H_∞ Functional Calculus of Second Order Elliptic Partial Differential Operators on L^p Spaces

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Abstract. Let L be a strongly elliptic partial differential operator of second order, with real coefficients on $L^p(\Omega)$, $1 < p < \infty$, with either Dirichlet, or Neumann, or "oblique" boundary conditions. Assume that Ω is an open, bounded domain with C^2 boundary. By adding a constant, if necessary, we then establish an H_∞ functional calculus which associates an operator m(L) to each bounded holomorphic function m so that

$$\| m(L) \| \leq M \| m \|_\infty$$

where M is a constant independent of m.

Under suitable asumptions on L, we can also obtain a similar result in the case of Dirichlet boundary conditions where Ω is a non-smooth domain.

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1. Introduction and Notation

We denote the sectors

$$S_\theta = \{ z \in \mathbb{C} \mid z = 0 \text{ or } |\arg z| \leq \theta \}$$
$$S^0_\theta = \{ z \in \mathbb{C} \mid z \neq 0 \text{ or } |\arg z| < \theta \}$$

A linear operator L is of type ω in a Banach space X if L is closed, densely defined, $\sigma(L)$ is a subset of $S_\omega \cup \{\infty\}$ and for each $\theta \in (\omega, \pi]$, there exists $C_\theta < \infty$ such that $\| (L-zI)^{-1} \| \leq C_\theta |z|^{-1}$ for all $z \notin S^0_\mu$, $z \neq 0$.

For $0 < \mu < \pi$, denote

$$H_\infty(S^0_\mu) = \{ f : S^0_\mu \to \mathbb{C} \mid f \text{ analytic and } \| f \|_\infty < \infty \}$$

where $\| f \|_\infty = \sup \{ |f(z)| \mid z \in S^0_\mu \}$,

$$\Psi(S^0_\mu) = \{ f \in H_\infty(S^0_\mu) \mid \exists s>0, c \geq 0 \text{ such that } |f(z)| \leq \frac{c|z|^s}{1+|z|^{2s}} \}.$$

Let Γ be the contour defined by

$$g(t) = \begin{cases} -t \exp(i\theta) & \text{for } -\infty < t < 0 \\ t \exp(i\theta) & \text{for } 0 \le t < +\infty \end{cases}$$

Assume that L is of type ω, $\omega < \theta < \pi$. We then define the bounded linear operator $\psi(L)$ by

$$\psi(L) = (2\pi i)^{-1} \int_\Gamma (L - zI)^{-1} \psi(z)\, dz \quad , \quad \psi \in \Psi(S_\mu^0).$$

If L is 1-1 with dense range, then

$$m(L) = (\psi(L))^{-1} (m\psi)(L)$$

where $\psi(z) = z(1+z)^{-2}$ and $m \in H_\infty(S_\mu^0)$

Details of these definitions can be found in [10].

We consider the following properties:

(I) The boundedness of the purely imaginary powers:

$\{L^{iy} \mid y \in R\}$ is a continuous group and

$\| L^{iy} \| \le C_\mu \exp(\mu|y|)$ where μ and C_μ are positive constants,

(II) The H_∞ functional calculus

$$\| m(L) \| \le c \| m \|_\infty \quad \text{for } m \in H_\infty(S_\mu^0),\ \mu > \omega ,$$

(III) The quadratic estimate when $X = L^p$:

$$\left\| \left\{ \int_0^\infty |\psi(tT) f(.)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_p \le K \| f \|_p \quad \text{for some } \psi \in \Psi(S_\alpha^0).$$

It is known that

(i) In Hilbert spaces, (I) $\Leftrightarrow$ (II) $\Leftrightarrow$ (III).

(ii) (II) $\Rightarrow$ (I) but the reverse is not true in Banach spaces.

