30 A Generalisation of Shiffman's Second Theorem

Shiffman's second theorem says that if a minimal annulus is bounded by circles in parallel planes, then every level set is a circle.

In [25], it is proved that the same conclusion is true if we replace the boundary circles in Theorem 29.2 by parallel straight lines and assume A is properly embedded.

Furthermore, Toubiana [78] has proved that if two non-parallel straight lines lie in distinct parallel planes then they cannot bound any proper minimal annulus in the slab bounded by the planes.

In this section we will give a generalization of the results stated above, with a unified proof.

Theorem 30.1 *Suppose $A \subset S(-1,1)$ is a minimal annulus in a slab and $A(1) = A \cap P_1$, $A(-1) = A \cap P_{-1}$ are straight lines or circles.*

1. *If both $A(1)$ and $A(-1)$ are circles, then $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$. In particular, A is embedded.*

2. *If at least one of the $A(1)$ and $A(-1)$ is a straight line and A is properly embedded, then $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$.*

Remark 30.2 The first part of Theorem 30.1 is exactly Shffiman's second theorem, Theorem 29.2. We will see that the second part of theorem 30.1 implies the results in [25] and [78].

Let $A \subset S(-1,1)$ be a proper minimal annulus such that $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are straight lines or circles and $\partial A = A(1) \cup A(-1)$. In the case that there is only one straight line, we will always assume that $A(1)$ is the straight line. Then the interior of A is conformally equivalent to the interior of

$$A_R = \{z \in \mathbf{C} : 1/R \leq |z| \leq R\},$$

for some $1 < R < \infty$. In fact the interior of A is conformally equivalent to

$$\{z \in \mathbf{C} : \rho < |z| < P, \ 0 \leq \rho < P \leq \infty\},$$

for some ρ and P. Since A has 1-dimensional boundary ∂A which is separated by the interior of A, it follows $0 < \rho$ and $P < \infty$. Hence if $R = \sqrt{P/\rho} > 1$ then $\mathrm{Int}(A) \cong \mathrm{Int}(A_R)$.

There is a conformal harmonic immersion

$$X : A_R - C \hookrightarrow S(-1,1),$$

where C is a subset of ∂A_R and $X(\{|z| = R\} - C) = A(1)$, $X(\{|z| = 1/R\} - C) = A(-1)$. If $A(1)$ and $A(-1)$ are both circles, then $C = \emptyset$; if only $A(1)$ is a straight line, then

$C \subset \{|z| = R\}$; if $A(1)$ and $A(-1)$ are both straight lines, then $C \cap \{|z| = R\} \neq \emptyset$ and $C \cap \{|z| = 1/R\} \neq \emptyset$. When $C \neq \emptyset$ we assume that X is a proper embedding.

The Enneper-Weierstrass representation of A is

$$X(z) = \Re \int_1^z (\omega_1, \ \omega_2, \ \omega_3) + V,$$

where $V = (a, b, 0) \in \mathbf{R}^3$, and

$$\begin{cases} \omega_1 = \frac{1}{2}(1 - g^2(z))f(z)dz, \\ \omega_2 = \frac{i}{2}(1 + g^2(z))f(z)dz, \\ \omega_3 = g(z)f(z)dz, \end{cases} \tag{30.162}$$

where g is the Gauss map and f is a holomorphic function. We first prove some facts about such a minimal immersion.

Lemma 30.3 *Suppose $X : \{1/R < |z| < R\} \to S(-1,1)$ is a properly immersed minimal annulus and is embedded in a neighbourhood of $\{|z| = R\} \cup \{|z| = 1/R\}$. Let $g : \{1/R < |z| < R\} \to \mathbf{C}$ be the Gauss map of X. Let $A = X(\{1/R < |z| < R\})$. Suppose that $\partial A \subset P_1 \cup P_{-1}$ and $A(1) = \partial A \cap P_1$, $A(-1) = \partial A \cap P_{-1}$ are circles or straight lines. Let $C \subset \{|z| = 1/R\} \cup \{|z| = R\}$ be the set such that $|X(z_n)| \to \infty$ whenever $z_n \to z \in C$, then $C \cap \{|z| = R\} = p$ and $C \cap \{|z| = 1/R\} = q$ if they are not empty sets. The Gauss map g can be extended to a neighbourhood of A_R such that the extended g at p and q has either zero or pole. Moreover, the Gauss map g has neither zero nor pole in a neighbourhood of A_R except at p and q.*

Furthermore, the third coordinate function X^3 can be extended to the whole A_R such that $X^3|_{|z|=1/R} = -1$ and $X^3|_{|z|=R} = 1$.

