

AN EXAMPLE IN THE THEORY OF SPECTRAL AND WELL-BOUNDED OPERATORS

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Abstract. An example is given of a linear transformation which defines a well-bounded operator on $L^p[0, 1]$ for $1 \leq p \leq \infty$. It is shown that the properties of the decomposition of the identity associated with this operator (and consequently the type of functional calculus that the operator admits) vary markedly depending on the domain space.

1. Introduction

As has been noted by several authors (for example, [Dun, p. 237]), some operators which are self-adjoint on L^2 fail to possess an integral representation with respect to a (countably additive) spectral measure on the other L^p spaces. Our aim in this paper is to show that the spectral behaviour of such operators can be highly dependent on the structure of the Banach spaces on which the operator acts. We begin with a very brief survey of the relevant parts of the theory of scalar-type spectral and well-bounded operators. For a full account of this theory, the reader is directed to [Dow] or [DS3].

A bounded operator T on a Banach space X is said to be *scalar-type spectral* if there exists a spectral measure μ taking values in $B(X)$ such that $T = \int_{\sigma(T)} \lambda \mu(d\lambda)$. Scalar-type spectral operators possess a weakly compact $C(\sigma(T))$ functional calculus and give rise in general to unconditional spectral expansions. Such an operator will be called *real scalar-type spectral* if $\sigma(T) \subset \mathbb{R}$.

An operator T is *well-bounded* if it admits a functional calculus for the absolutely continuous functions on some compact interval $[a, b]$ of the real line. Well-bounded operators give rise to increasing families of projections $\{F(\lambda)\}_{\lambda \in \mathbb{R}}$ acting on X^* for which

$$\langle Tx, x^* \rangle = b \langle x, x^* \rangle - \int_a^b \langle x, F(\lambda)x^* \rangle d\lambda$$

for all $x \in X$ and $x^* \in X^*$. These families, which satisfy certain natural properties are known as *decompositions of the identity for T* . Conversely, every decomposition of the identity defines a well-bounded operator. In general the decomposition of the identity associated with a well-bounded operator need not be unique. However, if T is *decomposable in X* (that is, each element $F(\lambda)$ of a decomposition of the identity associated with T is the adjoint of an operator $E(\lambda)$ acting on X), then the decomposition of the identity is uniquely determined. We shall call $E(\lambda)$ the *decomposition of the identity in X associated with T* . A well-bounded operator decomposable in X is said to be of *type (B)* if the function $\lambda \mapsto E(\lambda)$ is strongly right continuous and has a strong left limit at every point in \mathbb{R} .

This happens precisely when the absolutely continuous functional calculus for T is weakly compact. A decomposition of the identity in X , $\{E(\lambda)\}$, satisfying these extra continuity conditions is known as a *spectral family for T* and one can perform a Riemann-Stieltjes type integration with respect to $\{E(\lambda)\}$ to give $T = \int_{[a,b]}^{\oplus} \lambda dE(\lambda)$. A significant difference between well-bounded and scalar-type spectral operators is that the former give rise to conditional rather than unconditional spectral expansions. The following table shows the functional calculus that each of the above types of operator admits.

Operator type	Functional calculus
self-adjoint	isometric $BM(\sigma(T))$
scalar-type spectral	$BM(\sigma(T))$
well-bounded type (B)	$BV[a, b]$
well-bounded	$NBV_0[a, b]$ for T^*

Here $BM(\sigma(T))$ stands for the bounded Borel measurable functions on $\sigma(T)$, $BV[a, b]$ for the functions of bounded variation on $[a, b]$ and $NBV_0[a, b]$ for the subspace of all such functions f which can be written in the form $f = f_{ac} + f_b$ where $f_{ac} \in AC[a, b]$ and f_b is the limit of a uniformly convergent sequence of step functions. For an arbitrary well-bounded operator T , it may not be possible to extend the functional calculus for T to even this class of functions.

2. The example

We shall now show that the properties which the decomposition of the identity for a well-bounded operator exhibits depend crucially on the structure of the space chosen as the domain of the operator. The scalar field in what follows may be taken to be either the real or complex numbers. In the remainder of the paper, the notation $\|A\|_p$ will be used to denote the norm of an operator A acting on L^p .

