FOUR REMARKS CONCERNING THE FEYNMAN INTEGRAL

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REMARK ONE

The efforts of mathematicians in the area of functional integration are, from a certain point of view, focused too narrowly. That is, more specifically, the idea of a Feynman integral is not conceived in sufficient generality. This statement may seem ridiculous now when the Feynman integral attracts more attention than ever before, is generalized in various directions, and is used in contexts far beyond Feynman's original intention. (Instead of elaborating I would like to refer to the collection [3].) Therefore, to indicate what I have in mind I wish to invoke a historical analogy and to suggest an example.

As for the historical analogy, it may strike the listener (and the reader) as somewhat preposterous, but I do not have a better one: I wish to advert to the beginnings of the Integral Calculus. Some people, with a certain amount of justification, take for the origin of the Integral Calculus Archimedes' calculations of areas of some planar figures and volumes of some solid bodies. However, André Weil is right when he insists that crediting Archimedes with the invention of the Integral Calculus would be a historical nonsense. Indeed, we cannot yet speak of the Integral Calculus even some 2000 years later when Fibonacci calculated the area "under the curve $y = x^n$ in the interval $[0,1]$, for $n = 3,4,\ldots,9$, and not even after Fermat calculated this area for arbitrary integral $n \geq 1$. To be sure, this is not to belittle the ingenuity of Archimedes or that of Fibonacci or Fermat. On the contrary, we cannot speak of Integral Calculus at those stages precisely because each of the mentioned calculations was based on a particular "trick" exploiting the specificity of the considered problem and requiring ingenuity far exceeding that which is now needed for the calculation of such sophisticated integrals as presented at the Tripos, say. What was still missing was an underlying principle or a general theory, and that emerged only in the works of Barrow, Leibniz and Newton. Only in the light of such a theory can we understand why and how the specific "tricks" work and, eventually, dispose of them, or, alternatively, devise new ones almost *ad libitum.*

Let us now consider the example. Let $d \geq 1$ be an integer and let Δ be the Laplacian on $\mathbb{R}^d$.

Fixing conveniently some physically significant constants, letting

$$H_0 = \tfrac{1}{2}\Delta$$

and using loosely a mathematical rather than physical language we may say that, the original purpose for which Feynman devised "the Feynman integral" was to construct the semigroup $\exp(-itH)$, $t \geq 0$, where

$$H = H_0 + V$$

and V is a given real valued function on $\mathbb{R}^d$ interpreted as an operator on (a subspace of) $L^2(\mathbb{R}^d)$.

His and a majority of subsequent constructions depend heavily on the specific properties of the group $\exp(-itH_0)$, $t \in \mathbb{R}$. This group can be expressed explicitly. Namely,

$$(1) \qquad (\exp(-itH_0)\varphi)(x) = \int_{\mathbb{R}^d} k_t(x-y)\varphi(y)\,dy \;,$$

in the L^2-sense, for every $\varphi \in L^2(\mathbb{R}^d)$ and almost every $x \in \mathbb{R}^d$, where

$$(2) \qquad k_t(x) = (2\pi it)^{-d/2}\exp\left[\frac{i}{2t}\,|x|^2\right] , \quad x \in \mathbb{R}^d \;.$$

In terms of the Fourier-Plancherel transform,

$$(\exp(-itH_0)\varphi)^{\wedge}(\omega) = \exp(-1/2it|\omega|^2)\hat{\varphi}(\omega) \;,$$

for every $t \in \mathbb{R}$, $\varphi \in L^2(\mathbb{R}^d)$ and almost every $\omega \in \mathbb{R}^d$.

Let $t > 0$ and let $\Gamma_t = C([0,t], \mathbb{R}^d)$ be the space of all continuous maps (paths) $\gamma : [0,t] \to \mathbb{R}^d$. The specific form of the kernel (2) makes it possible to understand the Feynman formula

$$(\exp(-itH)\varphi)(x) = \int_{\Gamma_t} \exp\left[\frac{i}{2}\int_0^t (|\dot\gamma(s)|^2 - V(\gamma(s)))\,ds\right] \varphi(\gamma(0))\,\mathscr{D}\gamma \tag{3}$$

as

$$(\exp(-itH)\gamma)(x) = \lim_{n\to\infty} \left[\frac{2\pi i t}{n}\right]^{-dn/2} \overbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}}^{n \text{ times}} \exp\left[i\sum_{k=1}^{n}\left[\frac{n|x_k - x_{k-1}|^2}{2t} - V(x_k)\,\frac{t}{n}\right]\right]\gamma(x_0)\,dx_0\,dx_1\ldots dx_{n-1}\ , \tag{4}$$

with $x_n = x$.