Proof. Let $J = X(\{|z| = 1\})$ be the Jordan curve on A and let A_1 be the proper minimal annulus in A with boundary $A(1)$ and J. Suppose that $A(1)$ is a straight line, then let S be the rotation around $A(1)$ of angle π. By the Rotation Theorem (Theorem 8.2) and Extension Theorem (Theorem 4.2), $A_1 \cup S(A_1)$ is a smooth proper minimal surface with boundary $J \cup S(J)$. The conformal structure of $A_1 \cup S(A_1)$ is then $\{1 < |z| < R^2\} - C \cap \{|z| = R\}$ (with the mapping $Y(z) = X(z)$ for $z \in A_R - C$ and $Y(z) = S(X(R^2 z/|z|^2))$ for $z \in \{R < |z| < R^2\}$).

Since $\{|z| = R\} - C$ and $\{|z| = 1/R\} - C$ are homeomorphic to straight lines or circles, they are connected. It turns out that $C \cap \{|z| = R\}$ and $C \cap \{|z| = 1/R\}$ are also connected, hence simply connected as an interval.

Let $D \subset \{1 < |z| < R^2\}$ be a disk like neighbourhood of $C \cap \{|z| = R\}$ such that $z \in D$ if and only if $R^2 z/|z| \in D$ and ∂D is diffeomorphic to a circle, and the $Y(\partial D)$ is a Jordan curve on $A_1 \cup S(A_1)$ which bounds a properly embedded minimal annulus $\tilde{A} = Y(D - C \cap \{|z| = R\})$. Since $A_1 \cap S(A_1)$ is contained in the slab $S(-1, 3)$, by the Cone Lemma (Theorem 21.1), $\tilde{A}$ has finite total curvature. Then by Lemma 10.5, Propositions 10.7 and 10.6, this annular end has the conformal structure of a punctured disk, and the Gauss map of $\tilde{A}$ can be extended to the puncture. In particular, $C \cap \{|z| = R\}$ is a single point p and the Gauss map $g : D \to \mathbf{C}$ of Y can be extended to p, and $g(p)$ is either zero or ∞. Similarly, we can prove that $C \cap \{|z| = 1/R\} = \{q\}$ if it is not empty and $g(q)$ is either zero or ∞.

Since p corresponds to an embedded flat annular end, by Theorem 11.8 we know that there is a $\delta_1 > 0$ such that when $1 - \delta_1 < z < 1$, $P_z \cap A$ is compact. By Lemma 23.2, the tangent plane of A at any point of $A \cap P_z$ is not parallel to the xy-plane. In particular, $dX^3 \neq 0$ on $(X^3)^{-1}(z)$. Thus $(X^3)^{-1}(z)$ is a 1-dimensional submanifold of A_R consists of smooth loops. If it has more than one loop or any loop is homologically trivial, then using the maximum principle we can show that A is contained in a plane. Thus $(X^3)^{-1}(z)$ is a homologically non-trivial smooth Jordan curve. Similarly, if $A(-1)$ is a straight line, then there is a $\delta_2 > 0$ such that when $-1 < z < -1+\delta_2$, $(X^3)^{-1}(z)$ is a homologically non-trivial smooth Jordan curve. Let A'_z be the closed annulus bounded by $(X^3)^{-1}(z)$ and $(X^3)^{-1}(-z)$, for $0 < 1 - z < \min\{\delta_1, \delta_2\}$. Clearly A'_z is compact and $A'_z = X^{-1}(A \cap S(-z, z))$. Since $A \cap P_z$ is compact for $-1 < z < 1$, by Lemma 23.2, the extended Gauss map g of Y does not equal to zero or ∞ in a neighbourhood of A_R except at p or q.

