \neq k \\ 0 & \text{if } j = k \end{cases}$$

with j and k ranging from 0 to N . Then A is a self-adjoint operator with $\sigma(A) \subset [1, 2^N]$, B is a skew-adjoint operator with $\|B\| \leq \beta$,

$$AZ + ZA = BA,$$

and $\|Z\| \geq \beta(\frac{1}{7} \log N - 1)$. The inequality $\|B\| \leq \beta$ is a consequence of the fact that B is the $N \times N$ Toeplitz matrix corresponding to the function $b(\theta) =$

$i\beta(1-\pi^{-1}\theta)$ on $0 < \theta < 2\pi$. To obtain the lower bound on $\|Z\|$, estimate $\|Zu\|$ for $u = (1, 1, \dots, 1)$.

Note that Z is the only solution of the above operator equation.

4. Unbounded Projections

Our aim now is to construct a one-one operator T of type 0 in a Hilbert space \mathcal{H} with unbounded spectral projections E_{\pm} corresponding to the sectors $S_{\mu\pm}^{\circ}$ as defined in section 2. Such an operator T therefore does not have an $H_{\infty}(S_{\mu}^{\circ})$ functional calculus for any $\mu > 0$.

Theorem 3. *Let $\kappa > 1$. There exists a closed operator T of type 0 in a Hilbert space \mathcal{H} with real spectrum $\sigma(T)$ contained in the pair of intervals $(-\infty, -1] \cup [1, \infty)$, and with resolvent bounds $\|(T - \lambda I)^{-1}\| \leq \kappa |\operatorname{Im}(\lambda)|^{-1}$, for which the spectral projections $E_{\pm} = \chi_{\pm}(T)$ are unbounded.*

There also exists a one-one bounded operator S of type 0 on a Hilbert space \mathcal{H} with real spectrum $\sigma(S)$ contained in the interval $[-1, 1]$, and with resolvent bounds $\|(S - \lambda I)^{-1}\| \leq \kappa |\operatorname{Im}(\lambda)|^{-1}$, for which the spectral projections $E_{\pm} = \chi_{\pm}(S)$ are unbounded.

We remark that operators T and S which satisfy the above resolvent bounds with $\kappa = 1$ are necessarily self-adjoint and their spectral projections E_{+} and E_{-} have norm one.

To prove the theorem, it suffices to construct, for every natural number n , operators T_n on finite dimensional spaces $\mathcal{H}_n = \mathbb{C}^{2N+2}$ with real spectrum $\sigma(T_n)$ contained in the pair of intervals $[-2^N, -1] \cup [1, 2^N]$, and with resolvent bounds $\|(T_n - \lambda I)^{-1}\| \leq \kappa |\operatorname{Im}(\lambda)|^{-1}$, for which the spectral projections E_{\pm} satisfy $\|E_{\pm}\| \geq n$. We then take $T = \oplus T_n$ and $S = \oplus 2^{-N} T_n$ in the Hilbert space $\mathcal{H} = \oplus \mathcal{H}_n$.

So let us fix numbers n and κ greater than 1, and choose N large enough that $(\kappa - 1)(\frac{1}{7} \log N - 1) \geq n$. Then choose operators A and B on \mathbb{C}^{N+1} such that A is a self-adjoint operator with $\sigma(A) \subset [1, 2^N]$, B is an operator with $\|B\| \leq \kappa - 1$,

$$AZ + ZA = BA,$$

and $\|Z\| \geq (\kappa - 1)(\frac{1}{7} \log N - 1) \geq n$. This was shown to be possible in section 2.

