# FURTHER EXISTENCE RESULTS FOR TWO POINT BOUNDARY VALUE PROBLEMS ARISING IN ELECTRODIFFUSION

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## 1. INTRODUCTION.

In [**3**] the author discusses a two–point boundary value problem which arises naturally in the study of biology as for example in the study of nerve conduction. The physical problem is basically the study of two ions with the same valency diffusing and migrating across a liquid junction such as a membrane. The junction separates two comparatively large electrically neutral reservoirs each containing electrolyte composed of ion species such as sodium and chloride. The reservoirs are stirred to maintain different but constant concentrations and the ions species have different diffusion constants. As the diffusion constant and the concentration gradient determine the rate of diffusion of a given ion species across the junction an electric field $E$ results. This field varies in proportion to local concentration differences in the ion species. The electric field exerts a countervailing force on the ions. For large reservoirs, a steady state is reached in which macroscopically there is nett transfer of mass but not of charge and hence no electric current across the junction. Ion numbers are conserved. With two ion species this steady state model gives rise to a system of differential equations for the ionic concentrations and the electric field strength. Elimination of the ionic concentrations from the system leads to the following differential equation for the electric field:

$$\begin{aligned} y'' = y\{\lambda - (y(0)^2 - y^2)/2 + [l\lambda + (y(0)^2 - y(1)^2)/2]x\} - \\ [l\lambda + (y(0)^2 - y(1)^2)/2]D, \qquad x \in [0,1], \end{aligned} \tag{1}$$

where after scaling the liquid junction occupies the region $0 \leq x \leq 1$. Here $y$ is proportional to the electric field $E$ and the parameters $l$, $\lambda$ and $D$ are functions of the physical constants such as the diffusion constant. Electrical neutrality in the reservoirs yields the boundary conditions:

$$y'(0) = 0 = y'(1). \tag{2}$$

The parameter range of physical interest is $l, \lambda > 0$, and $-1 < D < 1$.

For detailed discussion of this model see Bass [**1,2**].

## 2. EXISTENCE OF SOLUTIONS.

For $l, D, \lambda$ positive the author proved the following.

COROLLARY 3.8 ([**3**]). *If there is a positive* $m$ *satisfying*

$$m(\lambda - m^2/2) - l\lambda D - m^2 D/2 > 0 \tag{3}$$

*then there is a solution of problem (1) and (2) satisfying* $0 < y < m$. *There is at most one solution in this range and it is strictly decreasing.*

The existence proof used upper and lower solutions together with the maximum principle to obtain the necessary a'priori bounds and Coincidence degree (see [4]); Coincidence degree requires

$$m(1 + l/2) - lD > 0, \tag{4}$$

however this follows from (3). The last part of Corollary 3.8 follows from the maximum principle; again the conditions required follow from (3).

Also in Theorem 4.2 of [3] the author used shooting and the implicit function theorem to prove existence of solutions when at least one of $l$ or $D$ is small.

Thus existence was established for a large range of the parameters of physical interest.

Solutions can be shown to exist for a bigger range of values of $l, D, \lambda > 0$.

THEOREM 1. *If* $m$ *is positive and satisfies,*

$$m\lambda - [l\lambda + m^2(1 - \frac{1}{(1+l)^2})/2]D \geq 0 \tag{5}$$

*then problem (1) and (2) has a strictly decreasing solution satisfying* $0 \leq y \leq m$.

We note that (5) holds iff

$$\lambda \geq 2l(1 - \frac{1}{(1+l)^2})D^2.$$

This improved existence result derives from the following better a'priori bounds on solutions established by the maximum principle.

THEOREM 2. *Let* $l, D, \lambda > 0$. *The boundary value problem (1) and (2) has no negative solutions and positive solutions are strictly decreasing and satisfy*

$$0 < y(1) \leq y(0) \leq (1 + l)y(1). \tag{6}$$

Also upper and lower solutions are used to modify the differential equation for values of $y, y(0)$ and $y(1)$ outside a certain region, in such a way that solutions of the modified differential equation lie in the region where the equation was not modified. Schauder degree theory in a suitable domain is used to prove existence of solutions of the modified equation.

