Minimal Surfaces in $\mathbf{R}^3$

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In this note all *surfaces* are *regular parametrised surfaces*, i.e., let $\Omega \subset \mathbf{R}^2$ be a domain and (u^1, u^2) be coordinates of $\mathbf{R}^2$, then a surface is a smooth mapping $X : \Omega \hookrightarrow \mathbf{R}^3$ such that

$$X_1 := \frac{\partial X}{\partial u^1} \text{ and } X_2 := \frac{\partial X}{\partial u^2}$$

are linear indepnedent on Ω. The induced metric on Ω by X is given by the *first fundamental form* $\mathbf{I} = (g_{ij})$,

$$g_{ij} = X_i \bullet X_j, \quad i,\, j = 1, 2,$$

where $\bullet$ is the inner product in $\mathbf{R}^3$. The *second fundamental form* $\mathbf{II} = (h_{ij})$ is given by

$$h_{ij} = X_{ij} \bullet N, \quad i,\, j = 1, 2,$$

where

$$N = \frac{X_1 \wedge X_2}{|X_1 \wedge X_2|}$$

is the *unit normal vector*, $\wedge$ is the cross product in $\mathbf{R}^3$, and of course

$$X_{ij} = \frac{\partial^2 X}{\partial u^i \partial u^j}.$$

Remember that the *mean curvature* is defined as

$$H := \frac{1}{2}\,\mathbf{trace}\left[(\mathbf{II})\mathbf{I}^{-1}\right] = \frac{1}{2} h_{ij} g^{ij},$$

where we write $\mathbf{I}^{-1} = (g^{ij})$.

Definition 1 The surface $X:\Omega \hookrightarrow \mathbf{R}^3$ is a *minimal surface* if $H \equiv 0$.

Let $G = \det \mathbf{I}$. Recall that the Laplacian on $(\Omega,\, \mathbf{I})$ is defined as

$$\triangle_{\mathbf{I}} := \frac{1}{\sqrt{G}}\frac{\partial}{\partial u^i}\left(g^{ij}\sqrt{G}\frac{\partial}{\partial u^j}\right).$$

We want to show that X is minimal if and only if $\triangle_{\mathbf{I}} X = \vec{0}$, i.e., if and only if each component of X is a harmonic function in the metric $\mathbf{I}$.

Let us first recall that from the Gauss equation we have

$$X_{ij} = \Gamma^k_{ij} X_k + h_{ij} N,$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right).$$

We calculate

$$\begin{aligned}
\triangle_{\mathbf{I}} X &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial u^i}(g^{ij}\sqrt{G}X_j) \\
&= g^{ij}X_{ij} + \frac{\partial g^{ij}}{\partial u^i}X_j + \frac{1}{\sqrt{G}}\frac{\partial \sqrt{G}}{\partial u^i} g^{ij} X_j \\
&= g^{ij}X_{ij} + \frac{\partial g^{ij}}{\partial u^i}X_j + \frac{1}{2G}\frac{\partial G}{\partial u^i} g^{ij} X_j.
\end{aligned}$$

Now we have an identity

$$\frac{1}{G}\frac{\partial G}{\partial u^i} = \mathbf{trace}\left(\mathbf{I}^{-1}\frac{\partial \mathbf{I}}{\partial u^i}\right) = g^{kl}\frac{\partial g_{kl}}{\partial u^i}.$$

Thus we have

$$\triangle_{\mathbf{I}} X = g^{ij}X_{ij} + \frac{\partial g^{ij}}{\partial u^i}X_j + \frac{1}{2}g^{ij}g^{kl}\frac{\partial g_{kl}}{\partial u^i}X_j.$$