Now define the operators T_n, P_{+} and P_{-} on $\mathcal{H}_n = \mathbb{C}^{N+1} \oplus \mathbb{C}^{N+1}$ by

$$T_n = \begin{bmatrix} A & BA \\ 0 & -A \end{bmatrix} \quad P_+ = \begin{bmatrix} I & Z \\ 0 & 0 \end{bmatrix} \quad P_- = \begin{bmatrix} 0 & -Z \\ 0 & I \end{bmatrix}.$$

Then T_n has spectrum $\sigma(T_n) = \sigma_+ \cup \sigma_-$ where $\sigma_+ = \sigma(A) = \{1, 2, 4, \dots, 2^N\}$ and $\sigma_- = \sigma(-A) = -\sigma_+$. Moreover $P_+ + P_- = I$, $P_+P_- = P_-P_+ = 0$, $P_+^2 = P_+$, $P_-^2 = P_-$,

$$P_+T_n = T_nP_+ = P_+ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} P_+ \quad \text{and} \quad P_-T_n = T_nP_- = P_- \begin{bmatrix} 0 & 0 \\ 0 & -A \end{bmatrix} P_-$$

and $\|P_\pm\| \geq \|Z\| \geq n$.

It is not too difficult to calculate resolvent bounds for T_n . Indeed, for all non-real numbers λ ,

$$(T_n - \lambda I)^{-1} = \begin{bmatrix} (A - \lambda I)^{-1} & (A - \lambda I)^{-1}BA(A + \lambda I)^{-1} \\ 0 & -(A + \lambda I)^{-1} \end{bmatrix}$$

and so

$$\|(T_n - \lambda I)^{-1}\| \leq |Im(\lambda)|^{-1} + \|(A - \lambda I)^{-1}BA(A + \lambda I)^{-1}\| \leq \kappa |Im(\lambda)|^{-1}$$

as required. Hence T_n is of type zero, so we can define the spectral projections $E_\pm = \chi_\pm(T)$ as in section 2. The proof is completed on observing that $E_\pm = P_\pm$ and hence that $\|E_\pm\| \geq \|Z\| \geq n$. //

5. Our last example

The operator constructed in section 4 is an operator of type zero (with respect to a double sector), whose H_∞ functional calculus is unbounded. Our aim now is to construct an operator T of type $\omega+$ (with respect to a single sector) whose H_∞ functional calculus is unbounded.

Theorem 4. *Suppose $0 < \omega < \mu < \frac{1}{2}\pi$. Then there exists an invertible closed operator T of type $\omega+$ in a Hilbert space \mathcal{H} which does not have a bounded $H_\infty(S_{\mu+}^\circ)$ functional calculus.*

This theorem is a consequence of the following lemma.

Lemma. *Let $\omega \in (0, \frac{1}{2}\pi]$ and let $n \in \mathbb{N}$. There exists an operator T_n of type ω on a finite dimensional space $\mathcal{H}_n = \mathbb{C}^{N+1}$, which has the form $T_n = (I+V)A^2$, where A is a self-adjoint operator with spectrum $\sigma(A)$ contained in the interval $[1, 2^N]$, $\|V\| \leq \sin \omega$, and $\|T_n^{\frac{1}{2}}v\| \geq n\|Av\|$ for some $v \in \mathcal{H}_n$.*

To see that theorem 4 is a consequence of the lemma, we first show that T_n is of type ω and that $\sigma(T_n - I) \subset S_{\mu+}^{\circ}$. Indeed

$$T_n - \lambda I = (I + VA^2(A^2 - \lambda I)^{-1})(A^2 - \lambda I)$$

is invertible when $\lambda - 1 \notin S_{\mu+}^{\circ}$, because

$$\|VA^2(A^2 - \lambda I)^{-1}\| \leq \begin{cases} \sin \omega & \text{if } \operatorname{Re}(\lambda) \leq 1 \\ \frac{|\lambda - 1| \sin \omega}{|\operatorname{Im}(\lambda)|} & \text{if } \operatorname{Re}(\lambda) \geq 1 \end{cases}$$

and then

$$\|(T_n - \lambda I)^{-1}\| \leq \begin{cases} \frac{1}{|\lambda - 1|(1 - \sin \omega)} & \text{if } \operatorname{Re}(\lambda) \leq 1 \\ \frac{1}{|\operatorname{Im}(\lambda)| - |\lambda - 1| \sin \omega} & \text{if } \operatorname{Re}(\lambda) \geq 1 \end{cases}$$

So T_n is of type ω and $\sigma(T_n - I) \subset S_{\mu+}^{\circ}$.