The following lemma is an easy consequence of [D1, Theorem 4.1].

Lemma 1. *Suppose that $(\Omega, \mathcal{A}, \nu)$ is a measure space and that $1 < p < \infty$. Suppose also that T is a well-bounded operator on $L^p(\Omega, \mathcal{A}, \nu)$ and that the decomposition of the identity associated with T satisfies $\|F(\lambda)\| \leq 1$ for all $\lambda \in \mathbb{R}$. Then T is real scalar-type spectral.*

Proof. It is well-known that for all polynomials g ,

$$\|g(T)\| \leq \sup_{\lambda \in [a,b]} \|F(\lambda)\| \left\{ |g(b)| + \int_a^b |g'(t)| dt \right\}.$$

Thus T has a contractive $AC[a, b]$ functional calculus and hence [D1, Theorem 4.1] is scalar-type spectral. ■

Remark. This swift proof rather obscures the reason why the lemma is true. In fact the lemma can be proved by a simplified version of the proof of [D1, Theorem 4.1]. The important point needed in constructing a countably additive measure is that if $0 = P_0, P_1, P_2, \dots$

is an increasing sequence of contractive projections on $L^p(\Omega, \mathcal{A}, \nu)$ and $\{\alpha_j\}$ is a sequence of scalars bounded in modulus by 1, then

$$\left\| \sum_{j=1}^{\infty} \alpha_j (P_j - P_{j-1}) \right\|_p \leq \max \{p, p/(p-1)\} - 1.$$

This rests on a characterisation of contractive projections on L^p spaces in terms of conditional expectation operators due to Ando and the fact that martingale transforms are bounded on these spaces (see [An,B1,B2,DO]).

Define the linear transformation T on the Lebesgue integrable functions on $[0, 1]$ by

$$(Tf)(t) = tf(t) + \int_0^1 \log(1 - \min\{u, t\})f(u) du.$$

Theorem 2. *The linear transformation T above defines a well-bounded operator on $L^p[0, 1]$ for $1 \leq p \leq \infty$. Furthermore*

- (i) T is self-adjoint on $L^2[0, 1]$;
- (ii) T is real scalar-type spectral on $L^p[0, 1]$ for $1 < p < \infty$;
- (iii) T is well-bounded of type (B) on $L^1[0, 1]$ - but is not scalar-type spectral;
- (iv) T is a well-bounded operator decomposable in X when $X = L^\infty[0, 1]$ - but is not of type (B) on this space.

In fact, on $C[0, 1]$, T defines a well-bounded operator which is not decomposable in X .

Proof. For $\lambda \in [0, 1)$ define $E(\lambda) \in B(L^p[0, 1])$ ($1 \leq p \leq \infty$) by

$$(E(\lambda)f)(t) = \begin{cases} f(t), & \text{for } t \in [0, \lambda); \\ 1/(1-\lambda) \int_\lambda^1 f(u) du & \text{for } t \in [\lambda, 1]. \end{cases}$$

If we set $E(\lambda) = 0$ for $\lambda < 0$ and $E(\lambda) = I$ for $\lambda \geq 1$, then it is not hard to see that $\{E(\lambda)\}$ forms a spectral family on $L^p[0, 1]$ for $1 \leq p < \infty$. Thus $\{E(\lambda)\}$ determines a well-bounded operator $S \in B(L^p[0, 1])$. For $1 \leq p < \infty$, fix $f \in L^p[0, 1]$ and $\phi \in L^q[0, 1] = L^p[0, 1]^*$. Then

$$\begin{aligned} \langle Sf, \phi \rangle &= \langle f, \phi \rangle - \int_0^1 \langle E(\lambda)f, \phi \rangle d\lambda \\ &= \int_0^1 f(t)\phi(t) dt - \int_0^1 \left\{ \int_0^\lambda f(t)\phi(t) dt + \int_\lambda^1 \frac{1}{1-\lambda} \int_\lambda^1 f(u) du \phi(t) dt \right\} d\lambda \\ &= \int_0^1 f(t)\phi(t) dt - \int_0^1 \int_0^\lambda f(t)\phi(t) dt d\lambda \\ &\quad + \int_0^1 \int_0^1 \int_0^1 \frac{1}{1-\lambda} \chi_{[\lambda, 1]}(u) \chi_{[\lambda, 1]}(t) f(u)\phi(t) du dt d\lambda. \end{aligned}$$