For any sequence $z_n \to p$, since $p \notin A'_z$ for $1 - \delta_1 < z < 1$, $z_n \notin A'_z$ for almost all z_n. Thus $X^3(z_n)$ must converge to 1. Similarly, for any sequence $z_n \to q$, $X^3(z_n)$ must converge to -1. Thus the third coordinate function X^3 can be continuously extended to the whole A_R such that $X^3|_{|z|=1/R} = -1$ and $X^3|_{|z|=R} = 1$. □

The harmonic third coordinate function X^3 satisfies $X^3|_{|z|=1/R} = -1$ and $X^3|_{|z|=R} = 1$ and $-1 < X^3|_{\mathrm{Int}(A_R)} < 1$. Hence we have

$$X^3 = \frac{1}{\log R} \log |z|,$$

and

$$\omega_3 = f(z)g(z)dz = 2\frac{\partial X^3}{\partial z} dz = \frac{d}{dz}\left(\frac{1}{\log R}\log z\right) dz = \frac{1}{\log R}\frac{1}{z}dz.$$

Thus

$$f(z) = \frac{1}{\log R}\frac{1}{zg(z)},$$

and

$$\begin{cases} \omega_1 = \frac{1}{\log R}\frac{1}{2z}\left(\frac{1}{g} - g\right) dz \\ \omega_2 = \frac{1}{\log R}\frac{i}{2z}\left(\frac{1}{g} + g\right) dz \\ \omega_3 = \frac{1}{\log R}\frac{1}{z}dz, \end{cases}$$

and X can be represented as

$$X(p) = \frac{1}{\log R}\Re\int_1^p \left(\frac{1}{2z}(1/g-g), \frac{i}{2z}(1/g+g), \frac{1}{z}\right) dz + V. \tag{30.163}$$

Let

$$g(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \frac{1}{g(z)} = \sum_{n=-\infty}^{\infty} b_n z^n. \tag{30.164}$$

Then by (27.126), (30.163) gives a minimal annulus if and only if

$$\Im(b_0) = \Im(a_0), \quad \Re(b_0) = -\Re(a_0). \tag{30.165}$$

Remark 30.4 Let S be the $180°$-rotation around the straight line $A(1)$ in $\mathbf{R}^3$, and $\mathcal{S} = A \cup S(A)$. Then

$$\int_A K dA = \frac{1}{2}\int_{\mathcal{S}} K dA, \tag{30.166}$$

where K is the Gauss curvature, and dA is the area element of A.

As in the proof of Theorem 27.2, $\kappa = r^{-1}\Lambda^{-1}\Re(zg'/g)$ in the interior of A_R. We must prove that κ is a non-zero constant on each $\{|z| = r\}$, $1/R < r < R$. This is equivalent to prove that $\kappa_\theta \equiv 0$. For that we calculate

$$\begin{aligned}
r\kappa_\theta &= (\Lambda^{-1})_\theta \Re\left(\frac{zg'}{g}\right) + \Lambda^{-1}\left[\Re\left(\frac{zg'}{g}\right)\right]_\theta \\
&= -\frac{1}{2}\Lambda^{-1}\Lambda^{-2}(\Lambda^2)_\theta \Re\left(\frac{zg'}{g}\right) + \Lambda^{-1}\Re\left[iz\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= -\Lambda^{-1}\Re\left(\Lambda^{-2}\frac{\partial\Lambda^2}{\partial z}iz\right)\Re\left(\frac{zg'}{g}\right) - \Lambda^{-1}\Im\left[z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= \Lambda^{-1}\Im\left(\frac{|g|^2-1}{1+|g|^2}\frac{zg'}{g}\right)\Re\left(\frac{zg'}{g}\right) - \Lambda^{-1}\Im\left[z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= \Lambda^{-1}\frac{|g|^2-1}{|g|^2+1}\Im\left(\frac{zg'}{g}\right)\Re\left(\frac{zg'}{g}\right) - \Lambda^{-1}\Im\left[z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= \frac{1}{2}\Lambda^{-1}\frac{|g|^2-1}{|g|^2+1}\Im\left(\frac{zg'}{g}\right)^2 - \Lambda^{-1}\Im\left[z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right] \\
&= \Lambda^{-1}\Im\left[\frac{1}{2}\frac{|g|^2-1}{|g|^2+1}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right)\right].
\end{aligned}$$

Let

$$\begin{aligned}
H(z) &= \frac{1}{2}\frac{|g|^2-1}{|g|^2+1}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right) \\
&= -\frac{1}{|g|^2+1}\left(z\frac{g'}{g}\right)^2 + \frac{1}{2}\left(z\frac{g'}{g}\right)^2 - z\frac{d}{dz}\left(z\frac{g'}{g}\right).
\end{aligned} \tag{30.167}$$

Note that
$$v := r\Lambda\kappa_\theta = \Im H. \tag{30.168}$$
Since $r\Lambda > 0$, to prove $\kappa_\theta \equiv 0$ we only need prove that $v \equiv 0$.