Define $T = \oplus T_n$ in the Hilbert space $\mathcal{H} = \oplus \mathcal{H}_n$. Then T is an invertible closed operator of type $\omega+$ in \mathcal{H} of the form $T = US$, where U is invertible and S is a self-adjoint operator with $\sigma(S) \subset [1, \infty)$. However T and S do not satisfy part (d) of theorem 1 with $\alpha = \frac{1}{2}$, and hence T does not have a bounded $H_{\infty}(S_{\mu+}^{\circ})$ functional calculus.

Let us now turn our attention to proving the lemma.

So let us fix $n \in \mathbb{N}$ and $\kappa = 1 + \sin \omega$, and choose operators A and B on a finite dimensional space $\mathcal{H}_n = \mathbb{C}^{N+1}$ and $u \in \mathcal{H}_n$ such that A is a self-adjoint operator with $\sigma(A) \subset [1, 2^N]$ and B is an operator with $\|B\| \leq \kappa - 1$, and such that the unique solution Z of the operator equation

$$AZ + ZA = BA$$

satisfies $\|Zu\| \geq (\kappa - 1) \left(\frac{1}{7} \log N - 1\right) \|u\| \geq n \|u\|$. This was shown to be possible in section 2.

For $z \in \mathbb{C}$, define $W_z = (I + zB)A^2$. Then W_z is of type μ for some $\mu < \frac{1}{2}\pi$ provided $|z| < (\kappa - 1)^{-1}$, and depends holomorphically on z , as does $W_z^{\frac{1}{2}}$. Of course $W_z^{\frac{1}{2}}W_z^{\frac{1}{2}} = (I + zB)A^2$, so on differentiating both sides with respect to z , setting $z = 0$, and substituting $W_0^{\frac{1}{2}} = A$, we obtain

$$\left(\frac{d}{dz}W_z^{\frac{1}{2}}\Big|_{z=0}\right)A + A\left(\frac{d}{dz}W_z^{\frac{1}{2}}\Big|_{z=0}\right) = BA^2,$$

or in other words $Z = \left(\frac{d}{dz}W_z^{\frac{1}{2}}\Big|_{z=0}\right)A^{-1}$.

We claim now that $\|W_z^{\frac{1}{2}}A^{-1}u\| \geq n\|u\|$ for some values of z satisfying $|z| = 1$. Recall that $\|Zu\| \geq n\|u\|$. Suppose to the contrary that $\|W_z^{\frac{1}{2}}A^{-1}u\| < n\|u\|$ whenever $|z| = 1$. Then

$$\|Zu\| = \left\| \left(\frac{d}{dz}W_z^{\frac{1}{2}}\Big|_{z=0}\right)A^{-1}u \right\| = \frac{1}{2\pi} \left\| \int_{|z|=1} \frac{1}{z^2} W_z^{\frac{1}{2}}A^{-1}u dz \right\| < n\|u\|$$

which is not possible. Hence $\|W_z^{\frac{1}{2}}A^{-1}u\| \geq n\|u\|$ for some values of z satisfying $|z| = 1$ as claimed.

To complete the proof of the lemma, take $V = zB$ for such a z , $T_n = W_z$ and $v = A^{-1}u$. //

This completes the proof of theorem 4. Two small problems remain open. The first is to make the example explicit by finding a specific value of z . The second is to determine whether or not a closed operator T of type $0+$ exists which does not have a bounded $H_\infty(S_{\mu+}^\circ)$ functional calculus. These problems are left for the amusement of the interested listener.