(iii) (II) $\Rightarrow$ (III) for suitable $\psi \in \Psi(S_\alpha^0)$ and (III) $\Rightarrow$ (II) for $m \in H_\infty(S_\mu^0)$ with suitable conditions on μ

The aim of this paper is to show that property (II), hence property (III), is satisfied by second order elliptic partial differential operators in L^p spaces, $1 < p < \infty$, under suitable assumptions on the smoothness of the coefficients and of the boundaries of domains.

2. Transference Methods and Multiplier Theorems.

The main technique that we employed in this paper to obtain H_∞ functional calculus is transference method. We define an operator A to be a subpositive contraction if there exists a positive contraction P such that $P+\mathrm{Re}\{e^{i\theta}A\}$ are positivity preserving for all q. The following theorem appeared in [2]:

THEOREM 1: Let X be a measure space and assume that

$$m(L)=\int_0^\infty e^{-tL}\,\phi(t)\,dt$$

is a linear operator from $L^p(X)$ into $L^p(X)$, $1\le p\le\infty$, where ϕ is integrable with compact support and e^{-tL} is a subpositive contraction semigroup, then

$$\| m(L)\| \le \|\phi *\|$$

where $\|\phi *\|$ is the norm of the convolution operator $f\rightarrow \phi * f$ on $L^p(\mathbb{R})$.

We now prove a multiplier theorem:

THEOREM 2: Let X be a measure space and L an operator of type ω, $\omega < \frac{\pi}{2}$, from $L^p(X)$ into $L^p(X)$, $1<p<\infty$. Assume that L generates a subpositive contraction semigroup e^{-tL}. Then there exists a constant $M>0$ such that

$$\| m(L)\| \le M\| m\|_\infty$$

for all $m\in H_\infty(S_\mu^0)$, $\mu > \frac{\pi}{2}$.

Proof: Let $m\in \Psi_1(S_\mu^0)$, $\mu > \frac{\pi}{2}$, where

$$\Psi_1(S_\mu^0)=\{g\in H_\infty(S_\mu^0) \mid |g(z)| \le \frac{c|z|^s}{1+|z|^{2s}},\ s>1\}$$

Let ϕ be defined by $m(it)=\hat{\phi}(t)$, $t\in\mathbb{R}$, where $\hat{\phi}$ denotes the Fourier transform of ϕ on $\mathbb{R}$. Then ϕ has its support included in $[0,\infty)$, and ϕ belongs to $L^2(\mathbb{R})$.

Let $v_k(t)=\max\{1-\frac{|t|}{k}, 0\}$, then $\hat{v}_k(\zeta)\ge 0$ and

$$\int_{-\infty}^{+\infty} \hat{v}_k(\zeta)\, d\zeta = 2\pi.$$

We fix an arbitrary constant $c_0 > 0$ and let $L_0 = L + c_0$. Then

$$\| e^{-tL_0} \| = \| e^{-t(L+c_0)} \| \leq e^{-c_0 t}$$

Consider

$$m_k(L_0)\, u = \int_0^{\infty} (e^{-tL_0} u)(v_k.\phi)(t)\, dt$$

$$= \int_0^{\infty} (e^{-tL} u)(v_k.\alpha)(t)\, dt \qquad \text{where } \alpha(t) = e^{-c_0 t}\phi(t)$$

and

$$m(L_0)\, u = \int_0^{\infty} (e^{-tL_0} u)\phi(t)\, dt = \int_0^{\infty} (e^{-tL} u)\, \alpha(t)\, dt$$