Let $h(u, t, \lambda) = (1 - \lambda)^{-1} \chi_{[\lambda, 1]}(u) \chi_{[\lambda, 1]}(t) f(u) \phi(t)$. Our next step will be to apply Fubini's theorem to the last two integrals. It is not immediately clear however that h is integrable, so we shall proceed by verifying this. By Tonelli's theorem then

$$\begin{aligned} \int_{[0, 1]^3} |h| &= \int_0^1 \left\{ \int_0^1 \left\{ \int_0^1 |h| \, du \right\} dt \right\} d\lambda \\ &= \int_0^1 \frac{1}{1 - \lambda} \left\{ \int_0^1 |f(u)| \chi_{[\lambda, 1]}(u) \, du \right\} \left\{ \int_0^1 |\phi(t)| \chi_{[\lambda, 1]}(t) \, dt \right\} d\lambda \\ &\leq \int_0^1 \frac{1}{1 - \lambda} \|f\|_p (1 - \lambda)^{1/q} \|\phi\|_q (1 - \lambda)^{1/p} d\lambda \\ &\quad \text{(by Hölder's inequality)} \\ &= \|f\|_p \|\phi\|_q < \infty. \end{aligned}$$

It follows that h is integrable, so applying Fubini's theorem gives that

$$\begin{aligned} \langle Sf, \phi \rangle &= \int_0^1 f(t) \phi(t) \, dt - \int_0^1 \int_t^1 f(t) \phi(t) \, d\lambda dt \\ &\quad - \int_0^1 \int_0^1 \int_0^{\min\{u, t\}} \frac{1}{1 - \lambda} f(u) \phi(t) \, d\lambda du dt \\ &= \int_0^1 f(t) \phi(t) \, dt - \int_0^1 (1 - t) f(t) \phi(t) \, dt \\ &\quad - \int_0^1 \int_0^1 (-\log(1 - \min\{u, t\})) f(u) \phi(t) \, du dt \\ &= \int_0^1 \left\{ t f(t) + \int_0^1 \log(1 - \min\{u, t\}) f(u) \, du \right\} \phi(t) \, dt. \end{aligned}$$

This implies that for $1 \leq p < \infty$, $S = T$, and so T is a bounded operator of type (B) on these spaces.

Checking that T is self-adjoint on $L^2[0, 1]$ is routine. Statement (ii) of the theorem will follow immediately from Lemma 1 if we can show that $\|E(\lambda)\|_p \leq 1$ for $\lambda \in \mathbb{R}$ and $1 < p < \infty$. But this is easily seen to be true (even for $p = 1$ and $p = \infty$), since for $\lambda \in [0, 1)$, $E(\lambda)$ is a conditional expectation operator.

The next step is to show that T is not scalar-type spectral on $L^1[0, 1]$. Since $L^1[0, 1]$ does not contain any subspace isomorphic to c_0 , [D2, Theorem 2] implies that T is scalar-type spectral if and only if the function $\lambda \mapsto \langle E(\lambda)f, \phi \rangle$ is of bounded variation for all $f \in L^1[0, 1]$ and all $\phi \in L^\infty[0, 1]$. Thus it suffices to show that there exist functions f and ϕ and an increasing sequence $0 \leq \lambda_1 < \lambda_2 < \dots < 1$ such that

$$\sum_{j=1}^n |\langle (E(\lambda_j) - E(\lambda_{j-1}))f, \phi \rangle| \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

It is slightly easier to do this indirectly. Let

$$S_n = \sum_{j=1}^n (-1)^j (E(\lambda_j) - E(\lambda_{j-1})).$$

If we can find $\phi \in L^\infty[0, 1]$ such that $\lim_{n \rightarrow \infty} \|S_n^* \phi\|_\infty = \infty$, then by the Principle of Uniform Boundedness there exists $f \in L^1[0, 1]$ such that $\sup_n |\langle S_n f, \phi \rangle| = \infty$. But

$$\begin{aligned} |\langle S_n f, \phi \rangle| &= \left| \langle \sum_{j=1}^n (-1)^j (E(\lambda_j) - E(\lambda_{j-1})) f, \phi \rangle \right| \\ &\leq \sum_{j=1}^n |\langle (E(\lambda_j) - E(\lambda_{j-1})) f, \phi \rangle|. \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \sum_{j=1}^n |\langle (E(\lambda_j) - E(\lambda_{j-1})) f, \phi \rangle| = \infty$.

