14 Examples of Complete Minimal Surfaces

We have discussed complete minimal surfaces without a single example. This section is designed to give some examples and their Enneper-Weierstrass representations. But first let us sum up what we have known, in order to simplify the following discussion.

Suppose now that $X : S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$ is a complete minimal immersion of finite total curvature. Then by the Enneper-Weierstrass representation, X is given by

$$X = \Re \int \left(\frac{1}{2}(1-g^2)\eta, \frac{i}{2}(1+g^2)\eta, g\eta\right) + C.$$

If we want X to be embedded, then g and η should satisfy certain conditions. Here is an important necessary condition.

Theorem 14.1 *Let S be a complete minimal immersion in $\mathbf{R}^3$ of finite total curvature, defined by $X : M \to \mathbf{R}^3$, where $M \cong S_k - \{p_1, \cdots, p_r\}$. Let g and the holomorphic 1-form η be the Enneper-Weierstrass data for X. Then $\eta \neq 0$ at a point $p \in M$ unless g has a pole at p, and if g has a pole at $p \in M$ of order m, then η has a zero of order $2m$ at p.*

Suppose that E_i is an embedded end corresponding to p_i. If g has a pole of order $k \geq 1$ at p_i, then η has a zero of order $2k-2$ at p_i. If g takes on a finite value at p_i, then η has a pole of order 2 at p_i. Furthermore, p_i is a branch point of g if and only if E_i is a flat end.

Proof. From $\Lambda^2 = \frac{1}{4}|f|^2(1+|g|^2)^2$, we see that to make $0 < \Lambda < \infty$ on M, the zeros and poles of g and η must be as stated in the theorem.

We have already seen that an end is embedded if and only if Λ has order 2, thus g and η has to satisfy the conditions stated in this theorem.

The last statement is Remark 11.13. □

Now let us see some examples. The first one, the *catenoid*, is quite a classical one, it was discovered in 1741 by Euler, see [61], page 5. By the way, one can find a very interesting history of minimal surfaces in [61].

Example 14.2 (Catenoid) *Let $M = \mathbf{C} - \{0\}$, $g(w) = w$, $\eta = \frac{dw}{w^2}$. The Enneper-Weierstrass representation of the catenoid is given by the three 1-forms:*

$$\omega_1 = \frac{1}{2}\frac{1}{w^2}(1-w^2)dw, \quad \omega_2 = \frac{i}{2}\frac{1}{w^2}(1+w^2)dw, \quad \omega_3 = \frac{1}{w}dw.$$

The total curvature of the catenoid is -4π since $g(w) = w$ has degree 1. It has genus zero and two ends. At 0 and ∞ we see that ϕ_1 and ϕ_2 have poles of order 2 and ϕ_3 has a pole of order 1, so the two ends are embedded and they are catenoid ends since $g(w) = w$ has no branch points. We can prove that the catenoid is embedded and is a rotation surface.

In fact, let $X(w) = (x, y, z)(w)$ and $X(1) = (-1, 0, 0)$. We have

$$x(w) = -1 + \frac{1}{2}\Re \int_1^w \left(\frac{1}{\zeta^2} - 1\right) d\zeta = \frac{1}{2}\Re\left(-\frac{1}{w} - w\right),$$

$$y(w) = -\frac{1}{2}\Im \int_1^z \left(\frac{1}{\zeta^2} + 1\right) d\zeta = -\frac{1}{2}\Im\left(-\frac{1}{w} + w\right),$$

so

$$x(w) - i\, y(w) = -\frac{1}{2}\left(\frac{1}{w} + \overline{w}\right).$$

It is obvious that $z(w) = \log|w|$ only depends on $|w|$. Note that when w is real, $y(w) = 0$. Moreover,

$$|x - i\,y|^2(w) = \frac{1}{4}\left(\frac{1}{w} + \overline{w}\right)\left(\frac{1}{\overline{w}} + w\right) = \frac{1}{4}\left(|w|^2 + \frac{1}{|w|^2} + 2\right) \geq 1,$$

and so X sends $|w| =$ constant to a circle centred at $(0, 0, z(w))$. Hence the surface is contained in a rotation surface with height function z. Since $|w| = \exp(z(w))$, we have

$$\begin{aligned} |x - i\,y|^2(w) &= \frac{1}{4}(\exp(2z(w)) + \exp(-2z(w)) + 2) \\ &= \left(\frac{\exp(z(w)) + \exp(-z(w))}{2}\right)^2 = \cosh^2(z(w)). \end{aligned}$$

Since the two ends are embedded, when $|w|$ is sufficiently large or small, $X(\{|w| = \text{constant}\})$ must be the circle centred at $(0, 0, \log|w|)$ with radius $\cosh(\log|w|)$. Hence the catenoid coincides with the rotation surface defined by

$$x^2 + y^2 = \cosh^2(z).$$

A little calculation shows that the rotation surface is minimal, and so by the extension theorem, the catenoid must be the same rotation surface. In particular, the catenoid is embedded.