For more detail and many illuminating comments, I wish to refer to the note "Feynman's paper revisited", by G. W. Johnson, in [3], pp. 249-270.

To make the specific role of the kernel (2) in the theory of the Feynman integral more apparent, let us change Feynman's problem and consider

$$H_0 = \sqrt{-\Delta}$$

instead. That is, for H_0 we take the positive operator such that $H_0^2\gamma = -\Delta\gamma$, for every γ in the domain of Δ. Then $\exp(-itH_0)$, $t \in \mathbb{R}$, is again a nice unitary group of operators. In terms of the Fourier-Plancherel transform,

$$(\exp(-itH_0)\varphi)\hat{\ }(\omega) = \exp(-it|\omega|)\varphi(\omega)\ ,$$

for every $t \in \mathbb{R}$, $\varphi \in L^2(\mathbb{R}^d)$ and almost every $\omega \in \mathbb{R}^d$. It can also be expressed in the form (1). Indeed, it suffices to define

$$k_t(x) = c_d\,\frac{it}{(|x|^2 - t^2)^{(d+1)/2}}\ , \tag{5}$$

for every $t \neq 0$ and every $x \in \mathbb{R}^d$ such that $|x| \neq |t|$, where

$$\frac{1}{c_d} = \int_{\mathbb{R}^d} \frac{dx}{(|x|^2+1)^{(d+1)/2}} .$$

As before, let V be a real or complex valued function on $\mathbb{R}^d$ and let $H = H_0 + V$. We may wish to construct the semi-group $\exp(-itH)$, $t \geq 0$. Attempts to produce a formula analogous to Feynman's formula (3) do not seem to have been made. The reasons seem to be in that the kernel (5) has a very different form from (2) and so, we cannot expect the "integrand" to be the composition of a functional on Γ_t with the exponential function. Of course, it is possible to consider an analogy to formula (4). For that purpose, we look at the integrand in (4) as a product of two factors, each being the exponential function of the appropriate sum, and replace the first factor by a new one. However, another difficulty then arises. Neither the kernel (2) nor the kernel (5) is integrable on the whole of $\mathbb{R}^d$, but, while the kernel (2) is continuous and bounded so that its singularity is concentrated near $|x| = \infty$, for every t , the singularity of the kernel (5) is concentrated around $|x| = |t|$ and so, it moves with time. Consequently, the usual methods for making sense of (3) cannot be used without further ado in this situation.

REMARK TWO

I wish now to present a context from which the example suggested in the first remark naturally arises, indicating thereby its significance and a possible, and perhaps desirable, extension of the idea of the Feynman integral.

The Fourier-Plancherel transform, $\hat{\varphi}$, of an element, φ , of the space $L^2(\mathbb{R}^d)$ is defined so that

$$\hat{\varphi}(\omega) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(-i\omega x)\varphi(x)\,dx ,$$

for every $\omega \in \mathbb{R}^d$ and every $\varphi \in L^1 \cap L^2(\mathbb{R}^d)$. Let the domain, $\mathscr{D}(\Delta)$, of the operator Δ consist of all elements, φ, of the space $L^2(\mathbb{R}^d)$ such that the function $\omega \to |\omega|^2\hat{\varphi}(\omega)$, $\omega \in \mathbb{R}^d$, belongs to $L^2(\mathbb{R}^d)$, and let

$$(\Delta\varphi)\hat{}\,(\omega) = -|\omega|^2\hat{\varphi}(\omega) ,$$

for almost every $\omega \in \mathbb{R}^d$ and every $\varphi \in \mathscr{D}(\Delta)$. The domain, $\mathscr{D}(\sqrt{-\Delta})$, of the operator $\sqrt{-\Delta}$ is assumed to consist of all elements, φ, of the space $L^2(\mathbb{R}^d)$ such that the function $\omega \to |\omega|\hat{\varphi}(\omega)$, $\omega \in \mathbb{R}^d$, also belongs to $L^2(\mathbb{R}^d)$, and the operator $\sqrt{-\Delta}$ itself is defined by