Since
$$\frac{1}{2}\left(z\frac{g'}{g}\right)^2 - z\frac{d}{dz}\left(z\frac{g'}{g}\right)$$
is holomorphic, we have
$$\begin{aligned}
\triangle H &= 4\frac{\partial^2 H}{\partial z \partial \bar{z}} = -4\frac{\partial^2}{\partial z\partial\bar{z}}\left[\frac{1}{1+|g|^2}\left(\frac{zg'}{g}\right)^2\right] = -4\frac{\partial}{\partial z}\frac{-g\overline{g'}}{(1+|g|^2)^2}\left(\frac{zg'}{g}\right)^2 \\
&= 4\frac{|g'|^2(1+|g|^2) - 2g'\bar{g}g\overline{g'}}{(1+|g|^2)^3}\left(\frac{zg'}{g}\right)^2 + 8\frac{g\overline{g'}}{(1+|g|^2)^2}\left(z\frac{g'}{g}\right)\frac{d}{dz}\left(z\frac{g'}{g}\right) \\
&= \frac{-8|g'|^2}{(1+|g|^2)^2}\left[\frac{1}{2}\frac{|g|^2-1}{|g|^2+1}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(z\frac{g'}{g}\right)\right].
\end{aligned}$$
By (7.28) and (7.30),
$$\Lambda^2 = \frac{1}{4}|f|^2\left(1+|g|^2\right)^2, \quad K = -\left[\frac{4|g'|}{|f|(1+|g|^2)^2}\right]^2,$$
hence we have
$$\frac{8|g'|^2}{(1+|g|^2)^2} = -\frac{1}{2}K|f|^2(1+|g|^2)^2 = -2K\Lambda^2.$$
Thus
$$\triangle H = 2K\Lambda^2 H. \tag{30.169}$$
Taking the imaginary part, we have
$$\triangle v = 2K\Lambda^2 v. \tag{30.170}$$
Remember that $\triangle_A = \Lambda^{-2}\triangle_{A_R} = \Lambda^{-2}\triangle$. If $\Gamma = A(1)$ and $A(-1)$ are straight lines or circles, then $\kappa_\theta \equiv 0$ on $\partial A_R - C$. Hence on A_R, v satisfies
$$\begin{cases} \triangle_A v - 2Kv = 0, \\ v|_{\partial A_R - C} = 0, \end{cases} \tag{30.171}$$

We want to prove that v is continuous on A_R and $v|_{\partial A_R} \equiv 0$, i.e., v is an eigenfunction corresponding to the eigenvalue zero. When $A(1)$ and $A(-1)$ are circles this is certainly true. The next lemma shows that it is always true.

Lemma 30.5 *Let A be as in Theorem 30.1, p, q be as in Lemma 30.3, and v be as defined in (30.168). Then v is continuous on A_R and $v|\partial A_R = 0$.*

Proof. Without loss of generality, we can assume that $p = R$. By Lemma 30.3, we can assume that the Gauss map g has limit zero at $p = R$ and g can be extended to a holomorphic function $\tilde{g}$. Let $\zeta = z - R$ on a disk D_ρ centered at $z = R$, we have

$$\tilde{g}(z) = (z - R)^n \tilde{h}(z) = \zeta^n h(\zeta),$$

where $\tilde{h}$ is a holomorphic function on D_ρ and $h(R) \neq 0$.

By definition, $v = \Im H$ and

$$H(z) = \frac{1}{2}\frac{|g|^2 - 1}{|g|^2 + 1}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right) = \left(1 - \frac{1}{|g|^2 + 1}\right)\left(\frac{zg'}{g}\right)^2 - \frac{1}{2}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right).$$