Exercise : 1. Prove that all rotation minimal surfaces are generated by functions as follows:

$$x(z) = a\cosh\left(\frac{z - z_0}{a}\right),$$

where $a > 0$ and z_0 are constants. Such a curve is called a *catenary*, the name "catenoid" comes from it. Of course we assume that the axis of rotation is parallel to the z-axis. All these rotation surfaces are homothetic to each other.

2. Use the formulas (7.34) to study the asymptotic and curvature lines of the catenoid.

Example 14.3 (Helicoid) *If we consider the conjugate surface of the catenoid, the forms will be*

$$\omega_1 = \frac{i}{2}\frac{1}{w^2}(1-w^2)dw, \quad \omega_2 = -\frac{1}{2}\frac{1}{w^2}(1+w^2)dw, \quad \omega_3 = i\frac{1}{w}dw.$$

Integrating them and taking real parts, we have a surface given by

$$(x,y,z)(w) = \left(-\Im\left(\frac{1}{w}-w\right), -\Re\left(\frac{1}{w}-w\right), -\arg w\right).$$

The third coordinate is not well defined on $\mathbf{C}-\{0\}$. If we pass to the universal covering $\exp : \mathbf{C} \to \mathbf{C}-\{0\}$, then we will get a well defined minimal surface on $\mathbf{C}$, called the *helicoid.* Moreover, being conjugate to the catenoid, the helicoid is locally isometric to the catenoid.

Let us derive the Enneper-Weierstrass representation of the helicoid. Let $w = e^{\zeta}$ for $\zeta \in \mathbf{C}-\{0\}$, then $dw = e^{\zeta}d\zeta$. Hence we have

$$g(\zeta) = e^{\zeta}, \quad \eta(\zeta) = e^{-\zeta}d\zeta,$$

and

$$\omega_1 = \frac{i}{2}(1-e^{2\zeta})e^{-\zeta}d\zeta = \frac{i}{2}(e^{-\zeta}-e^{\zeta})d\zeta = -i\sinh(\zeta)d\zeta,$$

$$\omega_2 = -\frac{1}{2}(1+e^{2\zeta})e^{-\zeta}d\zeta = -\cosh(\zeta)d\zeta,$$

$$\omega_3 = i\,d\zeta.$$

The helicoid is a ruled surface and is embedded. In fact, let $\zeta = u+iv$, then

$$\begin{aligned}
(x,y,z)(\zeta) &= (\Im\cosh(\zeta), -\Re\sinh(\zeta), -\Im\zeta) \\
&= \left(\Im\frac{1}{2}\left(e^u e^{iv} + e^{-u}e^{-iv}\right), -\Re\frac{1}{2}\left(e^u e^{iv} - e^{-u}e^{-iv}\right), -v\right) \\
&= \left(\frac{1}{2}\sin v(e^u - e^{-u}), -\frac{1}{2}\cos v(e^u - e^{-u}), -v\right) \\
&= (\sin v\sinh(u), -\cos v\sinh(u), -v).
\end{aligned}$$

Thus for fixed v, X maps the straight line $\zeta = u + i\,v$ one to one and onto the straight line generated by $(\sin v, -\cos v, 0)$ on the plane $z = -v$. Since $\mathbf{C}$ consists of all these straight lines, the helicoid is embedded and is a ruled surface.

If we change coordinates in $\mathbf{C} = \mathbf{R}^2$ by $(t,s) = (\sinh u, v)$, then we see that the helicoid is given by

$$X(t,s) = (t\sin s, -t\cos s, s) = t(\sin s, -\cos s, 0) - (0,0,s) = tY(s) + c(s).$$

When $t = 1$, the curve $(\sin s, -\cos s, s)$ is a *helix.*

Since the Gauss map $g(\zeta) = e^{\zeta}$ has an essential singularity at ∞, the helicoid has infinite total curvature.
Exercise : 1. Prove that a ruled minimal surface is a piece of the helicoid in the sense of homothety.

2. Prove that there is only one straight line on the helicoid which is also an asymptotic line, and that line is not in the family discussed above.

Example 14.4 (Enneper's Surface) *Let $M = \mathbf{C}$, $g(z) = z$, $\eta = dz$, then*

$$\omega_1 = \frac{1}{2}(1 - z^2)dz, \quad \omega_2 = \frac{i}{2}(1 + z^2)dz, \quad \omega_3 = zdz.$$

Since $\deg g = 1$, Enneper's surface has total curvature -4π. It has genus zero and one end. Since $\chi(M) = 1$ and there is only one end, by (13.62) we have

$$-2 = (\chi(M) + 1 - J_1) = (2 - J_1),$$

so $J_1 = 4$ and thus by Lemma 11.9 Enneper's surface is not embedded.