$$(\sqrt{-\Delta})\hat{}\,(\omega) = |\omega|\hat{\varphi}(\omega) ,$$

for every $\varphi \in \mathscr{D}(\sqrt{-\Delta})$. Then $\sqrt{-\Delta}$ is the positive operator whose square is equal to $-\Delta$; that is, $(\sqrt{-\Delta})^2\varphi = -\Delta\varphi$, for every $\varphi \in \mathscr{D}(\Delta)$.

Let $C > 0$. Let us re-write the wave equation, $\ddot{u} = C^2\Delta u$, in d space-dimensions as the system

$$\dot{u} = v, \; \dot{v} = C^2\Delta u ,$$

that is

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0, & 1 \\ C^2\Delta, & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} . \tag{6}$$

Let

$$S(t) = \exp\left[t \begin{bmatrix} 0, & 1 \\ C^2\Delta, & 0 \end{bmatrix}\right] , \; t \in \mathbb{R} , \tag{7}$$

be the fundamental solution of the equation (6). We introduce a space in which (7) represents a unitary group and give an explicit formula for $S(t)$, $t \in \mathbb{R}$.

Let E be the space of all pairs,

$$\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = (\varphi,\ \psi)'$$

such that $\varphi \in \mathscr{D}(\sqrt{-\Delta})$ and $\psi \in \sqrt{-\Delta}\,\mathscr{D}(\sqrt{-\Delta}))$. The relation $\psi \in \sqrt{-\Delta}\,\mathscr{D}(\sqrt{-\Delta}))$ means of course that there exists an element, χ, of $\mathscr{D}(\sqrt{-\Delta})$ such that $\hat{\psi}(\omega) = |\omega|\,\hat{\chi}(\omega)$ for almost every $\omega \in \mathbb{R}^d$. We shall write

$$\begin{bmatrix}\varphi\\ \psi\end{bmatrix}^{\wedge} = \begin{bmatrix}\hat{\varphi}\\ \hat{\psi}\end{bmatrix},\quad \begin{bmatrix}\varphi\\ \psi\end{bmatrix}(x) = \begin{bmatrix}\varphi(x)\\ \psi(x)\end{bmatrix},\ \text{etc.}$$

For every $\omega \in \mathbb{R}^d$, $\omega \neq 0$, let

$$\hat{\Pi}^{+}(\omega) = \frac{1}{2}\begin{bmatrix}1, & i(C|\omega|)^{-1}\\ -iC|\omega|, & 1\end{bmatrix} \text{ and } \hat{\Pi}^{-}(\omega) = \frac{1}{2}\begin{bmatrix}1, & -i(C|\omega|)^{-1}\\ iC\,|\omega|, & 1\end{bmatrix},$$

for every $\omega \in \mathbb{R}^d$ with $|\omega| \neq 0$. The matrices $\hat{\Pi}^{+}(\omega)$ and $\hat{\Pi}^{-}(\omega)$ represent the projections of the space $\mathbb{C}^2$ onto the subspaces

$$\{(\alpha,\ \beta)' : \beta = -iC|\omega|\,\alpha\} \text{ and } \{(\alpha,\ \beta)' : \beta = iC|\omega|\,\alpha\},$$

respectively. Using these matrices, we define the projections Π^{+} and Π^{-} on the space E by letting

$$(\Pi^{\pm}(\varphi,\ \psi)')^{\wedge}(\omega) = \hat{\Pi}^{\pm}(\omega)(\hat{\varphi}(\omega),\ \hat{\psi}(\omega))'$$

for almost every $\omega \in \mathbb{R}^d$ and every $(\varphi,\ \psi)' \in E$. Obviously, $\Pi^{+}\Pi^{-} = \Pi^{-}\Pi^{+} = 0$ and $\Pi^{+} + \Pi^{-} = I$. Let $E^{+} = \Pi^{+}E$ and $E^{-} = \Pi^{-}E$. Hence, every element, $(\varphi,\ \psi)'$, of the space E has a unique representation