6. Square Function Norms

We saw in theorem 1 that a one-one operator T of type $\omega+$ has a bounded $H_\infty(S_{\mu+}^\circ)$ functional calculus if and only if T satisfies quadratic estimates. Let us explore what happens when we replace the given norm by a new one defined by

$$\|u\|_\Psi = \left\{ \int_0^\infty \|\psi_t(T)u\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}$$

where $\psi_t(z) = \psi(tz)$.

Theorem 5. *Let T be a one-one operator of type $\omega+$ in \mathcal{H} . Let $\psi, \underline{\psi} \in \Psi(S_{\mu+}^\circ)$ where $\mu > \omega$, and suppose that $\underline{\psi}(\tau) > 0$ when $\tau > 0$. Then there exists c such that*

$$\left\{ \int_0^\infty \|(f\psi_t)(T)u\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \leq c \|f\|_\infty \left\{ \int_0^\infty \|\underline{\psi}_t(T)u\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}$$

for all $f \in H_\infty(S_{\mu+}^\circ)$ and all $u \in \mathcal{H}$.

Note that $f\psi_t \in \Psi(S_{\mu+}^\circ)$, and so $(f\psi_t)(T)$ is a bounded linear operator on \mathcal{H} .

The proof depends on the following estimates. There exist constants c_1 and c_2 such that

$$(i) \quad \|(f\underline{\psi}_t)(T)\| \leq c_1 \|f\|_\infty$$

for all $f \in H_\infty(S_{\mu+}^\circ)$ and all $t > 0$, and

$$(ii) \quad \left\{ \int_0^\infty \left\| \int_\alpha^\beta \underline{\psi}_t(T) \underline{\psi}_\tau(T) g(\tau) \frac{d\tau}{\tau} \right\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \leq c_2 \left\{ \int_\alpha^\beta \|g(t)\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}$$

for all continuous functions g from $[\alpha, \beta]$ to \mathcal{H} and all $0 < \alpha < \beta < \infty$.

These estimates can be proved by representing the operators as integrals over an unbounded contour $\delta = \{\lambda = re^{\pm iv} : r \geq 0\}$, $\omega < v < \mu$, parametrized clockwise around $S_{\omega+}$. The first is easy, and the second not much harder [M^cQ]. To derive it we observe that

$$\begin{aligned} \|\psi_t(T)\underline{\psi}_\tau(T)\| &\leq \left\| \frac{1}{2\pi i} \int_\delta (T - \lambda)^{-1} \psi_t(\lambda) \underline{\psi}_\tau(\lambda) d\lambda \right\| \\ &\leq \text{const.} \int_\delta \frac{1}{|\lambda|} \frac{|t\lambda|^s}{1+|t\lambda|^{2s}} \frac{|\tau\lambda|^s}{1+|\tau\lambda|^{2s}} |d\lambda| \\ &\leq \begin{cases} \text{const.} (t/\tau)^s (1 + \log(\tau/t)) & \text{if } 0 < t \leq \tau < \infty \\ \text{const.} (\tau/t)^s (1 + \log(t/\tau)) & \text{if } 0 < \tau \leq t < \infty \end{cases} \end{aligned}$$

and hence that

$$\left\{ \int_0^\infty \left\| \int_\alpha^\beta \psi_t(T) \underline{\psi}_\tau(T) g(\tau) \frac{d\tau}{\tau} \right\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}$$

$$\begin{aligned} &\leq \sup_{t > 0} \left\{ \int_{\alpha}^{\beta} \|\psi_t \underline{\psi}_{\tau}(T)\| \frac{d\tau}{\tau} \right\}^{\frac{1}{2}} \sup_{\tau > 0} \left\{ \int_0^{\infty} \|\psi_t \underline{\psi}_{\tau}(T)\| \frac{dt}{t} \right\}^{\frac{1}{2}} \left\{ \int_{\alpha}^{\beta} \|g(t)\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\ &\leq c_2 \left\{ \int_{\alpha}^{\beta} \|g(t)\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \end{aligned}$$

as required.