In terms of total curvature, we have

Theorem 14.5 *The catenoid and Enneper's surface are the only complete minimal surfaces of total curvature -4π.*

Proof. Let $X : M = S_k - \{p_1, \cdots, p_n\}$ be a complete minimal surface of finite total curvature -4π. Let g be the Gauss map. Then $\deg g = 1$ means $g : S_k \to S^2$ is a conformal diffeomorphism, thus $k = 0$. We have $\chi(M) = 2 - n$. By Corollary 13.6

$$-4\pi \leq 2\pi(\chi(M) - n) = 4\pi(1 - n),$$

so we have $n = 1$ or 2.

When $n = 2$, by (13.62), $-4\pi = 2\pi(\chi(M) + 2 - J_1 - J_2) = -2\pi(J_1 + J_2 - 2)$. We know that $J_i = 2$ since $J_i \geq 2$, and hence the two ends are embedded. Since $\deg g = 1$ means that $g' \neq 0$ everywhere on M, we can assume that $g \neq \infty$ on M and take $g(z) = z$ and $M = \mathbf{C} - \{z_0\}$. Then g has a pole of order 1 at ∞. The 1-form dz has a pole of order 2 at ∞ and no zeros in $\mathbf{C}$. The ends being embedded requires that η should have neither pole nor zero at ∞ (g has a pole of order 1 at ∞), and we have that $\eta = h(z)/(z - z_0)^2 dz$, where h is a holomorphic function which is bounded at ∞. Since η should have a pole of order 2 at z_0, we know that h is also bounded near z_0; thus h is a bounded entire holomorphic function and so h is a constant function, $h \equiv c \neq 0$. Let C be a circle centred at z_0. Since X is well defined,

$$0 = 2\Re \int_C \omega_1 + i\, 2\Re \int_C \omega_2 = \overline{\int_C \frac{c}{(z - z_0)^2} dz} + c \int_C \frac{z^2}{(z - z_0)^2} dz = 4cz_0\pi i$$

forces $z_0 = 0$.

Thus we get the Enneper-Weierstrass representation of the catenoid after a homothety, i.e., $g(z) = z$, $\eta = c\,dz/z^2$.

When $n = 1$, we can take $M = \mathbf{C}$ and $g(z) = z$. By $-4\pi = 2\pi(\chi(M) + 1 - J) = 2\pi(2 - J)$ we have $J = 4$. Let $\eta = f dz$. Since $(1 - z^2)dz$ and $(1 + z^2)dz$ have a pole of order 4 at ∞ and $z\,dz$ has a pole of order 3 at ∞, f can have neither pole nor zero at ∞. Being an entire holomorphic function, f must be a constant $c = re^{i\theta} \neq 0$. Thus we achieve the Enneper-Weierstrass representation of an associated Enneper's surface after a homothety. $\square$

Corollary 14.6 *The only embedded complete minimal surface of total curvature -4π is the catenoid.*

Proof. Enneper's surface is not embedded. $\square$

If these notes were written eleven years ago, then the catenoid, the helicoid and the plane would comprise all the known examples of embedded complete minimal surfaces of finite topology. In 1982, Costa [7] gave a pair of Weierstrass data on a torus with three punctures. Examination by the criteria in Theorem 14.1 shows that the surface is complete, has three embedded ends and total curvature -12π. It is a good candidate for an example of a new embedded complete minimal surface with finite total curvature. The trouble is, how to prove that it is embedded. Using computer graphics, David Hoffman observed that the surface has a lot of symmetries and seems is embedded. Together with William Meeks III, he eventually proved that the surface is embedded. In their proof [30] the symmetries play an important role.

Recently, more embedded complete minimal surfaces of finite topology type, with finite or infinite total curvature, have been discovered, see [80], [26], and [27] for example.

There are also examples of embedded, periodic minimal surfaces, both old and new, such as the classical one-parameter family of Riemann's examples. For these examples and their properties, see [50], [51], [39], and [40].

Here we only mention an infinite family with finite total curvatures. Which are the earliest examples after Costa's example. They were found by Hoffman and Meeks. The proof of their embeddedness is not an easy business, so let us only list their Enneper-Weierstrass representations. The reader is recommended to read the paper [31].

Example 14.7 (Hoffman-Meeks' Surfaces) *First we introduce the special genus k Riemann surfaces, k any positive integer, given by*

$$\overline{M_k} := \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2 \mid w^{k+1} = z^k(z^2 - 1)\}.$$

Let

$$p_0 = (0, 0), \quad p_{-1} = (-1, 0), \quad p_1 = (1, 0), \quad p_\infty = (\infty, \infty).$$

The surface we will consider is

$$M_k := \overline{M_k} - \{p_{-1}, p_1, p_\infty\}.$$

The Gauss map and the one form η will be

$$g = \frac{c_k}{w}, \quad \eta = \left(\frac{z}{w}\right)^k dz = \frac{w}{z^2 - 1} dz.$$

Here c_k is a positive constant to be determined. The determination of c_k is involved in the procedure of "killing the periods", i.e., to make (6.22) true.