We claim that $\triangle_{\mathbf{I}} X$ is perpendicular to the *tagent planes*, i.e, planes generated by (X_1, X_2). In fact, since $g_{ij}g^{jk} = \delta_{ik}$, we have

$$\begin{aligned}
\triangle_{\mathbf{I}} X \bullet X_m &= g^{ij}X_{ij}\bullet X_m + \frac{\partial g^{ij}}{\partial u^i}X_j \bullet X_m + \frac{1}{2}g^{ij}g^{kl}\frac{\partial g_{kl}}{\partial u^i}X_j \bullet X_m \\
&= g^{ij}\Gamma^k_{ij} g_{km} + \frac{\partial g^{ij}}{\partial u^i} g_{jm} + \frac{1}{2}g^{ij}g^{kl}\frac{\partial g_{kl}}{\partial u^i} g_{jm}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}g^{ij}g_{km}g^{kl}\left(\frac{\partial g_{il}}{\partial u^j}+\frac{\partial g_{jl}}{\partial u^i}-\frac{\partial g_{ij}}{\partial u^l}\right)-g^{ij}\frac{\partial g_{jm}}{\partial u^i}+\frac{1}{2}g^{kl}\frac{\partial g_{kl}}{\partial u^m}\\
&= \frac{1}{2}g^{ij}\left(\frac{\partial g_{im}}{\partial u^j}+\frac{\partial g_{jm}}{\partial u^i}-\frac{\partial g_{ij}}{\partial u^m}\right)-g^{ij}\frac{\partial g_{jm}}{\partial u^i}+\frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial u^m}\\
&= 0.
\end{aligned}
$$

Thus $\triangle_{\mathbf{I}}X$ is in the direction of N, and

$$\triangle_{\mathbf{I}}X=(\triangle_{\mathbf{I}}X\bullet N)N=(g^{ij}X_{ij}\bullet N)N=(g^{ij}h_{ij})N=2HN.$$

Since $N\neq\vec{0}$, we see that $\triangle_{\mathbf{I}}X\equiv\vec{0}$ if and only if $H\equiv 0$.

One important feature of 2-dimensional surfaces is the existence of *isothermal coordinates*, i.e, coordinates (u^1,u^2) such that

$$|X_1|=|X_2|=\Lambda>0,\quad\text{and}\quad X_1\bullet X_2=0.$$

Under this coordinate system, X is called *conformal.*

A classical theorem says that for any C^2 $X:\Omega\hookrightarrow\mathbf{R}^n$, we can always (by changing coordinate) find a good coordinate system (isothermal coordinates) such that X is conformal. Interested readers can see [1] for an elementary proof. From now on, we will assume all surfaces are conformal.

Under the isothermal coordinates, the metric $\mathbf{I}$ is very imp

$$g_{ij}=\Lambda^2\delta_{ij},\quad g^{ij}=\Lambda^{-2}\delta_{ij},\quad\text{and}\quad G=\Lambda^4.$$

Let

$$\triangle=\frac{\partial^2}{\partial u_1^2}+\frac{\partial^2}{\partial u_2^2}$$

be the usual Laplacian, we can check that

$$\triangle_{\mathbf{I}}=\Lambda^{-2}\,\triangle\,.$$

Thus we can also define that $X:\Omega\hookrightarrow\mathbf{R}^3$ is minimal if and only if X is conformal and $\triangle X=0$.

This definition has an advantage that since each component of X is a harmonic function, and hence is locally the real part of some holomorphic function, we can use the rich theory of holomorphic functions.

Let us take this advantage. We write $z = u + i\,v = u_1 + i\,u_2$ and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\,\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\,\frac{\partial}{\partial v}\right).$$

We can define a complex mapping

$$\phi := X_1 - i\,X_2 = 2\frac{\partial X}{\partial z} = 2\frac{\partial}{\partial z}\left(X^1, X^2, X^3\right).$$

We can calculate that

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = |X_1|^2 - |X_2|^2 - 2i\,X_1 \bullet X_2.$$

Thus X is conformal if and only if

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0. \tag{1}$$

Recall that a complex valued function f is holomorphic if and only if $\partial f/\partial \bar{z} = 0$. If X is minimal, then

$$\frac{\partial \phi}{\partial \bar{z}} = 2\frac{\partial^2 X}{\partial \bar{z}\partial z} = \frac{1}{2}\triangle X = 0.$$

On the other hand, if a conformal mapping X satisfies

$$\frac{\partial \phi}{\partial \bar{z}} = 0, \tag{2}$$

then we know that $\triangle X = 0$ and X is minimal.

Moreover, we can recovery the mapping X from ϕ, that is

$$X(z) = X(z_0) + \Re \int_{z_0}^{z} \phi(\zeta)d\zeta,$$

where $\Re$ means the real part. The integral on the right hand side is independent of path, since X is well defined.