Proof of theorem 5. Define $\psi_{\alpha, \beta} \in \Psi(S_{\mu+}^{\circ})$ by $\psi_{\alpha, \beta}(z) = k^{-1} \int_{\alpha}^{\beta} \underline{\psi}_{\tau}^3(z) \frac{d\tau}{\tau}$ where $k = \int_0^{\infty} \underline{\psi}^3(\tau) \frac{d\tau}{\tau}$. Let $u \in \mathcal{H}$. By the Convergence Lemma in section 2, $u_{\alpha, \beta} = \psi_{\alpha, \beta}(T)u \rightarrow u$ as $\alpha \rightarrow 0$ and $\beta \rightarrow \infty$. Also

$$\begin{aligned} \left\{ \int_0^{\infty} \|(f\psi_t)(T)u_{\alpha, \beta}\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} &= k^{-1} \left\{ \int_0^{\infty} \left\| \int_{\alpha}^{\beta} (f\psi_t)(T) \underline{\psi}_{\tau}^3(T) u \frac{d\tau}{\tau} \right\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\ &= k^{-1} \left\{ \int_0^{\infty} \left\| \int_{\alpha}^{\beta} (\psi_t(T) \underline{\psi}_{\tau}(T)) (f\underline{\psi}_{\tau}^2)(T) u \frac{d\tau}{\tau} \right\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\ &\leq k^{-1} c_2 \left\{ \int_0^{\infty} \|(f\underline{\psi}_t)(T) \underline{\psi}_t(T) u\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \quad \text{by (ii)} \\ &\leq k^{-1} c_2 c_1 \|f\|_{\infty} \left\{ \int_0^{\infty} \|\underline{\psi}_t(T) u\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \quad \text{by (i)}. \end{aligned}$$

The result follows on applying the dominated convergence theorem. //

Given a one-one T operator of type $\omega+$ in \mathcal{H} , let us choose a function $\psi \in \Psi(S_{\mu+}^{\circ})$ for some $\mu > \omega$ which satisfies $\psi(\tau) > 0$ when $\tau > 0$, and let $\mathcal{H}_0 = \{u \in \mathcal{H} : \|u\|_{\psi} < \infty\}$, where

$$\|u\|_\psi = \left\{ \int_0^\infty \|\psi_t(T)u\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}.$$

On applying the preceding theorem with $f = 1$, we deduce that different choices of μ and ψ which satisfy the same criteria, give rise to the same space \mathcal{H}_0 with equivalent norms. (To see that they are norms, we need to use the fact that T is one-one, and hence that $\psi_t(T)$ is one-one.) Moreover $\mathcal{D}(T^\alpha) \cap \mathcal{R}(T^\alpha) \subset \mathcal{H}_0$ for $\alpha > 0$, so \mathcal{H}_0 is dense in \mathcal{H} . The space \mathcal{H}_0 is an inner product space under

$$(u, v)_\psi = \int_0^\infty (\psi_t(T)u, \psi_t(T)v) \frac{dt}{t}.$$

Define the Hilbert space \mathcal{H}_T to be the completion of \mathcal{H}_0 , together with one of the equivalent inner products $(u, v)_\psi$ and norms $\|u\|_\psi$.

Given any linear operator S in \mathcal{H} , let $S_0 = S|_{\mathcal{D}(S_0)}$ where $\mathcal{D}(S_0) = \{u \in \mathcal{H}_0 : Su \in \mathcal{H}_0\}$, and let S^\sim be the closure of S_0 in \mathcal{H}_T if it exists.

Whenever B is a bounded linear operator on \mathcal{H} which commutes with $R_\lambda = (T - \lambda I)^{-1}$ when $\lambda \notin \sigma(T)$, then B commutes with $\psi_t(T)$, and consequently $B(\mathcal{H}_0) \subset \mathcal{H}_0$ and $\|Bu\|_\psi \leq \|B\| \|u\|_\psi$ for all $u \in \mathcal{H}_0$. So B^\sim is a bounded linear operator on \mathcal{H}_T and $\|B^\sim\|_\psi \leq \|B\|$, where $\|B^\sim\|_\psi$ denotes the operator norm of B^\sim on \mathcal{H}_T under $\|u\|_\psi$. If B_α is a uniformly bounded net of linear operators on \mathcal{H} which commutes with R_λ , and if B_α converges in the strong topology to B , then, by dominated convergence, $B_\alpha^\sim u \rightarrow B^\sim u$ in \mathcal{H}_T for all $u \in \mathcal{H}_0$ and hence for all $u \in \mathcal{H}_T$.