Theorem 14.8 *For $k > 0$, there is a unique $c_k > 0$ such that the above Enneper-Weierstrass data g and η give an embedded, complete minimal surface $X : M_k \hookrightarrow \mathbf{R}^3$ of three ends. It has the following properties:*

1. *The total curvature of M_k is $-4\pi(k+2)$;*
2. *M_k has two catenoid ends and one flat end;*
3. *M_k intersects the x_1x_2-plane in $k+1$ straight lines, which meet at equal angles at the origin;*
4. *Removal of the $k+1$ lines disconnects M_k. What remains is, topologically, the union of two open annuli;*
5. *The intersection of M_k with any plane parallel (but not equal) to the x_1x_2-plane is a single Jordan curve;*
6. *The symmetry group of M_k is the dihedral group with $4(k+1)$ elements generated by*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \multicolumn{2}{c}{\mathcal{R}_k} & 0 \\ & & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

   where $\mathcal{R}_k$ is the matrix of rotation by $\pi/(k+1)$ in the x_1x_2-plane;
7. *M_k may be decomposed into $4(k+1)$ congruent pieces, each a graph;*
8. *M_k is the unique properly embedded minimal surface of genus k with three ends, finite total curvature, and a symmetry group containing $4(k+1)$ or more elements.*

M_1 is the surface discovered by Costa, now is called the Costa-Hoffman-Meeks surface.