It is not difficult to check that the above integrals are absolutely convergent. It follows from theorem 1 that

$$\| m_k(L_0) \| \leq \| (v_k.\alpha)_* \|$$

On the other hand

$$\|(v_k.\alpha)_* u\|_p = \| \int_{-\infty}^{+\infty} \frac{1}{2\pi} \{ \int_{-\infty}^{+\infty} v_k(\zeta)\, e^{it\zeta}\, d\zeta \} \alpha(t)\, u(x-t)\, dt \,\|_p$$

$$\leq \frac{2}{\pi} \{ \int_{-\infty}^{+\infty} \hat{v}_k(\zeta)\, d\zeta \} \, \| \int_{-\infty}^{+\infty} \alpha(t)\, e^{i\zeta x} e^{-\zeta(x-t)}\, u(x-t)\, dt \,\|_p$$

$$\leq \| \chi_\zeta^{-1} (\alpha * \chi_\zeta u) \|_p \qquad \text{where } \chi_\zeta(\lambda) = e^{-i\zeta\lambda}$$

$$\leq \| \alpha * \| \, \| u \|_p$$

Therefore

$$\| m_k(L_0) \| \leq \| \alpha * \|$$

We also have

$$\| m_k(L_0) u - m(L_0) u \|_p = \| \int_0^{+\infty} (e^{-tL} u)\, (1 - v_k)(t) \alpha(t)\, dt \,\|_p$$

$$\leq \int_0^{+\infty} \|(e^{-tL}u)\|_p \, |(1- v_k)(t)\alpha(t)| \, dt$$

$$\leq \| u \|_p \left(\int_0^k \frac{t}{k} |\alpha(t)| \, dt + \int_k^{\infty} |\alpha(t)| \, dt \right)$$

It is clear that the right hand side $\to 0$ as $k \to \infty$. Hence

$$\| m(L_0) \| \leq \| \alpha * \|.$$

From the definitions of α and ϕ, we have

$$\hat{\alpha}(t) = m(c_0 + it),$$

and it follows from Mikhlin's multiplier theorem that

$$\| \alpha * \| \leq M \| m \|_{\infty}.$$

Hence

$$\| m(L_0) \| \leq M \| m \|_{\infty}.$$

Using Fubini's theorem, it can be shown that the definition of the operator $m(L_0)$ as above is consistent with the usual definition of functional calculus, e.g. that of section 1.

We also note that $\Psi_1(S^0_\mu)$ is dense in $H_\infty(S^0_\mu)$ in the sense of uniform convergence on compact subsets in the complex plane which do not contain 0. Thus

$$\| m(L_0) \| \leq M \| m \|_{\infty}$$

for all $m \in H_\infty(S^0_\mu)$, $\mu > \frac{\pi}{2}$

Since the constant M is independent of c_0, we let $c_0 \to 0$ and obtain the estimate

$$\| m(L) \| \leq M \| m \|_{\infty}$$

for all $m \in H_\infty(S^0_\mu)$, $\mu > \frac{\pi}{2}$.

We can always employ complex interpolation to improve the condition $\mu > \frac{\pi}{2}$. For example, with the additional assumption that L is positive self-adjoint in L^2, we have the following estimate which appereared in [4]:

THEOREM 3 : Suppose m is a bounded holomorphic function in the sector S^0_μ, where $0 < \mu < \frac{\pi}{2}$. Then the following estimate holds, provided that $|\frac{1}{p} - \frac{1}{2}| < \frac{\mu}{\pi}$

$$\| m(L) \| \le C \left(\frac{\mu}{\pi} - |\frac{1}{p} - \frac{1}{2}| \right)^{-5/2} \| m \|_\infty$$

where C is an absolute constant, $1 < p < \infty$.

3. H_∞ Functional Calculus of Second Order Elliptic Partial Differential Operators on C^2 Domains.

Let Ω be an open, bounded subset of $\mathbb{R}^n$. The operator L is a second order strongly elliptic operator in Ω, and B is a real boundary operator of order d (d=0 or d=1). We shall also make the following assumptions :

The boundary $\partial\Omega$ of Ω is of class C^2.

The operator L is in divergence form :

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j} u\right)$$

with $a_{ij} = a_{ji} \in C^{0,1}(\bar{\Omega})$ and there exists $\gamma > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \ge \gamma|\zeta|^2$$

for all $x \in \Omega$ and $\zeta \in \mathbb{R}^n$.