It is easy to check that

$$(E(\lambda)^* \phi)(t) = \begin{cases} \phi(t), & \text{for } t \in [0, \lambda); \\ 1/(1-\lambda) \int_\lambda^1 \phi(u) du & \text{for } t \in [\lambda, 1]. \end{cases}$$

For $j = 0, 1, 2, \dots$, let $\lambda_j = (3^j - 1)/3^j$. Define $\phi \in L^\infty[0, 1]$ by $\phi(t) = (-1)^k$ for $t \in [\lambda_{k-1}, \lambda_k)$. Then

$$\begin{aligned} \frac{1}{1-\lambda_j} \int_{\lambda_j}^1 &= 3^j \sum_{k=0}^{\infty} (-1)^{j+1+k} \frac{2}{3^{j+1}} \left(\frac{-1}{3}\right)^k \\ &= (-1)^{j+1} \frac{2}{3} \frac{3}{4} \\ &= (-1)^{j+1} / 2. \end{aligned}$$

Thus

$$(E(\lambda_j)^* \phi)(t) = \begin{cases} (-1)^m & \text{for } t \in [\lambda_{m-1}, \lambda_m), m < j; \\ (-1)^{j+1} / 2 & \text{for } t \in [\lambda_j, 1]. \end{cases}$$

Therefore, for $j \geq 1$,

$$((E(\lambda_j)^* - E(\lambda_{j-1}))\phi)(t) = \begin{cases} 0 & \text{for } t \in [0, \lambda_{j-1}); \\ (-1)^j / 2 & \text{for } t \in [\lambda_{j-1}, \lambda_j); \\ (-1)^{j+1} & \text{for } t \in [\lambda_j, 1]. \end{cases}$$

From this it follows that $\|S_n^*\|_\infty = n$. Thus we have shown that there exist functions f and ϕ such that the function $\lambda \mapsto \langle E(\lambda)f, \phi \rangle$ is not of bounded variation, and so T is not scalar-type spectral.

We next turn to the behaviour of T acting on $L^\infty[0, 1]$. It is readily verified that if $f \in L^1[0, 1]$ and $\phi \in L^\infty[0, 1]$, then $\langle Tf, \phi \rangle = \langle f, T\phi \rangle$. Thus if g is a polynomial

$$\begin{aligned} |\langle f, g(T)\phi \rangle| &= |\langle g(T)f, \phi \rangle| \\ &\leq \|f\|_1 \|\phi\|_\infty \left\{ |g(b)| + \int_0^1 |g'(t)| dt \right\} \end{aligned}$$

where the inequality follows because T has a contractive absolutely continuous functional calculus as an operator on $L^1[0, 1]$. As $L^1[0, 1]$ is a total subspace of $L^\infty[0, 1]^*$, we thus have that

$$\|g(T)\|_\infty \leq |g(b)| + \int_0^1 |g'(t)| dt$$

and so T is a well-bounded operator on $L^\infty[0, 1]$. Note that the operators $\{E(\lambda)\}$ above do not form a spectral family acting on $L^\infty[0, 1]$. The dual of $L^\infty[0, 1]$ may be identified with the space $ba[0, 1]$ of all bounded finitely additive functions on the Lebesgue measurable subsets of $[0, 1]$ which are zero on sets of Lebesgue measure zero. The norm of such a function μ is its total variation. Further details of the space $ba[0, 1]$ may be found in [DS1, IV].