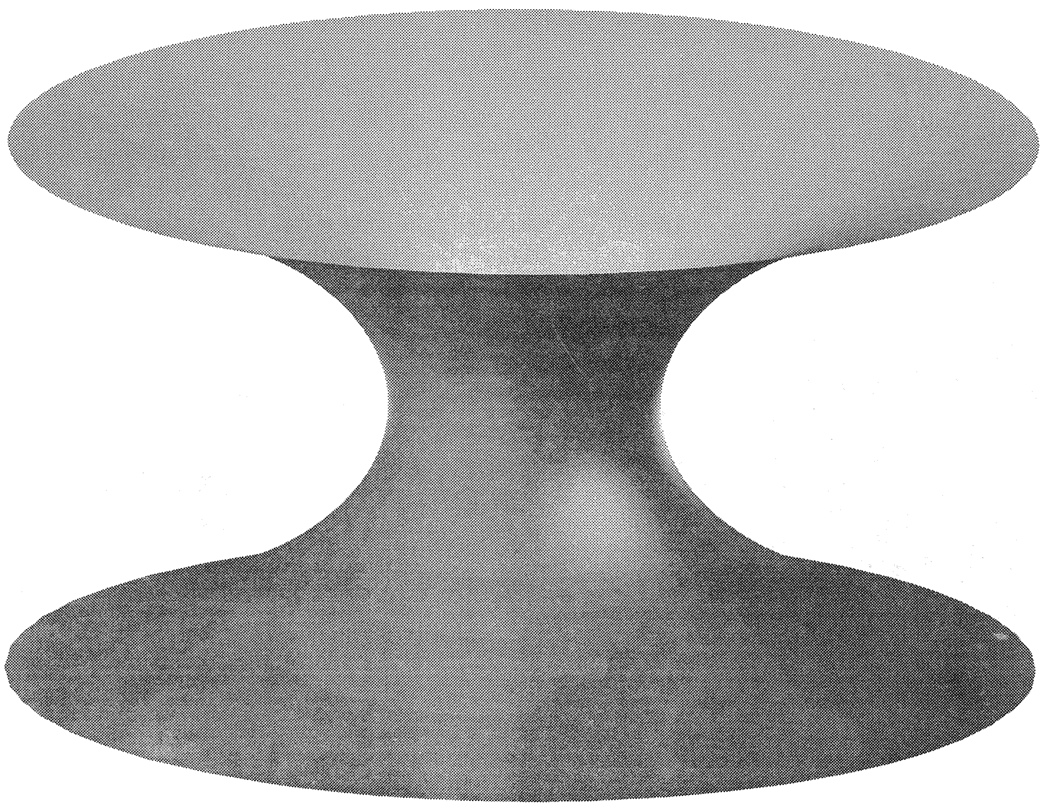

Figure 3
Catenoid, a rotation surface

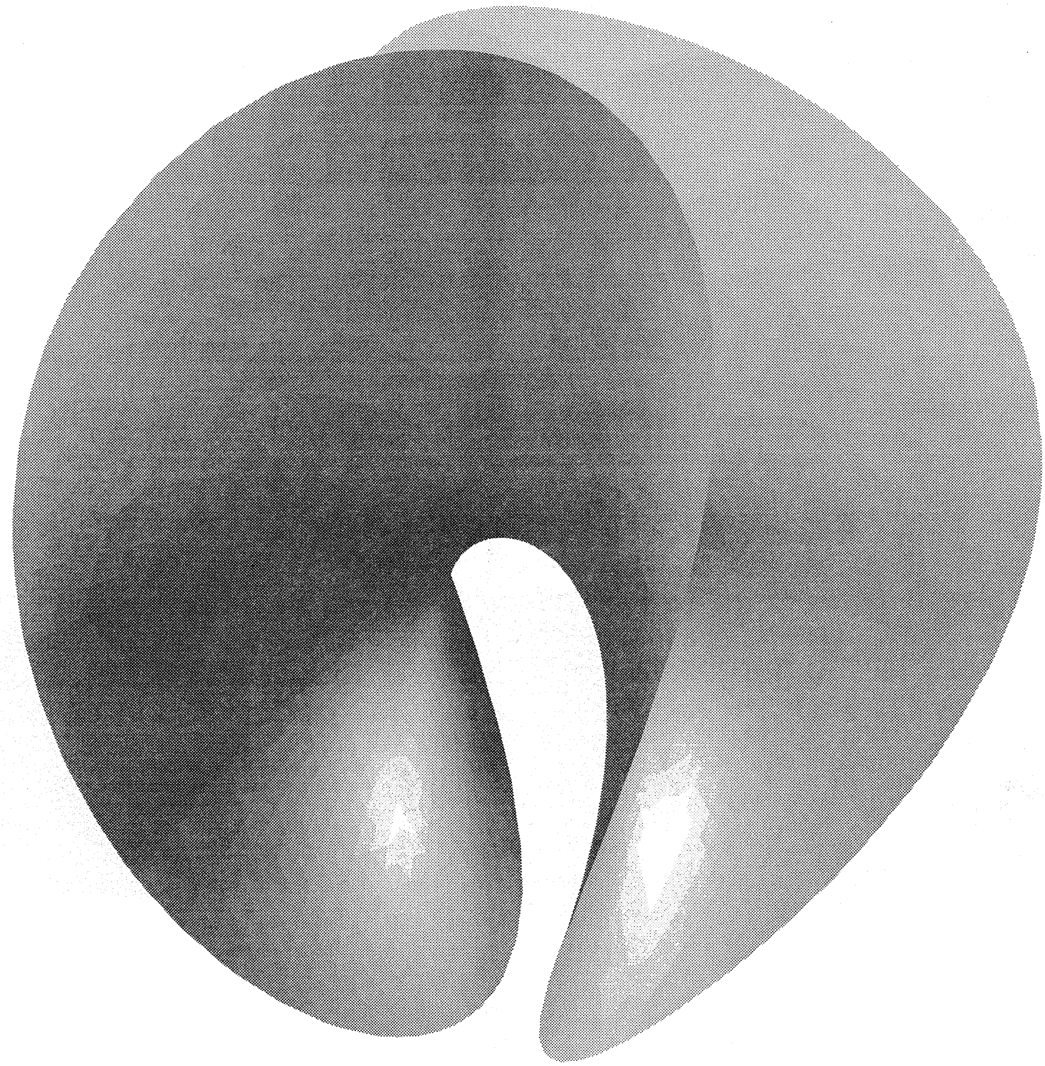

Figure 4
Enneper's Surface