B is either the identity operator (thus d=0) or

$$Bu = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} u$$

with $b_i \in C^{0,1}(\bar{\Omega})$, $1 \le i \le n$, (then d=1) and $\sum_{i=1}^{n} b_i\nu^i \ne 0$ everywhere on $\partial\Omega$ (In other words, $\partial\Omega$ is nowhere characteristic for B).

We define

$$D(L) = \{u \in W^{2,p}(\Omega) \mid Bu = 0 \text{ on } \Omega\}$$

where $W^{2,p}(\Omega) = \{u \in L^p(\Omega) \mid \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}$ belong to $L^p(\Omega)$ in distribution sense for i,j =1,...,n }

<u>THEOREM 4 :</u>

(i) L generates a positivity-preserving contraction semigroup e^{-tL} on $L^p(\Omega)$, $1 < p < \infty$

(ii) L has an H_∞ functional calculus as in theorem 2, i.e. there exists a constant $M > 0$ such that

$$\| m(L) \| \leq M \| m \|_\infty$$

for $m \in H_\infty(S_\mu^0)$, $\mu > \frac{\pi}{2}$.

<u>Proof</u> :

(ii) follows directly from (i) and theorem 2. For simplicity, we present the proof of (i) where L is positive and self-adjoint. The general case of " oblique " boundary conditions can be handled in the same way as in [9], chapter 2, inequality (2,3,1,11) with some modifications for complex-valued functions.

Let $u \in C^2(\bar{\Omega})$, which is a dense set of D(L).

Let $u_\varepsilon^* = (u\bar{u} + \varepsilon)^{(p-2)/2}\, \bar{u} \in L^q(\Omega)$, $\varepsilon > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. We have :

$$\langle Lu, u_\varepsilon^* \rangle = \int_\Omega - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij} \frac{\partial}{\partial x_j} u)\, (u\bar{u} + \varepsilon)^{(p-2)/2}\, \bar{u}\, dx$$

$$= \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i} [(u\bar{u} + \varepsilon)^{(p-2)/2}\, \bar{u}\,]\, dx$$

$$= \int_\Omega \sum_{i,j=1}^n a_{ij} \{ (u\bar{u} + \varepsilon)^{(p-2)/2} \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_i} + \bar{u} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i} (u\bar{u} + \varepsilon)^{(p-2)/2} \}\, dx$$

Note that $\frac{\partial}{\partial x_i}(u\bar{u} + \varepsilon)^{(p-2)/2} = \frac{1}{2}(p-2)\, (u\bar{u} + \varepsilon)^{(p-4)/2} (\bar{u} \frac{\partial}{\partial x_i} + u \frac{\partial \bar{u}}{\partial x_i})$. We denote

$$| u |^{(p-4)/2}\, \bar{u} \frac{\partial u}{\partial x_i} = \alpha_i + \beta_i$$

and $| u |^{(p-4)/2}\, \bar{u} \frac{\partial u}{\partial x_j} = \alpha_j + \beta_j$.

Let $\varepsilon \to 0$ and it follows from the Lebesgue dominated convergence theorem that for $u^* = (u\bar{u})^{(p-2)/2}\, \bar{u}$, we have

$$\langle Lu, u^* \rangle = \int_\Omega \sum_{i,j=1}^n a_{ij} \{ (p-1)\alpha_i\alpha_j + \beta_i\beta_j + i((p-1)\alpha_i\beta_j - \alpha_j\beta_i) \}\, dx$$

Hence $\mathrm{Re}\, \langle Lu, u^* \rangle = \int_\Omega \sum_{i,j=1}^n a_{ij} \{ (p-1)\alpha_i\alpha_j + \beta_i\beta_j \}\, dx \geq 0$ from the ellipticity of L.