For convenience, we will write g and h instead of $\tilde{g}$ and $\tilde{h}$. Note that

$$\zeta^2\left(z\frac{g'(z)}{g(z)}\right)^2$$

is holomorphic on D_ρ and since $|g|^2 = |z - R|^4|h(z)|^2 = |\zeta|^4|h(z)|^2$,

$$\frac{1}{\zeta^2}\left(1 - \frac{1}{1 + |g|^2}\right) = \frac{1}{\zeta^2}\sum_{k=1}^{\infty}(-1)^{k+1}|g|^{2k} = \overline{\zeta^2}\sum_{k=1}^{\infty}(-1)^{k+1}|\zeta|^{4(k-1)}|h(z)|^{2k}$$

is a C^∞ complex function in a neighborhood of R. Thus

$$\Psi(z) := \left(1 - \frac{1}{|g|^2 + 1}\right)\left(\frac{zg'}{g}\right)^2 = \frac{1}{\zeta^2}\left(1 - \frac{1}{1 + |g|^2}\right)\zeta^2\left(z\frac{g'(z)}{g(z)}\right)^2$$

is a C^∞ complex valued function near $z = R$. If we can prove that

$$\Phi(z) := -\frac{1}{2}\left(\frac{zg'}{g}\right)^2 - z\frac{d}{dz}\left(\frac{zg'}{g}\right)$$

is holomorphic in a neighbourhood of R, then H is a C^∞ complex valued function in a neighbourhood of R. In particular, $v = \Im H$ is C^∞ in a neighbourhood of R, and thus $v(R) = 0$ since on $|z| = R$ and $z \neq R$ we already know that $v(z) = 0$.

Since R corresponds to an embedded flat end, and that end intersects P_1 at a straight line, we have $n = 2$ by Proposition 11.14. Hence

$$z\frac{g'(z)}{g(z)} = \frac{2R}{z - R} + 2 + z\frac{h'(z)}{h(z)}, \quad \text{or} \quad z\frac{g'(z)}{g(z)} = \frac{a_{-1}}{\zeta} + \sum_{k=0}^{\infty} a_k\zeta^k,$$

where

$$a_{-1} = 2R \quad \text{and} \quad a_0 = 2 + R\frac{h'(R)}{h(R)}.$$

Moreover

$$\left(z\frac{g'(z)}{g(z)}\right)^2 = \frac{a_{-1}^2}{\zeta^2} + \frac{2a_{-1}a_0}{\zeta} + \sum_{k=0}^{\infty} b_k\zeta^k,$$

$$z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) = -\frac{a_{-1}R}{\zeta^2} - \frac{a_{-1}}{\zeta} + (\zeta+R)\sum_{k=1}^{\infty} ka_k\zeta^{k-1},$$

and

$$\begin{aligned} & -\frac{1}{2}\left(z\frac{g'(z)}{g(z)}\right)^2 - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) \\ = & -\frac{1}{2}\frac{a_{-1}^2 - 2a_{-1}R}{\zeta^2} - \frac{a_{-1}a_0 - a_{-1}}{\zeta} - \frac{1}{2}\sum_{k=0}^{\infty} b_k\zeta^k - (\zeta+R)\sum_{k=1}^{\infty} ka_k\zeta^{k-1}. \end{aligned}$$

Since $a_{-1} = 2R$,

$$a_{-1}^2 - 2a_{-1}R = 0.$$

We would like to prove that $a_{-1}a_0 - a_{-1} = 0$ and thus Φ is holomorphic near $z = R$.

The a_0 can be calculated as follows. The Weierstrass representation for the extended surface $\mathcal{S}$ is

$$\begin{cases} \omega_1 = \frac{1}{\log R}\frac{1}{2z}\left(\frac{1}{g} - g\right)dz \\ \omega_2 = \frac{1}{\log R}\frac{i}{2z}\left(\frac{1}{g} + g\right)dz \\ \omega_3 = \frac{1}{\log R}\frac{1}{z}dz, \end{cases}$$

as commented after Lemma 30.3. Let C be a loop around $z = R$ in a small disk. Then since $X : \{z : 1/R < |z| < R^3\} - \{R\} \to \mathbf{R}^3$ is well defined and

$$X(z) = \Re\int_{p_0}^{z}(\omega_1, \omega_2, \omega_3),$$

we must have

$$\Re\int_C \frac{1}{2z}\left(\frac{1}{g(z)} - g(z)\right)dz = 0, \quad -\Im\int_C \frac{1}{2z}\left(\frac{1}{g(z)} + g(z)\right)dz = 0,$$

and

$$\int_C \frac{1}{zg}dz = \overline{\int_C \frac{g}{z}dz} = 0,$$

since $g(z)/z$ is holomorphic at $z = R$. Hence we know that the residue of $1/zg(z)$ at $z = R$ is zero. Hence we have

$$\begin{aligned} 0 &= \lim_{z\to R}\left(\frac{(z-R)^2}{zg(z)}\right)' = \lim_{z\to R}\left(\frac{1}{zh(z)}\right)' \\ &= \lim_{z\to R}\left(-\frac{1}{z^2h(z)} - \frac{h'(z)}{zh^2(z)}\right) \\ &= -\frac{1}{R^2h(R)} - \frac{h'(R)}{Rh^2(R)}. \end{aligned}$$