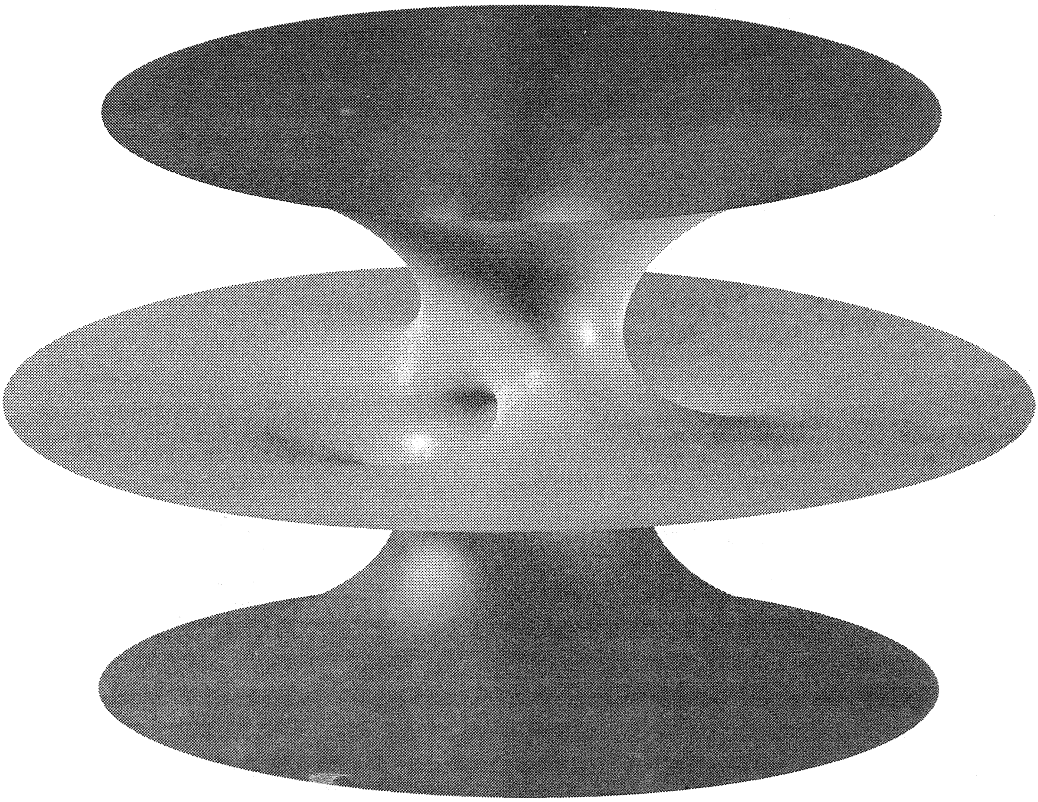

Figure 5
Costa-Hoffman-Meeks Surface

Figure 6
Genus 2 Hoffman-Meeks Surface

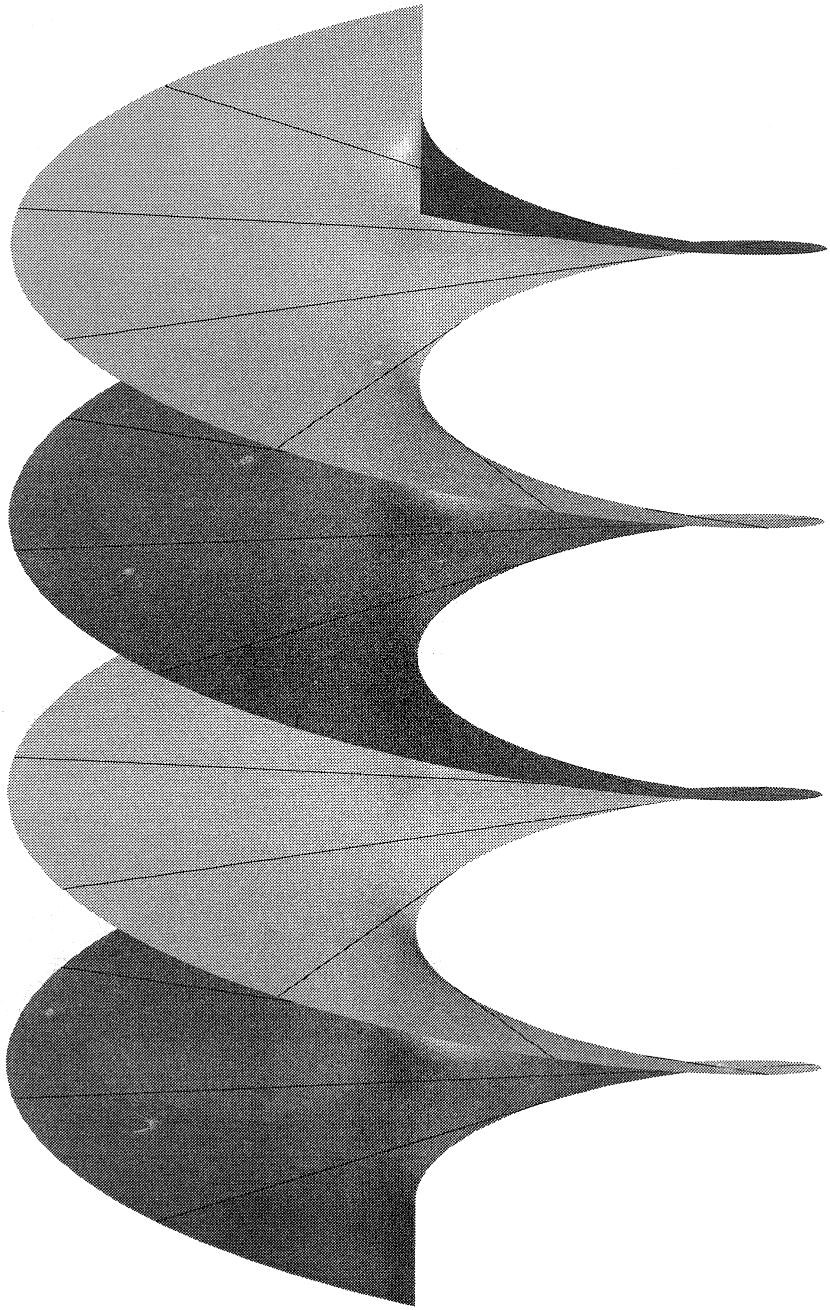

Figure 7