Therefore, for $\lambda > 0$:

$$| \langle (L+\lambda)u, u^* \rangle | \geq \mathrm{Re} \langle Lu, u^* \rangle + \lambda \langle u, u^* \rangle$$

$$\geq \lambda\langle u, u^*\rangle = \lambda \| u \|_p^p$$

Thus

$$\lambda \| u \|_p^p \leq \left| \int_\Omega ((L+\lambda)u)\, |u|^{p-2}\, \bar{u}\ dx \right|$$

$$\leq \{ \int_\Omega |(L+\lambda)u|^p\, dx \}^{1/p} \{ \int_\Omega (|u|^{p-1})^q\, dx \}^{1/q}$$

$$= \| (L+\lambda)u \|_p \, \| u \|_p^{p/q}$$

Thus we obtain $\lambda \| u \|_p \leq \|(L+\lambda)u \|_p$. This inequality and the semi-Fredholm property of L shows that L generates a contraction semigroup on $L^p(\Omega)$.

The positivity-preserving property of the semigroup follows from that of the resolvent $(L+\lambda)^{-1}$ which is a consequence of maximum principle.See [12], chapter 2 for details of various forms of maximum principle for second order elliptic partial differential operators. Hence the theorem is proved.

We now extend a result of purely imaginary powers of operators in [6] to obtain H_∞ functional calculus of elliptic operators which are not in divergence form. Let the operator T be defined by

$$Tu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j} u\right) + \sum_{i=1}^{n} k_i \frac{\partial}{\partial x_i} u + cu$$

$$= Lu + \sum_{i=1}^{n} k_i \frac{\partial}{\partial x_i} u + cu$$

where k_i and c are real-valued functions and belong to $L^\infty(\Omega)$.With the same domain and boundary conditions as for L, we have the following result :

<u>THEOREM 5</u>: There exist constants c_0, N, μ_0, ($\mu_0 < \pi$), such that

$$\| m(T+c_0) \| \leq N \| m \|_\infty$$

for all $m \in H_\infty(S_\mu^0)$, $\mu > \mu_0$.

Proof: That $(T+c_0)$ is of type $\omega = \mu_0$ for sufficiently large c_0 is well-known, e.g. [1]. Since we can always assume that $0 \in \rho(T+c_0)$, there exists $K > 0$ such that $\| (L-\lambda I)^{-1} \| \leq K(1+|\lambda|)^{-1}$ for $\lambda \in \rho(T+c_0)$. Denote $A = (T+c_0 I) - L$, we have :

$$\| m(L+A)\, u - m(L)u \|_p \leq \| \frac{1}{2\pi i} \int_\Gamma m(\lambda)[(L+A-\lambda I)^{-1}u - (L-\lambda I)^{-1}u]\, d\lambda \|_p$$

$$\leq \| \frac{1}{2\pi i} \int_\Gamma m(\lambda)[(L+A-\lambda)^{-1} LA^{-\theta}A^{\theta}(L-\lambda)^{-1}u]\, d\lambda \|_p$$

for a fixed θ, $\frac{1}{2} < \theta < 1$

$$\leq K \| m \|_\infty \| u \|_p \int_0^\infty \frac{r^{\theta-1}}{r+1}\, dr$$

$$\leq K_1 \| m \|_\infty \| u \|_p$$

Hence $\| m(T+c_0) \| = \| m(L+A) \| \leq N \| m \|_\infty$

Note that in the above estimate, we have employed the boundedness of the operator $LA^{-\theta}$ and the momentum inequality.

Remark:

1) Under suitable boundary conditions so that L is positive self-adjoint in $L^2(\Omega)$ we obtain an H_∞ functional calculus as that of theorem 3 with $\mu > \pi \left| \frac{1}{p} - \frac{1}{2} \right|$

2) When the boundary condition is Dirichlet condition, we can reduce the smoothness of the boundary $\partial\Omega$ to being $C^{1,1}$.