We can further analyse the holomorphic mapping ϕ. We can rewrite (1) as

$$(\phi_1 + i\,\phi_2)(\phi_1 - i\,\phi_2) = -\phi_3^2.$$

If $\phi_3 \equiv 0$, then clearly $X(\Omega)$ is contained in a plane. After a rotation, we can assume that $\phi_3 \not\equiv 0$. Let

$$g = \frac{\phi_3}{\phi_1 - i\,\phi_2} \not\equiv 0, \quad f = \phi_1 - i\,\phi_2.$$

Then

$$g^2 = \frac{\phi_3^2}{(\phi_1 - i\,\phi_2)^2} = -\frac{\phi_1 + i\,\phi_2}{\phi_1 - i\,\phi_2}.$$

A little calculation shows that

$$\phi_1 = \frac{1}{2} f\left(1 - g^2\right), \quad \phi_2 = \frac{i}{2} f\left(1 + g^2\right), \quad \phi_3 = fg. \tag{3}$$

Note that f and g are both holomorphic functions. Thus we conclude that a minimal surface is given by a pair of holomorphic functions f and g by

$$X(z) = X(z_0) + \int_{z_0}^{z} \left(\frac{1}{2} f(1 - g^2),\, \frac{i}{2} f(1 + g^2),\, fg\right)(\zeta) d\zeta. \tag{4}$$

Equation (4) is called the *Enneper-Weierstrass Representation* of the minimal surface $X : \Omega \hookrightarrow \mathbf{R}^3$, the functions g and f are called the *Enneper-Weierstrass data* of X.

We will give the geometric data, such as the Gauss map, the first and second fundamental forms, the principal and Gauss curvatures, etc., of a minimal surface via its Enneper-Weierstrass representation.

One important fact is that the meromorphic function g in the Enneper-Weierstrass representation corresponds to the *Gauss map* N. For this we first recall that the Gauss map $N : M \to S^2$ of an immersion $X : M \hookrightarrow \mathbf{R}^3$ is defined as

$$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v) : M \to S^2.$$

Let $\tau : S^2 - \{\mathcal{N}\} \to \mathbf{C}$ be stereographic projection, where $\mathcal{N}$ is the north pole. Then

$$\tau(x, y, z) = \frac{x + iy}{1 - z}, \quad \tau^{-1}(w) = \frac{1}{1 + |w|^2}(2\Re w, 2\Im w, |w|^2 - 1),$$

where $\Re$ and $\Im$ are the real and imaginary parts. We claim that

$$g = \tau \circ N : M \to \mathbf{C}.$$

In fact,

$$\tau^{-1}\circ g = \frac{1}{1+|g|^2}\left(2\Re g,\ \ 2\Im g,\ \ |g|^2-1\right).$$

By $\phi = X_1 - i X_2$ and (3)

$$\begin{aligned} X_u &= \Re\left(\frac{1}{2}f(1-g^2),\ \ \frac{i}{2}f(1+g^2),\ \ fg\right), \\ X_v &= -\Im\left(\frac{1}{2}f(1-g^2),\ \ \frac{i}{2}f(1+g^2),\ \ fg\right), \end{aligned}$$

thus

$$X_u \wedge X_v = \begin{pmatrix} -\Re\frac{i}{2}f(1+g^2)\Im fg + \Re fg\Im\frac{i}{2}f(1+g^2) \\ \Re\frac{1}{2}f(1-g^2)\Im fg - \Re fg\Im\frac{1}{2}f(1-g^2) \\ -\Re f(1-g^2)\Im\frac{i}{4}f(1+g^2) + \Re\frac{i}{4}f(1+g^2)\Im f(1-g^2) \end{pmatrix}$$

$$= \begin{pmatrix} \Im[\frac{i}{2}f(1+g^2)\overline{fg}] \\ \Im[\frac{1}{2}\overline{f(1-g^2)}fg] \\ \Im\left[\frac{-i}{4}\overline{f(1+g^2)}f(1-g^2)\right] \end{pmatrix} = \begin{pmatrix} \frac{1}{2}|f|^2\Re(\overline{g}+|g|^2g) \\ \frac{1}{2}|f|^2\Im(g-|g|^2\overline{g}) \\ \frac{1}{4}|f|^2\Re(|g|^4-1-\overline{g}^2+g^2) \end{pmatrix}$$

$$= \frac{|f|^2(1+|g|^2)^2}{4(1+|g|^2)}\begin{pmatrix} 2\Re g \\ 2\Im g \\ |g|^2-1 \end{pmatrix} = \frac{1}{4}|f|^2(1+|g|^2)^2\tau^{-1}\circ g.$$

