Complex Variables: Single v Several

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0. PRELIMINARIES

Let D be a domain in the complex plane, i.e. an open and connected set in $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is said to be *holomorphic* (or *analytic*) on D, if it is $\mathbb{C}$-differentiable at any point $z_0 \in D$, i.e. the following limit exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

The n-dimensional complex space $\mathbb{C}^n$ is the product of n copies of the complex plain $\mathbb{C}$. It consists of n-tuples of the form $z = (z_1, \dots, z_n)$, $z_j \in \mathbb{C}$ for all j. We denote by $||z||$ the length of a vector z: $||z|| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$.

For a domain D lying in $\mathbb{C}^n$ we say that a function $f : D \to \mathbb{C}$ is holomorphic on D, if for every j and fixed $z_1, \dots, z_{j-1}, z_{j+1}, z_n$ it is holomorphic as a function of z_j (note the difference with real differentiability!).

A mapping $(f_1, \dots, f_n)$ between two domains in $\mathbb{C}^n$ is a holomorphic mapping, if each its component f_j is a holomorphic function.

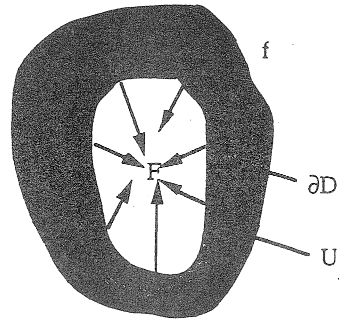
Maximum Principle. If a function f is holomorphic on a domain $D \subset \mathbb{C}^n$ and continuous on $\overline{D}$, then

$$\max_{z \in \overline{D}} |f(z)| = \max_{z \in \partial D} |f(z)|.$$

1. CONTINUATION PHENOMENON

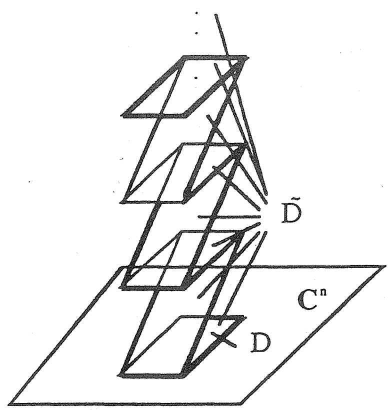
The most impressive fact from complex analysis is the phenomenon of the continuation of functions (Hartogs, 1906; Poincaré, 1907). We elucidate its significance by an example. If a function $f(z_1, \dots, z_n)$ is defined and holomorphic in a neighbourhood of the boundary of a ball in the n-dimensional complex space $\mathbb{C}^n$, $n \geq 2$, then it turns out that $f(z_1, \dots, z_n)$ can be continued to a function holomorphic on the whole ball.

Analogously, for an arbitrary bounded domain D whose complement is connected, any function f holomorphic in a neighbourhood U of the boundary ∂D admits a holomorphic continuation F to the domain itself.



Let us emphasise that this holds only for $n \geq 2$. In the one-dimensional case, i.e. on the complex plane, this phenomenon does not occur. Indeed, for the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ the function $f(z) = \frac{1}{z}$ is holomorphic near the boundary $\partial\Delta = \{z \in \mathbb{C} : |z| = 1\}$, but can not be holomorphically continued to the origin.

This discovery marked the beginning of the systematic study of functions of several complex variables. Two fundamental concepts originating in connection with this property of holomorphic functions are "envelope of holomorphy" and " domain of holomorphy". Let D be a domain in $\mathbb{C}^n$. The *envelope of holomorphy* $\tilde{D}$ of D is the "largest" set to which all functions holomorphic on D extend holomorphically. The envelope of holomorphy of a domain in $\mathbb{C}^n$ is a domain which in general "cannot fit" into $\mathbb{C}^n$, but rather is a multi-sheeted domain over $\mathbb{C}^n$ (Thullen, 1932).