$$\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \begin{bmatrix}\varphi_{+}\\ \psi_{+}\end{bmatrix} + \begin{bmatrix}\varphi_{-}\\ \psi_{-}\end{bmatrix}, \tag{8}$$

with $(\varphi_{+},\ \psi_{+}) \in E^{+}$ and $(\varphi_{-},\ \psi_{-}) \in E^{-}$. That is, E is a direct sum of its subspaces E^{+} and E^{-}. Consequently, we can define

$$\|(\varphi,\psi)'\|^2 = \int_{\mathbb{R}^d} (|\varphi_{+}(x)|^2 + |\varphi_{-}(x)|^2 + |\psi_{+}(x)|^2 + |\psi_{-}(x)|^2)\,dx\ ,$$

for every $(\varphi,\psi)' \in E$ expressed in the form (8). The functional $(\varphi,\psi)' \to \|(\varphi,\psi)'\|$, $(\varphi,\psi)' \in E$, is a norm which makes of E a Hilbert space in which the subspaces E^+ and E^- are orthogonal to each other.

Now , by a direct calculation, we obtain that

$$\exp\left[t\begin{bmatrix}0\,, & 1\\ -C^2|\omega|^2, & 0\end{bmatrix}\right] = \exp(-itc|\omega|)\hat{\Pi}^+(\omega) + \exp(itc|\omega|)\hat{\Pi}^-(\omega) \quad ,$$

for every $t \in \mathbb{R}$ and $\omega \neq 0$. Hence,

(9) $\quad (S(t)(\varphi,\psi)')\hat{}\,(\omega) = (\exp(-itc|\omega|)\hat{\Pi}^+(\omega) + \exp\,(itc|\omega|)\hat{\Pi}^-(\omega))(\hat{\varphi}(\omega),\hat{\psi}(\omega))'$,

for almost every $\omega \in \mathbb{R}^d$ and every $(\varphi,\psi)' \in E$.

For every $t \in \mathbb{R}$, let

$$S^{\pm}(t) = \exp(\mp itc\sqrt{-\Delta}) \ .$$

Then $S^+(t)$ and $S^-(t)$, $t \in \mathbb{R}$, are unitary groups of transformations of $L^2(\mathbb{R}^d)$. In fact, for every $t \in \mathbb{R}$ and every $\varphi \in L^2(\mathbb{R}^d)$, we have

$$(S^{\pm}(t)\varphi)\hat{}\,(\omega) = \exp(\mp itc|\omega|)\hat{\varphi}(\omega)$$

for almost every $\omega \in \mathbb{R}^d$. It is obvious that $S^{\pm}(t)\mathscr{D}(\Delta) = \mathscr{D}(\Delta)$ and $S^{\pm}(t)\mathscr{D}(\sqrt{-\Delta}) = \mathscr{D}(\sqrt{-\Delta})$, for every $t \in \mathbb{R}$. Also, if we define

$$S^{\pm}(t)\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \begin{bmatrix}S^{\pm}(t)\varphi\\ S^{\pm}(t)\psi\end{bmatrix} \quad ,$$

for every $\varphi \in L^2(\mathbb{R}^d)$ and $\psi \in L^2(\mathbb{R}^d)$, then $S^{\pm}(t)E = E$, $S^{\pm}(t)E^+ = E^+$ and $S^{\pm}(t)E^- = E^-$, for every $t \in \mathbb{R}$. Hence, by (9),

$$S(t) = S^+(t)\Pi^+ + S^-(t)\Pi^- \quad ,$$

for every $t \in \mathbb{R}$. Because the projections Π^+ and Π^- are othogonal, it follows that $S(t)$, $t \in \mathbb{R}$, is a unitary group of transformations on the space E.

So, the groups $S^+(t)$ and $S^-(t)$, $t \in \mathbb{R}$, naturally arise in connection with the wave equation. What's more, by solving the Feynman-type problems of constructing the semigroups $\exp(\mp it(C\sqrt{-\Delta} + V))$, $t \geq 0$, we produce the fundamental solutions of certain modifications of the wave equation. However, a more interesting is the task of finding the solutions of the equation

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0, & 1 \\ C^2\Delta, & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} U, & V \\ W, & X \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} ,$$

with "arbitrary" functions U , V , W and X on $\mathbb{R}^d$. This equation includes as a special case the higher dimensional extension of the telegrapher's equation.