Let $LM[0, 1]$ denote the set of Lebesgue measurable subsets of $[0, 1]$ and let Λ denote Lebesgue measure on these sets. For $\lambda \in [0, 1]$, it is easily checked that the adjoint $F(\lambda) \in B(ba[0, 1])$ of the operator $E(\lambda) \in B(L^\infty[0, 1])$ is given by

$$(F(\lambda)\mu)(A) = \mu(A \cap [0, 1]) + \frac{\mu(\lambda, 1]}{1 - \lambda} \Lambda(A \cap (\lambda, 1])$$

for $A \in LM[0, 1]$. Showing that $\{F(\lambda)\}$ is a decomposition of the identity is a non-trivial task. That it is concentrated on $[0, 1]$, is naturally ordered and is uniformly bounded follows from the properties of $\{E(\lambda)\}$. The final three properties (see [Dow, p. 288]) are less obvious however. One has to show that

(i) for all $\phi \in L^\infty[0, 1]$ and all $\mu \in ba[0, 1]$ the function

$$\lambda \mapsto \langle \phi, F(\lambda)\mu \rangle = \int_0^\lambda \phi(t) \mu(dt) + \frac{\mu[\lambda, 1]}{1 - \lambda} \int_\lambda^1 \phi(t) dt$$

is Lebesgue measurable;

(ii) if $\phi \in L^\infty[0, 1]$, $\mu \in ba[0, 1]$ and $0 \leq s < 1$ and if the function

$$t \mapsto \int_0^t \langle \phi, F(\lambda)\mu \rangle d\lambda$$

is right differentiable at s , then the right derivative at s is $\langle \phi, F(s)\mu \rangle$;

(iii) if $\phi_0 \in L^\infty[0, 1]$ and $f \in L^1[0, 1]$ are fixed and μ_α ($\alpha \in \mathcal{A}$), $\mu \in ba[0, 1]$ are such that

$$\int_0^1 \phi(t) \mu_\alpha(dt) \rightarrow \int_0^1 \phi(t) \mu(dt)$$

for all $\phi \in L^\infty[0, 1]$, then

$$\int_0^1 f(\lambda) \int_0^1 \phi_0(t) (F(\lambda)\mu_\alpha)(dt) d\lambda \rightarrow \int_0^1 f(\lambda) \int_0^1 \phi_0(t) (F(\lambda)\mu)(dt) d\lambda.$$

All three of these follow from the fact that the function $\lambda \mapsto \langle \phi, F(\lambda)\mu \rangle$ is continuous. We omit the details.

Finally we shall prove the assertion concerning T acting on the space $C[0, 1]$. It is not hard to see that Tf is continuous whenever f is, so the fact that T defines a well-bounded operator on $L^\infty[0, 1]$ implies that T also defines a well-bounded operator on $C[0, 1]$. Moreover, as above, the decomposition of the identity for T (this time acting on $C[0, 1]^* = rca[0, 1]$, the regular countably additive scalar-valued set functions on $[0, 1]$ (see [DS1, IV.6])) is given by

$$(F(\lambda)\mu)(A) = \mu(A \cap [0, \lambda]) + \frac{\mu[\lambda, 1]}{1 - \lambda} \Lambda(A \cap (\lambda, 1]).$$

However, here $F(\lambda)$ is not the adjoint of any operator acting on $C[0, 1]$. ■

It should be noted that despite the fact that T is not of type (B) on $L^\infty[0, 1]$, it does possess a $BV[0, 1]$ functional calculus this space. This is obtained by duality from the $BV[0, 1]$ functional calculus which T possesses as an operator on $L^1[0, 1]$. However, T cannot possess a $BM[0, 1]$ (or even a $C[0, 1]$) functional calculus on either of these spaces. Indeed the methods of [D2] show that when X does not contain a subspace isomorphic to c_0 , then an operator S on X is scalar-type spectral if and only if it possesses a $C(\sigma(S))$ functional calculus. This immediately shows that T cannot have a $C[0, 1]$ functional calculus on $L^1[0, 1]$. If T possessed such a functional calculus on $L^\infty[0, 1]$ then we would have that

$$\|g(T)\|_\infty \leq K \sup_{\lambda \in [0, 1]} |g(\lambda)|$$

for all polynomials g . Since $g(T)$ on $L^\infty[0, 1]$ is the adjoint of $g(T)$ on $L^1[0, 1]$, this would imply that

$$\|g(T)\|_1 \leq K \sup_{\lambda \in [0, 1]} |g(\lambda)|$$

for all polynomials, and hence all continuous functions on $[0, 1]$, contradicting the fact that T has no $C[0, 1]$ functional calculus on $L^1[0, 1]$.

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