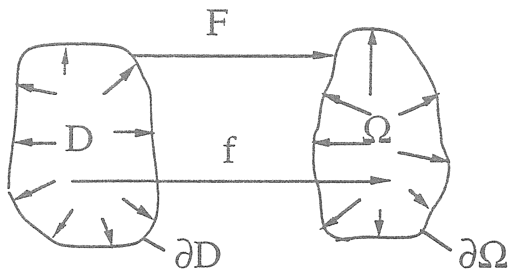
A domain $D \subset \mathbb{C}^n$ is called a *domain of holomorphy* if $\tilde{D} = D$, i.e. if there exists a holomorphic function on D which cannot be continued to any larger domain.

The theorem on disks (Hartogs, 1909) gives an idea helpful in constructing the envelope of holomorphy of a domain: if a sequence of analytic disks lying in a domain D converges towards a disk whose boundary lies in D, then the entire limit disk lies in the envelope of holomorphy $\tilde{D}$. An *analytic disk* is a holomorphic one-to-one image of a closed disk in the complex plane.

2. Holomorphic Mappings. Classification Problems

By the Riemann Mapping Theorem, in $\mathbb{C}$ any two proper simple connected domains are holomorphically equivalent, i.e. there is a holomorphic one-to-one (biholomorphic) mapping between them. In $\mathbb{C}^n$ the situation is substantially different. For example, a ball and a polydisk (a product of n planar disks) are not equivalent (Reinhardt, 1921). Moreover, almost any two randomly chosen domains turn out to be non-equivalent.

For certain pairs of domains D and Ω (e.g. for so-called "strictly pseudoconvex domains") any biholomorphic mapping f extends to a biholomorphic correspondence F between their boundaries (e.g. Fefferman, 1974; Pinchuk, 1975), and the classification problem for such domains reduces to that of classifying real hypersurfaces.



There are two approaches to the problem of classification of real hypersurfaces. The first is geometric; the hypersurface is characterised by a system of differential-geometric invariants (E. Cartan, 1934; Tanaka, 1967; Chern, 1974). In the second approach the characterisation is via a special form for the equation of the hypersurface, the "normal form"(Moser, 1974). Both these constructions enable one to distinguish an infinite-dimensional space of pairwise non-equivalent hypersurfaces.

3. Approximation of Functions

Let K be a compact subset of $\mathbb{C}$. Suppose that K has a connected complement, i.e. $K^c := \mathbb{C} \setminus K$ is connected. Classical Runge's Theorem then states that any function holomorphic in a neighbourhood of K can be uniformly approximated on K by holomorphic polynomials, i.e. functions of the form $P(z) = \sum_{k=0}^{N} a_k z^k$, $a_0, \dots a_N \in \mathbb{C}$. Conversely, if this approximation property holds, the complement K^c is connected. Therefore, Runge's Theorem completely characterises compact sets on the complex plane having the above property.

In $\mathbb{C}^n$ the situation is more complicated. Instead of the condition for a compact set to have a connected complement, one needs a different condition called "polynomial convexity". This means that for any point $w \notin K$ there exists a holomorphic polynomial $P(z_1, \dots, z_n)$ such that $|P(w)| > \max_{z \in K} |P(z)|$. On the complex plane polynomial convexity coincides with the connectedness of K^c.

The polynomial convexity of a compact set K is sufficient to approximate any function holomorphic near K by holomorphic polynomials (Weil, 1932). However, there exist sets which are not polynomially convex, but still admit polynomial approximation for holomorphic functions. For example, the "spherical shell" $\{z \in \mathbb{C}^n : 1 < ||z|| < 2\}$ is not polynomially convex by the maximum principle, but has the approximation property by the continuation phenomenon.

A much more complicated approximation problem is the one for function holomorphic only at the interior points of a compact set K (if there are any) and continuous on K. The criterion for K to have the polynomial approximation property for this class of functions on the plane is formulated in terms of the *continuous analytic capacity*:

$$\alpha(K) = \sup_f \left| \lim_{z \to \infty} z f(z) \right| ,$$

where the supremum is taken over all functions f continuous on $\mathbb{C}$, bounded in modulus by 1 and holomorphic on K^c (Vitushkin, 1966). In $\mathbb{C}^n$ not very much is known except results concerning very special types of compact sets, and certainly there is no description of sets having the approximation property in terms of a single characteristic.