Interestingly enough, solutions of the telegrapher's equation and its extensions to higher dimensions do not seem to have been produced by constructions analogous to the Feynman integral. Instead, probabilistic methods initiated by Mark Kac have been used for this purpose. As a nice and typical example of an early work in this direction, let us mention the note [1]. This approach leads to the theory of random evolution to which is devoted already a rather extensive literature. In the following remark, an approach based on an analogy of the Feynman integral is suggested.

REMARK THREE

Let $N \geq 1$ be an integer. Elements of the space $\mathbb{C}^N$ will be written as column-vectors, that is, matrices with N rows and one column, and elements of $\mathbb{C}^{N\times N}$ as square matrices with N rows and columns, so that the product Aa , with $A \in \mathbb{C}^N$ and $a \in \mathbb{C}^N$, has the standard meaning as a product of matrices and so, is an element of $\mathbb{C}^N$. A fixed norm on $\mathbb{C}^N$, such as the Euclidean norm, and the corresponding matrix norm on $\mathbb{C}^{N\times N}$, are both denoted as modulus.

Let E be a Banach space whose elements are (equivalence classes of) $\mathbb{C}^N$-valued functions on $\mathbb{R}^d$ written as column-vectors,

$$\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{bmatrix} = (\varphi_1,\varphi_2,\dots,\varphi_N)' ,$$

having $\mathbb{C}$-valued functions, $\varphi_1,\varphi_2,\dots,\varphi_N$, for the components. A $\mathbb{C}^{N\times N}$-valued function,

$$\Phi = \begin{bmatrix} \Phi_{11}, \Phi_{12}, \dots, \Phi_{1N} \\ \Phi_{21}, \Phi_{22}, \dots, \Phi_{2N} \\ \Phi_{N1}, \Phi_{N2}, \dots, \Phi_{NN} \end{bmatrix} ,$$

where $\Phi_{\iota\kappa}$, $\iota,\kappa = 1,2,\dots,N$, are $\mathbb{C}$-valued functions on $\mathbb{R}^d$, is said to be an E-multiplier if $\Phi\varphi \in E$ and $\|\Phi\varphi\| \le k\|\varphi\|$, for some $k \ge 0$ and every $\varphi \in E$. Here, of course,

$$\Phi\varphi = \left[\sum_{\kappa=1}^{N} \Phi_{1\kappa}\varphi_\kappa , \sum_{\kappa=1}^{N} \Phi_{2\kappa}\varphi_\kappa , \dots, \sum_{\kappa=1}^{N} \Phi_{N\kappa}\varphi_\kappa \right]$$

and the multiplication and addition of $\mathbb{C}$-valued functions are point-wise. Hence, the point-wise multiplication (from the left) by a multiplier, Φ, represents a bounded linear operator on E which is also denoted by Φ. The space of all E-multipliers will be denoted by $\mathcal{M}(E)$.

Let $L(E)$ be the space of all bounded linear operators on E. Let $S : [0,\infty) \to L(E)$ be a (strongly) continuous semigroup of operators.

For any $t > 0$, let Γ_t be the space of all continuous maps (paths) $\gamma : [0,t] \to \mathbb{R}^d$. If $0 < s \le t$ and $\gamma \in \Gamma_t$, then the restriction of γ to the interval $[0,s]$ is denoted by $\gamma | [0,s]$.

Let $\varphi \in E$. For every $t > 0$, let $\mathcal{L}(t,\varphi)$ be a vector space of $\mathbb{C}^{N\times N}$-valued functions on Γ_t, $M_{t,\varphi} : \mathcal{L}(t,\varphi) \to E$ a linear map and $\rho_{t,\varphi}$ a seminorm on $\mathcal{L}(t,\varphi)$ such that the following requirements are satisfied.

(i) $\|M_{t,\varphi}(f)\| \le \rho_{t,\varphi}(f)$, for every $f \in \mathcal{L}(t,\varphi)$.