Further application of the maximum principle shows that solutions $y$ which change sign, if they do exist are bounded in terms of their boundary conditions. This information may be useful in a shooting argument applied to a modified equation since solutions of the initial value problems for the unmodified equation do not always exist. Using the above information and the maximum principle applied to the differentiated and twice differentiated equations shows that if large positive solutions exist they are asymptotically linear.

## 3. SOLUTIONS BY SHOOTING.

In [**3**] the author obtained the following existence results using shooting and the implicit function theorem.

THEOREM 4.2 ([**3**]). *Let* $l_0 D_0 = 0$ *and* $\lambda_0 > 0$ *then there is* $\delta > 0$ *such that for* $|l - l_0| + |D - D_0| + |\lambda - \lambda_0| < \delta$ *there exists a solution of problem (1)and (2) with* $y(i) = y_i(l, D, \lambda), |y(i)| < \delta, i = 0, 1$.

THEOREM 4.3 ([**3**]). *If* $l_0 D_0 \neq 0$ *and* $\lambda_0 = 0$ *there is a solution of problem (1) and (2) with* $y(0) = y_0(y(1), l, D), \lambda = \lambda(y(1), l, D)$ *continuously differentiable in a neighborhood of* $(0, l_0, D_0)$, $y_0(0, l_0, D_0) = 0 = \lambda(0, l_0, D_0)$. *Moreover this is the only solution in a neighborhood of* $(y(0), y(1), l, D, \lambda) = (0, 0, l_0, D_0, 0)$.

These solutions are positive for $l$ or $D$ small, $\lambda, l, D > 0$.

The solutions obtained in Theorem 4.2 are $\lambda = 0, y$ identically constant in a neighborhood of $y(0) = y(1) = \lambda_0 = 0$. Moreover, $\lambda = 0, y$ identically constant are the only solutions in a neighborhood of $y(0) = c = y(1), \lambda_0 = 0$.

As $y$ identically constant are the only solutions of problem (1) and (2) which do not change sign additional solutions cannot be obtained by this approach.

If $lD = 0$, then $y$ identically zero is the only solution of problem (1) and (2) which does not change sign and again additional positive solutions cannot be obtained by this approach.

For $l$ or $D$ small enough (5) is satisfied and these solutions are those obtained in Theorem 1.

If $y$ is such a solution for $l, D, \lambda > 0$, then $-y$ is a solution for $l, -D, \lambda$.

It would be interesting to know if solutions exist for other parameter values of physical interest and if solutions are unique.

## 4. IONS OF DIFFERENT VALENCIES.

Leuchtag [**5**] extended the above model in two directions by allowing multiple ions and allowing different valencies; of course he allows ions of different mobilities.

Our results extend to Leuchtag's case of two ions with different valencies. We very briefly derive the equations for this case. We follow the notation of Leuchtag [**5**]. Thus the liquid junction occupies the region $0 \leq t \leq \delta$, $\epsilon$ denotes the dielectric constant, $k$ the Boltzmann constant, $T$ the temperature, $E$ the electric field, $N_0$ an arbitary unit of ionic density, $q_0$ the charge of a proton, $q_\pm$ the charge of the ions, $N_\pm$ the their densities, $u_\pm$ their mobilities, and, according to the Einstein relations, $D_\pm = u_\pm kT$ their diffusion coefficients. Set $\nu_\pm = q_\pm/q_0$, $n_\pm = N_\pm/N_0$, the Debye length $\lambda = [(\epsilon kT)/(4\pi q_0^2 N_0)]^{1/2}$, $p = [(q_0\lambda)/(kT)]E$ and $n = n_+ + n_-$.