Since X is conformal, the first fundamental form is given by $g_{12} = 0$ and

(5) $$g_{11} = g_{22} = \Lambda^2 = |X_u||X_v| = |X_u \wedge X_v| = \frac{1}{4}|f|^2(1+|g|^2)^2,$$

where the last equality comes from $|\tau^{-1}\circ g| = 1$. Thus

(6) $$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v) = \frac{1}{1+|g|^2}\left(2\Re g, 2\Im g, |g|^2-1\right) = \tau^{-1}\circ g,$$

as we claimed.

Later we will *also* call the function $g = \tau\circ N$ the *Gauss map of the immersion* $X : M \hookrightarrow \mathbf{R}^3$. We have seen that if X is a minimal surface then g is a meromorphic function. The converse is also true, i.e., X is minimal if and only if $g = \tau\circ N$ is meromorphic. For a proof, the readers can confer [4], pages 107–110.

We can also calculate the second fundamental form of X via the Enneper-Weierstrass representation. Recall that

$$X_1 - iX_2 = X_u - iX_v = (\phi_1,\, \phi_2,\, \phi_3)$$

are holomorphic functions of $z = u + iv$. Hence

$$X_{11} - iX_{12} = X_{uu} - iX_{uv} = (\phi_1',\, \phi_2',\, \phi_3').$$

Because X is harmonic, the data of the second fundamental form then must be

$$h_{11} = X_{11} \bullet N = \Re(\phi_1',\, \phi_2',\, \phi_3') \bullet N, \qquad h_{22} = -h_{11},$$

$$h_{12} = X_{12} \bullet N = -\Im(\phi_1',\, \phi_2',\, \phi_3') \bullet N.$$

By (3),

$$\begin{aligned}
X_{11} \bullet N &= \Re(\phi_1',\, \phi_2',\, \phi_3') \bullet N \\
&= \Re\left[\left(\frac{1}{2}f'(1-g^2), \frac{i}{2}f'(1+g^2), f'g\right) + (-fgg',\, ifgg',\, fg')\right] \bullet N \\
&= \frac{1}{1+|g|^2}\big(\Re f'(1-g^2)\Re g - \Im f'(1+g^2)\Im g + \Re f'g(|g|^2-1) \\
&\quad -2\Re fgg'\Re g - 2\Im fgg'\Im g + \Re fg'(|g|^2-1)\big) \\
&= \frac{1}{1+|g|^2}\big(\Re f'\Re g - \Re f'g^2\Re g - \Im f'\Im g - \Im f'g^2\Im g \\
&\quad +\Re f'g(|g|^2-1) - 2\Re fgg'\overline{g} + \Re fg'(|g|^2-1)\big) \\
&= \frac{1}{1+|g|^2}\big(\Re f'g - \Re f'g^2\overline{g} + \Re f'g(|g|^2-1) - 2|g|^2\Re fg' + \Re fg'(|g|^2-1)\big) \\
&= \frac{1}{1+|g|^2}\big(-\Re fg'(|g|^2+1)\big) = -\Re fg'.
\end{aligned}$$

Similarly, we have $h_{12} = \Im fg'$. From these we see that for a minimal surface,

$$h_{11} - ih_{12} = -fg' \tag{7}$$

is a holomorphic function.