Helicoid, a ruled surface

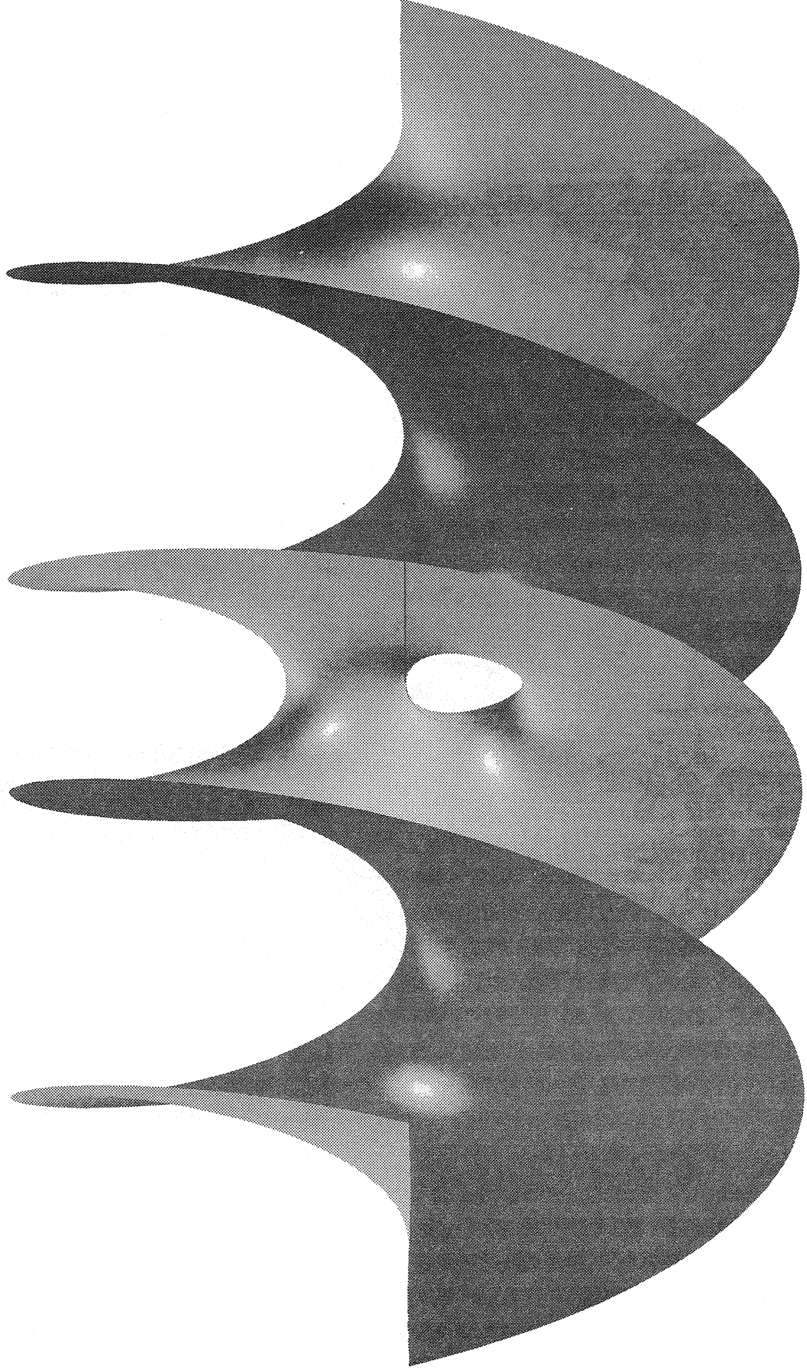

Figure 8
Hoffman-Karcher-Wei's Genus 1 Helicoid, or Helicoid with a hole

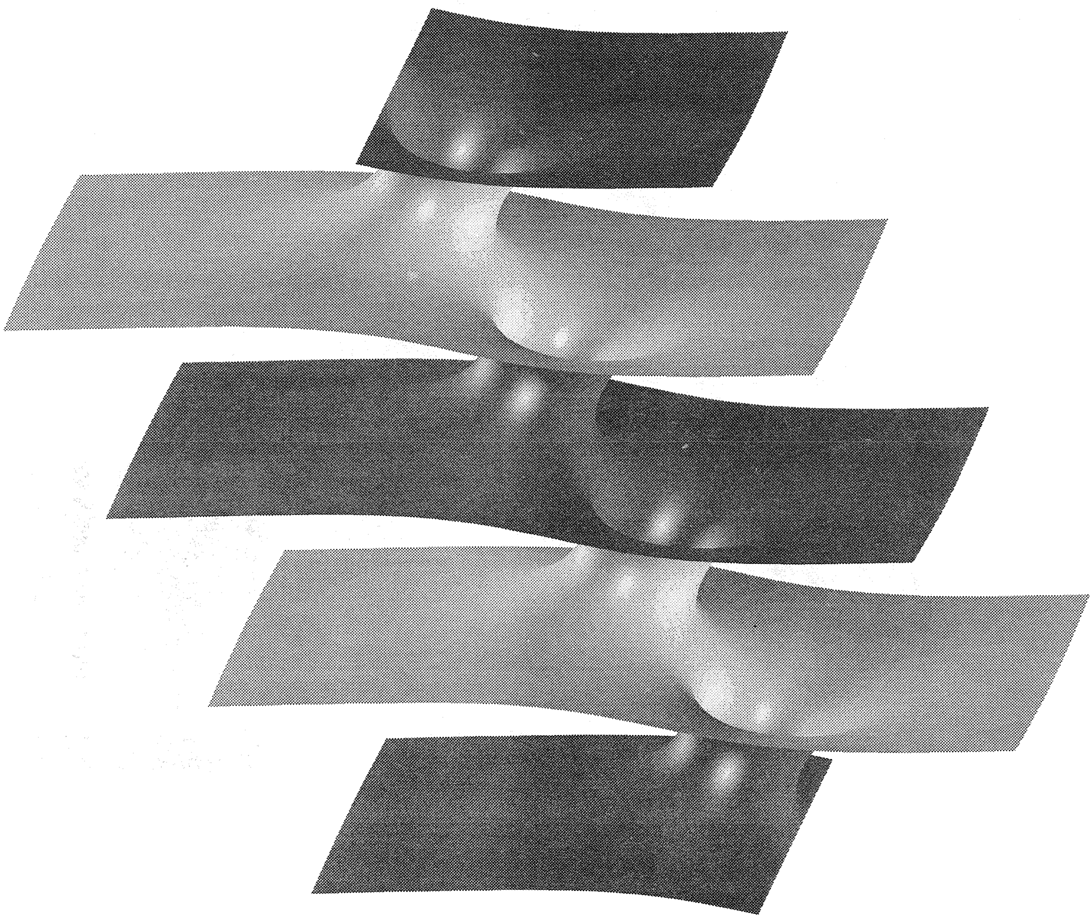