Integrating the steady state equations for the conservation of ions one obtains the Nernst-Planck equations

$$n'_- = \nu_- p n_- - c_- \tag{7}$$

$$n'_+ = \nu_+ p n_+ - c_+, \tag{8}$$

where the current induced by the ions is given by $J_\pm = (\frac{q_0 N_0}{\lambda})\nu_\pm D_\pm c_\pm$. Set $c = c_+ + c_-$. Gauss's equation has the form

$$p' = \nu_+ n_+ + \nu_- n_-. \tag{9}$$

Adding (7) and (8), using (9) to substitute for $p(\nu_+ n_+ + \nu_- n_-)$ and integrating one obtains

$$n_+ + n_- - n(0) = (p^2 - p(0)^2)/2 - ct \tag{10}$$

This is the corrected equation (20) of Leuchtag [5]. Using (10) to substitute for $n_+$ in (9) one obtains

$$p' = (\nu_- - \nu_+)n_- + \nu_+(p^2 - p(0)^2)/2 - \nu_+ ct + \nu_+ n(0). \tag{11}$$

Differentiating, using (7) to substitute for $n'_-$, and (11) to substitute for $n_-$ one obtains

$$p'' = (\nu_+ + \nu_-)pp' - \nu_-\nu_+[p\{n(0) + (p^2 - p(0)^2)/2 - ct\} + \frac{D_+ - D_-}{\nu_+ D_+ - \nu_- D_-}c]. \tag{12}$$

Using no nett current in the junction one obtains $\nu_- D_- c_- + \nu_+ D_+ c_+ = 0$, solve (10) for $c$ when $t = \delta$ and noting that stirring in the resevoirs results in $n_\pm$ constant at 0 and $\delta$ one eliminates $c$ from (12). Also electrical neutrality in the reservoirs together with (9) gives $p'(0) = 0 = p'(1)$. Setting $x = t\delta$, $D = \sqrt{-\nu_-\nu_+}\frac{D_+ - D_-}{\nu_+ D_+ - \nu_- D_-}$, $l = \delta\frac{n(1) - n(0)}{n(0)}$, $\lambda = -\delta^2\nu_-\nu_+ n(0)$, $\chi = \frac{\nu_- + \nu_+}{\sqrt{-\nu_-\nu_+}}$ and $y = \delta\sqrt{-\nu_-\nu_+}p$ one obtains

$$\begin{aligned} y'' = \chi yy' + y\{\lambda - (y(0)^2 - y^2)/2 + [l\lambda + (y(0)^2 - y(1)^2)/2]x\} \\ - [l\lambda + (y(0)^2 - y(1)^2)/2]D \qquad x \in [0, 1]. \end{aligned}$$

Note that $\nu_+\nu_- < 0$ and in the case $n(1) > n(0)$ and $\nu_- = -\nu_+$ one obtains (1). The results for problem (1) and (2) carry over to this equation.

## 5. INTERPRETATION OF RESULTS.

From Theorem 1 solutions exist if $\lambda \geq 2l(1-\frac{1}{(1+l)^2})D^2$, that this, if

$$\delta^2 n(0) \geq 2\delta\frac{n(1)-n(0)}{n(0)}(\frac{D_+-D_-}{\nu_+D_+-\nu_-D_-})^2\frac{\delta\frac{n(1)-n(0)}{n(0)}+(\delta\frac{n(1)-n(0)}{n(0)})^2}{(1+\delta\frac{n(1)-n(0)}{n(0)})^2}.$$

Thus for $\delta\frac{n(1)-n(0)}{n(0)}$ large enough, solutions exist if $2\frac{n(1)-n(0)}{\delta n(0)^2}(\frac{D_+-D_-}{\nu_+D_+-\nu_-D_-})^2 < 1$ while for $\delta\frac{n(1)-n(0)}{n(0)}$ small enough, solutions exist if $4\frac{(n(1)-n(0))^2}{n(0)^3}(\frac{D_+-D_-}{\nu_+D_+-\nu_-D_-})^2 < 1$.

It would be interesting to prove existence for the higher order systems arising in Leuchtag [**5**]. It would be interesting to consider the coupled system of partial differential differential equations which arise in the transient state and show whether or not there is uniqueness for the steady state problem, even for the two ion model.

## REFERENCES.

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