Again let $dz = du + i\,dv$ and $(dz)^2 = (du)^2 - (dv)^2 + 2i\,du\,dv$. The second fundamental form of X can be written as

$$h_{11}(du)^2 + 2h_{12}\,du\,dv + h_{22}(dv)^2 = -\Re(fg')((du)^2-(dv)^2) + 2\Im(fg')\,du\,dv$$

$$= -\Re(fg')\Re(dz)^2 + \Im(fg')\Im(dz)^2 = -\Re(fg'(dz)^2) = -\Re(f\,dg\,dz).$$

Let $V \in T_pM$ be a unit tangent vector and write $V = \Lambda^{-1}(\cos\theta, \sin\theta) = \Lambda^{-1}e^{i\theta}$ in complex form; then

$$\mathbf{II}(V,V) = -\Lambda^{-2}\Re(fg'e^{2i\theta})$$

by the previous formulae. Thus the two principal curvatures (eigenvalues of the second fundamental form $\mathbf{II}$) are

$$\kappa_1 = \max_{0\le\theta\le 2\pi} -\Lambda^{-2}\Re(fg'e^{2i\theta}) = \Lambda^{-2}|fg'| = \frac{4|g'|}{|f|(1+|g|^2)^2}, \tag{8}$$

$$\kappa_2 = \min_{0\le\theta\le 2\pi} -\Lambda^{-2}\Re(fg'e^{2i\theta}) = -\Lambda^{-2}|fg'| = -\frac{4|g'|}{|f|(1+|g|^2)^2}. \tag{9}$$

Then from $K = \kappa_1\kappa_2$ we get

$$K = -\left[\frac{4|g'|}{|f|(1+|g|^2)^2}\right]^2. \tag{10}$$

Now let $r(t) = r_1(t) + ir_2(t)$ be a curve on M and $r'(t) = r_1'(t) + ir_2'(t)$; then

$$\begin{aligned} \mathbf{II}(r'(t), r'(t)) &= -\Re\{f[r(t)]\,g'[(r(t)]\,[r'(t)]^2\}(dt)^2 \\ &= -\Re\{d[g(r(t)]f[r(t)]dr(t)\}. \end{aligned} \tag{11}$$

Remember that a regular curve r is an *asymptotic line* on a surface M if $\mathbf{II}(r'(t), r'(t)) \equiv 0$; a curve r is a *curvature line* if and only if $r'(t)$ is in a principal direction, if and only if $|r'(t)|^{-2}\,\mathbf{II}(r'(t), r'(t))$ takes either maximum or minimum value of $\mathbf{II}(v,v)$ for all unit tangent vectors in $T_{r(t)}M$. We have the following criteria:

1. A regular curve r is an asymptotic line if and only if $f[r(t)]\,g'[r(t)]\,[r'(t)]^2 \in i\mathbf{R}$.

2. A regular curve r is a curvature line if and only if $f[r(t)]\,g'[r(t)]\,[r'(t)]^2 \in \mathbf{R}$.

The last assertion comes from the fact that $-\Re\{f[r(t)]g'([(t)][r'(t)]^2\}$ achieves its maximum or minimum for all directions v at $r(t)$ only if $f[r(t)]g'[r(t)][r'(t)]^2$ is real.

Finally, we should mention that if Ω is a two-dimensional manifold instead of a plane domain, then the holomorphic mapping ϕ is no longer well defined, but the 1-form

$$\omega = \phi\,dz$$

is. In this case we get

$$g = \frac{\omega_3}{\omega_1 - i\,\omega_2}$$

a meromorphic function and

$$\eta = \omega_1 - i\,\omega_2$$

a meromorphic 1-form, the Enneper-Weierstrass of X is given by

$$X(p) = X(p_0) + \Re \int_{p_0}^{p} \omega, \tag{12}$$

where

$$\omega_1 = \frac{1}{2}(1 - g^2)\eta, \quad \omega_2 = \frac{i}{2}(1 + g^2)\eta, \quad \omega_3 = g\eta. \tag{13}$$

On the other hand, if we have a meromorphic function g and a meromorphic 1-form η, we can get a minimal surface by (12) if

$$\Re \int_C \omega = \vec{0}, \tag{14}$$

for any loop C in Ω. Thus although locally any pair of g and η gives a minimal surface, if we want the surface is globally well defined, g and η have to match each other well such that (14) is satisfied.

We call g and η as the Enneper-Weierstrass data of X and (12) the Enneper-Weierstrass representation of X.

Now let us give some examples of minimal surfaces. We only give the Enneper-Weierstrass data.

1. Catenoid:

$\Omega = \mathbf{C} - \{0\}$, $g(z) = z$, $\eta = dz/z^2$.