Thus

$$\frac{h'(R)}{h(R)} = -\frac{1}{R}, \quad \text{and} \quad a_0 = 2 + R\frac{h'(R)}{h(R)} = 1.$$

This shows that $a_{-1}a_0 - a_{-1} = 0$.

Note that by orientability, if $g(p) = 0$ then $g(q) = \infty$. Using

$$\frac{(1/g)'}{1/g} = -\frac{g'}{g},$$

we can prove that Φ is holomorphic near q exactly as above. □

Now by (30.170) and Lemma 30.5, v is a Jacobi field. Moreover, v satisfies

$$\begin{cases} \triangle_A v - 2Kv = 0, \\ \\ v|_{\partial A_R} = 0. \end{cases} \tag{30.172}$$

Recall that $v = \Im H = r\Lambda\kappa_\theta$.

If $v \not\equiv 0$, then the zero set of v divides A_R into connected subdomains, called *nodal domains*. As mentioned in Section refsec, any proper subdomain of a nodal domain is stable. Thus by Theorem 20.3, the total curvature of each nodal domain is less than or equal to -2π. Suppose that there are k nodal domains; the total curvature of A must be less than or equal to $-2k\pi$.

By our hypothesis that A is embedded and the proof of Lemma 30.3, $A(t)$ is a Jordan curve for $-1 < t < 1$. By the four-vertex-theorem, see [36], which says that if $\kappa_\theta \not\equiv 0$ then the zero set of κ_θ divides each $A(t)$ into at least four components, we know that there are at least four nodal domains. Thus if $v \not\equiv 0$, then $K(A) \leq -8\pi$.

The next lemma shows that in fact, $K(A) \geq -4\pi$. This contradiction then shows that $v \equiv 0$, which is equivalent to κ being constant along each $A(t)$ for $-1 < t < 1$. Since $A(t)$ is a Jordan curve, we know that $A(t)$ must be a circle.

Lemma 30.6 *Suppose that $A \subset S(-1,1)$ is a proper minimal annulus, and $\partial A = A(1) \cup A(-1)$. If $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are circles or straight lines, then*

$$\int_A K dA \geq -4\pi.$$

Proof. If $A(1)$ and $A(-1)$ are both circles, then by Theorem 27.4 the Gauss map g is one-one onto a sphere domain. Hence $\int_A K dA > -4\pi$.

Now assume that $A(1)$ is a straight line and $C \cap \{|z| = R\} = \{p\}$, then $A(-1)$ is a circle. We will use the extended surface $\mathcal{S}$ in the proof of Lemma 30.3 to calculate the total curvature of A. Notice that $\mathcal{S}$ has an embedded flat annular end corresponding to the point p. Since the end is embedded, the order of Λ at that end is 2. Let

$D_\rho \subset \{1/R < |z| < R^3\}$ be a disk centred at p and radius ρ. Note that $\chi(\{1/R < |z| < R^3\} - \{p\}) = -1$. By the Gauss-Bonnet theorem

$$K(\mathcal{S}) = -2\pi - \int_{|z|=1/R} \kappa_g ds - \int_{|z|=R^3} \kappa_g ds - \int_{\partial D_\rho} \kappa_g ds.$$

Using the same argument as in Theorem 23.1 we have

$$\lim_{\rho \to 0} \int_{\partial D_\rho} \kappa_g ds = 2\pi.$$

Notice that the other two integrals are larger than -2π because $A(-1)$ and $R(A(-1))$ are circles and

$$\int_{|z|=1/R} \kappa_g ds = \int_{A(-1)} \kappa_g ds, \quad \int_{|z|=R^3} \kappa_g ds = \int_{R(A(-1))} \kappa_g ds.$$

We have

$$\int_S K dA > -8\pi.$$

By (30.166), we conclude that the total curvature of A is larger than -4π.