Figure 9
A Riemann's example, it is fibered by circles and straight lines

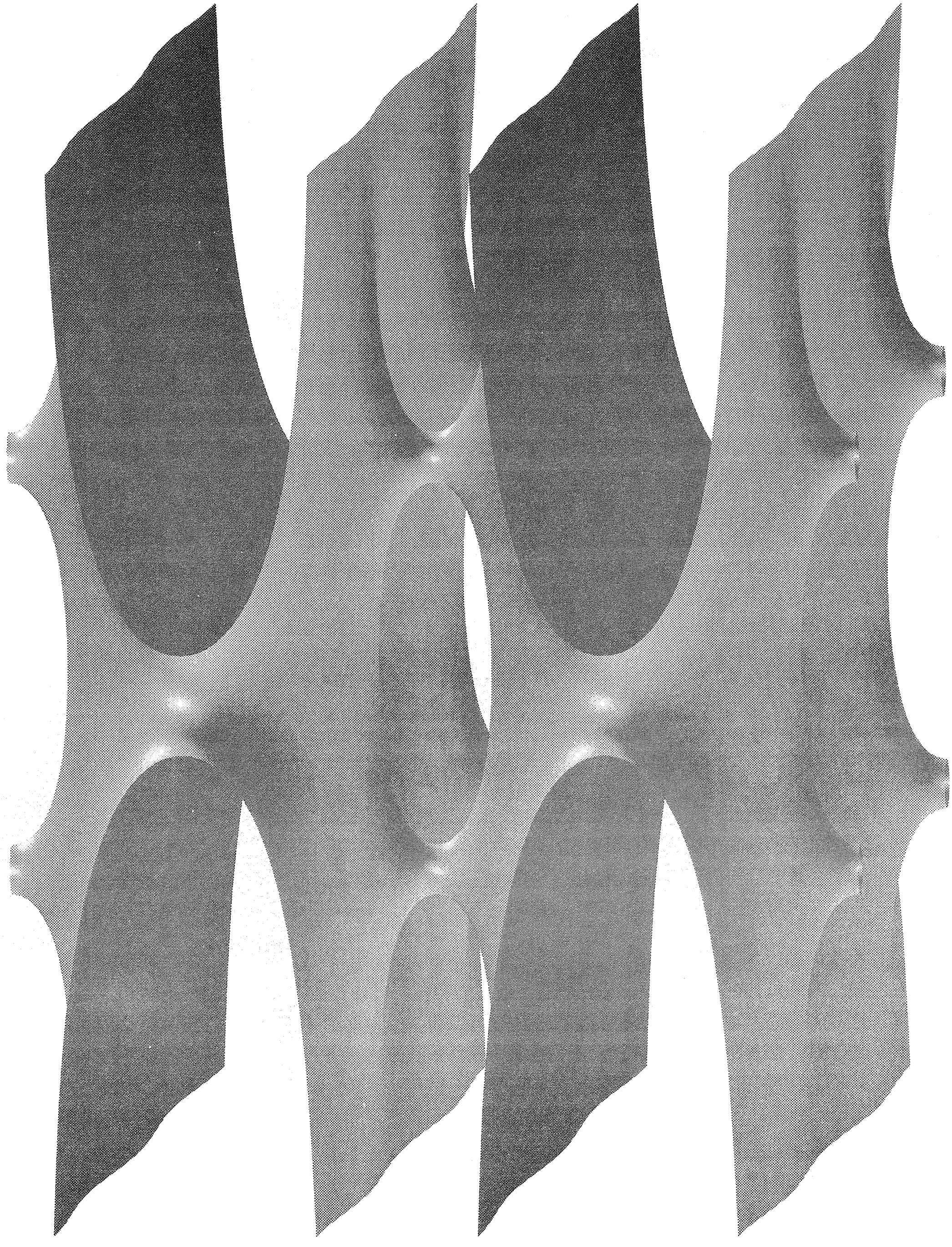

Figure 10
Wei's doubly periodic surface