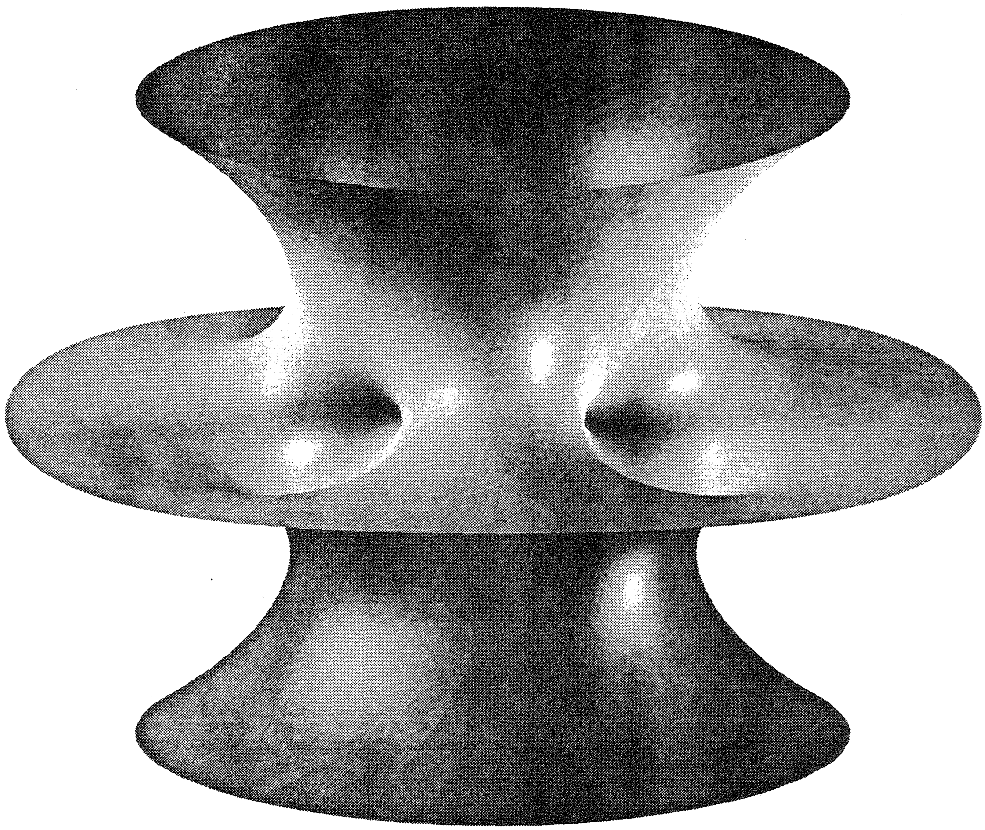
2. Helicoid:

$\Omega = \mathbf{C}$, $g(z) = e^z$, $\eta = e^{-z}dz$.

3. Enneper's Surface:

$\Omega = \mathbf{C}$, $g(z) = z$, $\eta = dz$.

4. Hoffman-Meeks' Surfaces:



Genus 2 Hoffman-Meeks Surface

These surfaces were discovered in 1985. Let

$$\overline{M_k} := \{(z,w) \in (\mathbf{C} \cup \{\infty\})^2 \,|\, w^{k+1} = z^k(z^2-1)\}.$$

$\overline{M_k}$ is a *genus k Riemann surface*, roughly speaking, a sphere with k handles and a special complex structure. Let

$$p_0 = (0,0), \;\; p_{-1} = (-1,0), \;\; p_1 = (1,0), \;\; p_\infty = (\infty,\infty) \in \overline{M_k}.$$

The surfaces we will consider are defined on

$$\Omega_k := \overline{M_k} - \{p_{-1}, p_1, p_\infty\},$$

and

$$g = \frac{c_k}{w}, \quad \eta = \left(\frac{z}{w}\right)^k dz = \frac{w}{z^2-1} dz.$$

It has been proved in [3] that for each integer $k > 0$, there is a unique $c_k > 0$ such that (14) is satisfied. The procedure of looking for c_k to satisfy (14) is called "killing periods".

For further readings in the theory of classical minimal surfaces in $\mathbf{R}^3$, we recommend [5], [2], and [4].

References

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