In particular, when $\lambda \notin \sigma(T)$, then R_λ^\sim is a bounded linear operator on \mathcal{H}_T and $\|R_\lambda^\sim\|_\psi \leq \|R_\lambda\|$. Note also that I^\sim is the identity map and 0^\sim is the zero map on \mathcal{H}_T . Now, for $t > 0$, tR_{-t} converges strongly to I as $t \rightarrow \infty$, and tR_{-t} converges strongly to 0 as $t \rightarrow 0$ (since T is a one-one operator of type $\omega+$ in \mathcal{H}), and therefore tR_{-t}^\sim converges strongly to I^\sim as $t \rightarrow \infty$, and tR_{-t}^\sim converges strongly to 0^\sim as $t \rightarrow 0$.

We shall now prove that the operator T_0 has a closure T^\sim in \mathcal{H}_T and that $R_\lambda^\sim = (T^\sim - \lambda I^\sim)^{-1}$ when $\lambda \notin \sigma(T)$. To do this it suffices to show that R_λ^\sim is one-one. Now $(\tau + \lambda)R_{-\tau}R_\lambda = R_\lambda - R_{-\tau}$ when $\tau > 0$, so $(\tau + \lambda)R_{-\tau}^\sim R_\lambda^\sim = R_\lambda^\sim - R_{-\tau}^\sim$ also. Hence, if $R_\lambda^\sim u = 0$, then $0 = \tau R_{-\tau}^\sim u \rightarrow u$ as $\tau \rightarrow \infty$, and so $u = 0$. We conclude that R_λ^\sim is one-one as claimed.

Theorem 6. Let T be a one-one operator of type $\omega+$ in \mathcal{H} , and let T^\sim be the associated operator in \mathcal{H}_T which was defined above. Then T^\sim is a one-one operator of type $\omega+$ in \mathcal{H}_T which has a bounded $H_\infty(S_{\mu+}^\circ)$ functional calculus when $\mu > \omega$, and thus satisfies the equivalent statements of theorem 1 (with \mathcal{H} replaced by \mathcal{H}_T).

Moreover, $f(T^\sim) = f(T)^\sim$ for all $f \in H_\infty(S_{\mu+}^\circ)$, and so $\|f(T^\sim)\|_\psi \leq \|f(T)\|$ when $f(T)$ is bounded, where the operator norm $\|f(T^\sim)\|_\psi$ is defined with respect to any function $\psi \in \Psi(S_{\nu+}^\circ)$ with $\nu > \omega$ and $\psi(\tau) > 0$ when $\tau > 0$.

Proof. We have already seen that $\sigma(T^\sim) \subset \sigma(T)$ and $\|(T^\sim - \lambda I)^{-1}\|_\psi \leq \|(T - \lambda I)^{-1}\|$ whenever $\lambda \notin \sigma(T)$, so T^\sim is an operator of type $\omega+$ in \mathcal{H}_T . Suppose $T^\sim u = 0$ for some $u \in \mathcal{H}_T$. Then $u = \tau R_{-\tau}^\sim u \rightarrow 0$ as $\tau \rightarrow 0$, and so $u = 0$. Thus T^\sim is one-one.