4. H_∞ functional calculus of elliptic operators on Lipschitz domains :

We first recall a well-known result which characterizes the generators of contraction semigroups. Let E be a complex Banach space and let $p: E \to \mathbb{R}^+$ be a seminorm on E, i.e. $p(f+g) \leq p(f)+p(g)$ and $p(\lambda f) \leq |\lambda| p(f)$ for all $f,g \in E$ and $\lambda \in \mathbb{C}$. The subdifferential $dp(f)$ of p at $f \in E$ is defined by

$$dp(f) = \{ \phi \in E^* \mid \mathrm{Re}\langle f,g\rangle \leq p(g) \text{ for all } g \in E \text{ and } \langle f,\phi\rangle = p(f) \}$$

We assume in addition that p is continuous.Then it follows from the Hahn-Banach theorem that $dp(f)$ is not empty for any $f \in E$.

A linear operator L is called p-accretive if for all $f \in D(L)$, there exists $\phi \in dp(f)$ such that $\mathrm{Re}\langle Lf,\phi \rangle \geq 0$. Then it can be proved that an operator L is p-accretive if and only if $p((tL-1)f) \geq p(f)$ for all $f \in D(L)$ and $t > 0$.

A linear operator L is called accretive if it is p-accretive for the norm $p(f) = \| f \|$, $f \in E$.

THEOREM 6 (Lumer-Phillips): Let L be a densely defined operator on a complex Banach space E. The following assertions are equivalent:

(i) L is closable and the closure of L generates a contraction semigroup e^{-tL}.

(ii) L is accretive and $(L-\lambda I)$ has dense range for some $\lambda < 0$.

Proof of this theorem can be found in [11].

We now assume Ω to be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j} u\right)$$

with $a_{ij} = a_{ji} \in C^{1,\alpha}(\Omega) \cap C(\bar{\Omega})$, $0 < \alpha < 1$, and there exists $\gamma > 0$ such that

$$\sum_{i\, j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \geq \gamma|\zeta|^2$$

for all $x \in \Omega$ and $\zeta \in \mathbb{R}^n$.

Let $D(L) = \{ u \in C^2(\Omega) \cap C(\bar{\Omega}) \mid u = 0 \text{ on } \partial\Omega \text{ and } Lu \in L^p(\Omega) \}$

Following is the main result of this section:

THEOREM 7: The operator L is closable, the closure $\bar{L}$ of L generates a positivity-preserving contraction semigroup in $L^p(\Omega)$. Thus $\bar{L}$ has an H_∞ functional calculus:

$$\| m(\bar{L}) \| \leq M \| m \|_\infty$$

for all $m \in H_\infty(S_\mu^0)$ with $\mu > \pi\left|\frac{1}{p} - \frac{1}{2}\right|$.

Proof:

It is obvious that D(L) is dense in $L^p(\Omega)$.

That $(L-\lambda I)D(L)$, $\lambda < 0$, is dense in $L^p(\Omega)$ is well-known in the theory of partial differential equations, e.g. Theorem 6.13, [8].

The inequality $\|(\lambda L - I)f\| \geq \|f\|$ can be verified as in (i), theorem 4.

The positivity-preserving property of the semigroup follows from the maximum principle.

Thus theorem 7 follows from theorem 6.

Remark: Let

$$Tu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j} u\right) + \sum_{i=1}^{n} k_i \frac{\partial}{\partial x_i} u) + cu$$

$$= Lu + \sum_{i=1}^{n} k_i \frac{\partial}{\partial x_i} u + cu$$

where k_i and c are real-valued functions and belong to $C^{0,\alpha}(\Omega)$, $0 < \alpha < 1$, with the domain

$$D(T) = \{ u \in C^2(\Omega) \cap C(\bar{\Omega}) \mid u = 0 \text{ on } \partial\Omega \text{ and } Tu \in L^p(\Omega) \}$$

It is not difficult to see from perturbation theory that there exists a constant c_0 such that $(\bar{T} + c_0 I)$ is of type ω for some $\omega < \pi$. A similar proof to that of theorem 5 shows that $(\bar{T} + c_0 I)$ has an H_∞ functional calculus as in theorem 5.

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