(ii) If $f_j \in \mathcal{L}(t,\varphi)$, $j = 1,2,\ldots,$

$$\sum_{j=1}^{\infty} \rho_{t,\varphi}(f_j) < \infty$$

and f is a $\mathbb{C}^{N\times N}$-valued function on Γ_t such that

$$f(\gamma) = \sum_{j=1}^{\infty} f_j(\gamma)$$

for every $\gamma \in \Gamma_t$ for which

$$\sum_{j=1}^{\infty} |f_j(\gamma)| < \infty ,$$

then $f \in \mathcal{L}(t,\varphi)$ and

$$\rho_{t,\varphi}(f) \le \sum_{j=1}^{\infty} \rho_{t,\varphi}(f_j) .$$

(iii) Constant functions belong to $\mathcal{L}(t,\varphi)$ and $M_{t,\varphi}(1) = S(t)\varphi$, for every $t > 0$. If $0 < s \le t$, $g \in \mathcal{L}(s,\varphi)$, $\Phi \in \mathcal{M}(E)$ and $f(\gamma) = \Phi(\gamma(s))g(\gamma\,|\,[0,s])$, for every $\gamma \in \Gamma_t$, then $f \in \mathcal{L}(t,\varphi)$ and $M_{t,\varphi}(f) = S(t-s)\Phi M_{s,\varphi}(g)$.

The requirement (iii) implies that every function, f , for which there exist an integer, $k \ge 1$, numbers, t_j , and E-multipliers, Φ_j , $j = 1,2,\ldots,k$, such that $0 < t_1$, $t_{j-1} < t_j \le t$, for $j = 2,\ldots,k$, and

$$f(\gamma) = \prod_{j=1}^{k} \Phi_j(\gamma(t_j)) ,$$

for every $\gamma \in \Gamma_t$, belongs to $\mathcal{L}(t,\varphi)$ and

$$M_{t,\varphi}(f) = S(t-t_k)\Phi_k S(t_k-t_{k-1})\Phi_{k-1}\ldots\Phi_2 S(t_2-t_1)\Phi_1 S(t_1)\varphi .$$

To make the statements and formulas more transparent, we shall use the terminology and notation of the integration theory. In particular, we shall write

$$\int_{\Gamma_t} f(\gamma)M_t(d\gamma)\varphi = M_{t,\varphi}(f) ,$$

for every $f \in \mathcal{L}(t,\varphi)$.

Let λ be the one-dimensional Lebesque measure (and integral). Let $t > 0$. By $\mathcal{L}(\lambda \otimes \rho_{t,\varphi})$ is denoted the family of all functions, f , on $[0,t] \times \Gamma_t$ for which there exist λ-integrable functions g_j on $[0,t]$ and functions $h_j \in \mathcal{L}(t,\gamma)$, $j = 1,2,\ldots,$ such that

$$\sum_{j=1}^{\infty} \lambda(|g_j|)\rho_{t,\varphi}(h_j) < \infty$$

and

$$f(s,\gamma) = \sum_{j=1}^{\infty} g_j(s)h_j(\gamma) ,$$

for every $s \in [0,t]$ and $\gamma \in \Gamma_t$ such that

$$\sum_{j=1}^{\infty} |g_j(s)||h_j(\gamma)| < \infty .$$

Now, let V be a suitable $\mathbb{C}^{N\times N}$-valued function on $\mathbb{R}^d$ and let

$$e_t(\gamma) = \exp\left[\int_0^t V(\gamma(r))\,dr\right] , \tag{10}$$

for every $\gamma \in \Gamma_t$ for which the integral exists. If $e_t \in \mathcal{L}(t,\varphi)$, let

$$u(t) = u_\varphi(t) = \int_{\Gamma_t} e_t(\gamma)M_t(d\gamma)\varphi .$$

Let $t > 0$. Let

$$f(s,\gamma) = V(\gamma(s))\exp\left[\int_0^s V(\gamma(r))\,dr\right] , \tag{11}$$

for every $s \in [0,t]$ and $\gamma \in \Gamma_t$ for which the integral exists.