Suppose for the moment that $f \in \Psi(S_{\mu+}^\circ)$. Then the bounded linear operators $f(T^\sim)$ and $f(T)^\sim$ on \mathcal{H}_T are equal. This is a consequence of the contour integral representations of these operators and the above facts about resolvents and convergence. Hence $\|f(T^\sim)\|_\psi \leq \|f(T)\|$. Further, by theorem 5 with $\underline{\psi} = \psi$, $\|f(T^\sim)u\|_\psi \leq c\|f\|_\infty\|u\|_\psi$ for all $u \in \mathcal{H}_0$ and thus for all $u \in \mathcal{H}_T$.

Now every function $f \in H_\infty(S_{\mu+}^\circ)$ is the limit of a sequence of functions $f_n \in \Psi(S_{\mu+}^\circ)$ which converges to f uniformly on sets of the form $\{z \in S_{\mu+}^\circ : 0 < \delta \leq |z| \leq \Delta < \infty\}$, and which satisfies $\|f_n\|_\infty \leq \|f\|_\infty$ for all n . Therefore, by the Convergence Lemma, $f(T^\sim)$ is bounded on \mathcal{H}_T with $\|f(T^\sim)\|_\psi \leq \sup \|f_n(T^\sim)\|_\psi \leq c\|f_n\|_\infty \leq c\|f\|_\infty$. That is, T^\sim has a bounded $H_\infty(S_{\mu+}^\circ)$ functional calculus.

To conclude, suppose that $f \in H_\infty(S_{\mu+}^\circ)$. We need to show that $f(T)u = f(T^\sim)u$ for all $u \in \mathcal{D}(f(T)_0)$ and that $\mathcal{D}(f(T)_0)$ is dense in \mathcal{H}_T , for then $f(T)^\sim = f(T^\sim)$. Note that, if $u \in \mathcal{D}(f(T)) \cap \mathcal{H}_0$, then, by theorem 5, $f(T)u \in \mathcal{H}_0$, so $\mathcal{D}(f(T)_0) = \mathcal{D}(f(T)) \cap \mathcal{H}_0$. For such a u , and for the functions $\psi_{\alpha,\beta}$ used in the proof of theorem 5, we have that $u_{\alpha,\beta} = \psi_{\alpha,\beta}(T)u = \psi_{\alpha,\beta}(T^\sim)u \rightarrow u$ in \mathcal{H}_T as $\alpha \rightarrow 0$, $\beta \rightarrow \infty$. Also $f(T^\sim)u_{\alpha,\beta} = f(T^\sim)\psi_{\alpha,\beta}(T^\sim)u = (f\psi_{\alpha,\beta})(T^\sim)u = (f\psi_{\alpha,\beta})(T)u = \psi_{\alpha,\beta}(T)f(T)u = \psi_{\alpha,\beta}(T^\sim)f(T)u$, so, on taking limits in \mathcal{H}_T , we have $f(T^\sim)u = f(T)u$ as required. To prove density, take $v \in \mathcal{H}_0$ and observe that $\psi_{\alpha,\beta}(T)v \in \mathcal{D}(f(T)) \cap \mathcal{H}_0$ and that $\psi_{\alpha,\beta}(T)v = \psi_{\alpha,\beta}(T^\sim)v \rightarrow v$ in \mathcal{H}_T . Hence $\mathcal{D}(f(T)_0) = \mathcal{D}(f(T)) \cap \mathcal{H}_0$ is dense in \mathcal{H}_0 (in the \mathcal{H}_T topology), and therefore is dense in \mathcal{H}_T itself. //

We have discovered that, although there exist one-one operators of type $\omega+$ in a Hilbert space \mathcal{H} which do not have a bounded $H_\infty(S_{\mu+}^\circ)$ functional calculus when $\mu > \omega$, nevertheless all one-one operators of type $\omega+$ do have a bounded $H_\infty(S_{\mu+}^\circ)$ functional calculus in the closely related Hilbert space \mathcal{H}_T .

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School of Mathematics, Physics, Computing, and Electronics
Macquarie University, N.S.W. 2109, AUSTRALIA

Department of Mathematics, Himeji Institute of Technology
2167 Shosha, Himeji, Hyogo 671-22, JAPAN