Assume $\{p\} = C \cap \{|z| = R\}$ and $\{q\} = C \cap \{|z| = 1/R\}$, i.e., $A(1)$ and $A(-1)$ are both straight lines. Then let D^1_ρ and D^2_ρ be two disks centered at p and q with radii ρ and let $M_\rho = A_R - (D^1_\rho \cup D^2_\rho)$. Since p and q correspond to embedded ends, Λ has order 2 at p and q. Thus

$$\int_{M_\rho} K\, dA + \int_{\partial M_\rho} \kappa_g\, ds + \sum_i (\alpha_i + \beta_i) = 0,$$

where α_i and β_i are the exterior angles at $\partial D^i_\rho \cap \partial A_R$, and obviously

$$\lim_{\rho \to 0} (\alpha_i + \beta_i) = \pi.$$

Again by the same argument as in Theorem 13.4, noting that Λ has poles at p and q, we have

$$\lim_{\rho \to 0} \int_{\partial D^i_\rho \cap A_R} \kappa_g\, ds = \pi.$$

Since $A(1)$ and $A(-1)$ are straight lines,

$$\lim_{\rho \to 0} \int_{\partial M_\rho - \cup \partial D^i_\rho} \kappa_g\, ds = \int_{\partial A_R} \kappa_g\, ds = 0.$$

Thus we have

$$K(A) = \lim_{\rho \to 0} \int_{M_\rho} K\, dA = -4\pi.$$

□

The proof of theorem 30.1 is complete.

Note that the proof of $K(A) \geq -4\pi$ only used the fact that A is embedded in a neighbourhood of the straight line boundary. Thus we see immediately that

Corollary 30.7 *Suppose that $A \subset S(-1,1)$ is a proper minimal annulus. If $A(1) = A \cap P_1$ is a straight line and A is embedded in a neighborhood of $A(1)$, and $A(-1) = A \cap P_{-1}$ is a circle, then each $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$. In particular, A is embedded.*

Proof. We only need point out that we can still use the four-vertex theorem, even though some level sets $A(t)$ may not be Jordan curves. It is shown in [36], that all curves which have exactly two vertices are curves which have exactly two simple loops, on each loop the curvature is positive or negative and hence its total curvature must be 0. Note that $A(-1)$ has total curvature 2π. Since $A(t)$ is a closed curve for $-1 \leq t < 1$, by continuity every $A(t)$ has total curvature 2π. Hence the four-vertex theorem is applicable to $A(t)$ for $-1 \leq t < 1$. □

Corollary 30.8 *Suppose that $A \subset S(-1,1)$ is a proper minimal annulus. If $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are straight lines and A is embedded in neighbourhoods of $A(1)$ and $A(-1)$, then each $A(t) = A \cap P_t$ is a circle for $-1 < t < 1$. In particular, A is embedded.*

Proof. We have $X^3 = \log|z|/\log R$. Let $\epsilon > 0$ such that on $\{R - \epsilon < |z| \leq R\}$ X is an embedding. Then $A(t)$ is a Jordan curve when $\log(R-\epsilon)/\log R < t < 1$. Thus we can still use the four-vertex theorem. □

Remark 30.9 Corollaries 30.7 and 30.8 are slightly better than Corollary 1 in [17]. There do exist properly immersed minimal annuli in $S(-1,1)$ whose level sets are not circles, see [78].

Since all minimal surfaces foliated by circles must be a part of a Riemann's example, we have proved that:

Corollary 30.10 *Let $L_1 \subset P_1$, $L_{-1} \subset P_{-1}$ be two parallel straight lines. If $\Gamma = L_1 \cup L_{-1}$ is the boundary of a properly embedded minimal annulus A in $S(-1,1)$, then A is one of Riemann's examples.*

Finally, we have a non-existence theorem:

Corollary 30.11 *Let $L_1 \subset P_1$, $L_{-1} \subset P_{-1}$ be two non-parallel straight lines. Then $\Gamma = L_1 \cup L_{-1}$ cannot bound a properly embedded minimal annulus in $S(-1,1)$.*

Corollary 30.10 is the main theorem of [25], in which it is proved via elliptic function theory. Corollary 30.11 is a result of Toubiana [78]. The proof of Theorem 30.1 is adapted from [17].