If this function, f, belongs to $\mathcal{L}(\lambda \otimes \rho_{\varphi,t})$, then $f(s,.) \in \mathcal{L}(t,\varphi)$, for λ-almost every s, and

$$u(t) = S(t)\varphi + \int_0^t S(t-s)\,Vu(s)\,ds \tag{12}$$

in the Bochner sense.

For the proof, we note that

$$\int_0^t f(s,\gamma)\,ds = e_t(\gamma) - 1 ,$$

for every $\gamma \in \Gamma_t$ such that the function $f(\cdot,\gamma)$ is λ-integrable. Also,

$$\int_{\Gamma_t} f(z,\gamma) M_t(d\gamma)\varphi = S(t-s)\,Vu(s) ,$$

for every $s \in [0,t]$ such that $f(s,.) \in \mathcal{L}(t,\varphi)$. Therefore, by Theorem 5.11 in [2],

$$u(t) - S(t)\varphi = \int_{\Gamma_t} (e_t(\gamma)-1) M_t(d\gamma)\varphi =$$

$$= \int_{\Gamma_t} \left[\int_0^t f(s,\gamma)\,ds\right] M_t(d\gamma)\varphi - \int_0^t \left[\int_{\Gamma_t} f(s,\gamma) M_t(d\gamma)\varphi\right] ds =$$

$$= \int_0^t S(t-s)\,Vu(s)\,ds .$$

The requirement that the function (11) belong to $\mathcal{L}(\lambda \otimes \rho_{\varphi,t})$, for every $\varphi \in E$, may prove to be rather strong. None-the-less, it may happen that $e_t \in \mathcal{L}(t,\varphi)$, for every $\varphi \in E$, and the map $\varphi \mapsto u_\varphi(t)$, $\varphi \in E$, is a bounded linear operator on E. In that case, we write $U(t)\varphi = u_\varphi(t)$, for every $\varphi \in E$, where $U(t) \in L(E)$, and

$$U(t) = \int_{\Gamma_t} e_t(\gamma) M_t(d\gamma) .$$

Moreover, if $t \mapsto U(t)$, $t \geq 0$, turns out to be a strongly continuous group, then the relation (12) indicates that its infinitesimal generator is an extension of the operator $\dot{S}(0) + V$, where $\dot{S}(0)$ is the infinitesimal generator of the semigroup S. The last remark is devoted to the description of a case in which this situation occurs.

REMARK FOUR

Using the notation of the preceding remark, we take $N = 1$ and $E = L^2(\mathbb{R}^d)$. Let the kernel k_t be defined by (2) and let $S : \mathbb{R} \to L(E)$ be the unitary group such that $S(0) = I$ and

$$(S(t)\varphi)(x) = \int_{\mathbb{R}^d} k_t(x-y)\varphi(y)dy ,$$

for every $t \neq 0$, $\varphi \in L^2 \cap L^1(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$.

Let w be the Wiener measure on Γ_t with unit variance per unit of time and with the initial distribution standard normal. That is, w is a probability measure such that

$$w(Z) = \left[(2\pi)^{dn} \prod_{j=1}^{n-1} (t_j - t_{j-1})\right]^{-1/2} \times$$

$$\times \int_{B_{n-1}} \cdots \int_{B_1} \int_{B_0} \exp\left[-\frac{x_0^2}{2} - \sum_{j=1}^{n-1} \frac{|x_j - x_{j-1}|^2}{2(t_j - t_{j-1})} \right] dx_0 dx_1 \ldots dx_{n-1} ,$$

for every set

$$Z = \{\gamma \in \Gamma_t : \gamma(t_j) \in B_j , \ j = 0,1,\ldots,n-1\} ,$$

where B_j are Borel sets in $\mathbb{R}^d$, $j = 1,2,\ldots,n-1$, $n = 1,2,\ldots$. Let us note that, instead of w, some other probability measures on Γ_t could be used equally well. The integral (expectation) of an integrable function, is denoted by $w(f)$. By $w(f|F)$ is denoted the conditional expectation of such a function, f, with respect to a measurable map, $F : \Gamma_t \to \Omega$, of Γ_t into a measurable space Ω. Hence, $w(f|F)$ is a measurable function on Ω.

For any integer $n \geq 1$, let $\pi_{t,n}$ be the projection of Γ_t onto the space $(\mathbb{R}^d)^n$ defined by

$$\pi_{t,n}(\gamma) = (\gamma(0), \gamma(t/n), \ldots, \gamma((n-1)t/n)) ,$$

for every $\gamma \in \Gamma_t$.

For any $r > 0$, let $B(r) = \{x \in \mathbb{R}^d : |x| \leq r\}$ be the closed ball in $\mathbb{R}^d$ of radius r centred at 0.

Let $n \geq 1$ be an integer and $r > 0$ a real number. Given a w-integrable function, f, let

$$(M_t^{n,r}(f)\varphi)(x) = \int_{B(r)}\int_{B(r)}\cdots\int_{B(r)} \varphi(x_0)\, w(f|\pi_{t,n})(x_0,x_1,\ldots,x_{n-1}) \times$$
$$\times \left[\prod_{j=1}^{n} k_{t/n}(x_j - x_{j-1})\right] dx_0\, dx_1 \ldots dx_{n-1} ,$$

for every $\varphi \in L^2(\mathbb{R}^d)$.

Let LIM be a Banach limit on the space, $\ell^\infty = \ell^\infty(\mathbb{N} \times (0,\infty))$, of all bounded complex valued functions on $\mathbb{N} \times (0,\infty)$ endowed with the sup-norm. That is, LIM is a continuous linear functional on ℓ^∞ assigning nonnegative real values to nonnegative real elements of ℓ^∞ such that

$$\mathrm{LIM}\,\xi = \lim_{r\to\infty,\ n\to\infty} \xi(n,r) ,$$

for every $\xi \in \ell^\infty$ for which the limit on the right exists in the usual sense. If convenient, we write

$$\lim_{r\to\infty,\ n\to\infty} \xi(n,r) = \mathrm{LIM}\,\xi .$$

By the same symbol, LIM, we denote the continuous linear map from the space, $\ell^\infty(\mathbb{N} \times (0,\infty), E)$, of all bounded E-valued functions on E into E defined by

$$<\mathrm{LIM}\xi\ ,\ \psi> \ = \ \underset{r\to\infty\ ,\ \ n\to\infty}{\mathrm{LIM}} \ <\xi(n,r),\ \psi> \ ,$$

for every $\psi \in E' = E = L^2(\mathbb{R}^d)$.

Let $\varphi \in L^2(\mathbb{R}^d)$. Let $\mathcal{L}(t,\varphi)$ be the family of all w-integrable functions, f, such that $M_t^{n,r}(f)\varphi \in L^2(\mathbb{R}^d)$, for every integer $n \geq 1$ and every number $r > 0$, and

$$\rho_{t,\varphi}(f) \ = \ w(|f|) + \sup\{\|M_t^{n,r}(f)\varphi\| : r > 0,\ n = 1,2,\ldots\} \ < \ \infty \ .$$

For every $f \in \mathcal{L}(t,\varphi)$, let

$$M_t(f)\varphi \ = \ \underset{r\to\infty\ ,\ \ n\to\infty}{\mathrm{LIM}} \ \mathrm{M}_t^{n,r}(f)\varphi \ .$$

It can be established now that, for every $\varphi \in E$ and $t > 0$, the requirements (i), (ii) and (iii) from Remark Three are satisfied. Moreover, for a large class of functions, V, on $\mathbb{R}^d$, the function e_t, defined by (10), belongs to $\mathcal{L}(t,\varphi)$, for every $\varphi \in L^2(\mathbb{R}^d)$, and the resulting map $\mathrm{t} \mapsto M_t(e_t)$, $\mathrm{t} \geq 0$, is a continuous semigroup of operators. In that case, we can write $M_t(e_t) = \exp(-it\bar{H})$, $t \geq 0$, where $\bar{H}$ is a self-adjoint extension of the operator $H = -1/2\Delta - iV$. Since the operators $M_t(e_t)$, $\mathrm{t} > 0$, depend on the choice of the Banach limit LIM, we have a method of "producing" many self-adjoint extensions of the operator H.

This construction can be adapted to the case of the wave equation. Details will be published elsewhere.

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AMS(MOS) Subject Classification: 28C20, 81C35