# LECTURES ON GEOMETRIC EVOLUTION EQUATIONS

*Klaus Ecker*

*Department of Mathematics*
*Monash University*

## Summary

We present a number of evolution equations which arise in differential geometry starting with the linear heat equation on a Riemannian manifold and proceeding to the curve shortening flow, mean curvature flow and Hamilton's Ricci flow for metrics.
We shall first show that a solution of the heat equation on a compact Riemannian manifold converges smoothly to its average value as $t \to \infty$, using only techniques which carry over to the nonlinear evolution equations presented in the lectures.
We will then concentrate mainly on curve shortening and mean curvature flow which exhibit many of the features particular to a variety of nonlinear parabolic equations.

## The Linear Heat Equation

Let $(M, g)$ be a Riemannian manifold and let $\Delta = \Delta_M$ denote the Laplace-Beltrami operator on $M$ defined by

$$\begin{aligned}\Delta f &= g^{ij}\nabla_i\nabla_j f \\ &= g^{ij}\Big(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\Big) \\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\Big(\sqrt{g}\, g^{ij}\frac{\partial f}{\partial x^j}\Big)\end{aligned}$$

where $(g^{ij})$ denotes the inverse metric, $\Gamma_{ij}^k$ are the Christoffel symbols, $g = \det g_{ij}$ and $\nabla_i\nabla_j f$ is the Hessian operator acting on $f$. Let $f : M \times (0,T) \to \mathbf{R}$, $T > 0$ be a function solving the heat equation

$$\frac{\partial f}{\partial t} = \Delta f \tag{1}$$

on $M$ subject to the initial condition $f(\cdot, 0) = f_0$.

For $M = S^2$ parametrized by spherical coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta \ \cos\varphi \\ \sin\theta \ \cos\varphi \\ \sin\varphi \end{pmatrix}$$

where $0 < \theta < 2\pi$, $0 < \varphi < \pi$ equation (1) becomes

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial \varphi^2} + \cos^2\varphi \frac{\partial^2 f}{\partial \theta^2} - \tan\varphi \frac{\partial f}{\partial \varphi}.$$

For $M = \mathbf{R}^n$, an explicit solution of (1) is given by

$$f(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbf{R}^n} f_0(y) e^{\frac{-|x-y|^2}{4t}} dy.$$

For a solution of the heat equation any smooth limiting function $f(\infty) = \lim_{t_k \to \infty} f(t_k)$ is *harmonic* on $M$, that is it satisfies the equation

$$\Delta f_\infty = 0.$$

Note that the only harmonic functions on a compact Riemannian manifold are the constant functions. Indeed, integration by parts yields (see also R. Bartnik's notes in these proceedings)

$$0 = \int_M f\Delta f = -\int_M |\nabla f|^2.$$

Hence $\nabla f = 0$ on $M$ (since $f$ is smooth) whence $f \equiv$ const. Therefore the following theorem seems very natural:

**Theorem.** *On a smooth compact Riemannian manifold $M$, the heat equation*

$$\frac{\partial f}{\partial t} = \Delta f$$

*for any reasonable initial data $f_0$ has a unique smooth solution for all positive time. Its average value $\bar{f} = \frac{1}{\mathrm{vol}(M)} \int_M f$ is time independent and $f$ converges smoothly to this constant as $t \to \infty$.*

We shall later present a complete proof of this theorem using only techniques which carry over to a large number of nonlinear evolution equations.

## Harmonic Map Heat Flow

Consider two Riemannian manifolds $(M, g_{ij}, \Gamma_{ij}^k)$ and $(N, \gamma_{ab}, K_{ab}^c)$ with metrics and connections as indicated. We consider a mapping $F : M \times (0,T) \to N$ satisfying the evolution equation

$$\frac{\partial F}{\partial t} = \tau(F) \tag{2}$$

with

$$\tau(F)^a = g^{ij}\left(\frac{\partial^2 F^a}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial F^a}{\partial x^k} + K_{bc}^a \frac{\partial F^b}{\partial x^i}\frac{\partial F^c}{\partial x^j}\right)$$

where $x^i$ denotes coordinates on $M$ and the superscript $a$ is with respect to to coordinates $y^a$ on $N$. Note that $\tau(F)^a$ can be interpreted as the tangential projection of $\Delta_M F^a$ onto $TN$. The first summand is $\Delta_M F^a$ while the second summand relates to the second fundamental form of $N$ in case $N$ is embedded isometrically into some Euclidean space.

For instance, in the case where $M = \mathbf{R}^n$ and $N = S^n$, (2) becomes

$$\frac{\partial F}{\partial t} = \Delta F - |DF|^2 F.$$

Mappings $F$ which satisfy $\tau(F) = 0$ are called *harmonic mappings*. They arise as critical maps for the energy functional

$$\int_M |DF|^2 = \int_M \gamma_{ab} g^{ij} \frac{\partial F^a}{\partial x^i}\frac{\partial F^b}{\partial x^j}\, d\mathrm{vol}_M.$$

**Examples:**
(i) $F : S^1 \to N$ is harmonic if and only if $F(S^1)$ is a geodesic in $N$ parametrized proportional to arclength.
(ii) $F : S^2 \to N$ is harmonic if and only if $F(S^2)$ is a minimal surface parametrized conformally.

The following theorem due to Eells & Sampson ([ES]) established the heat equation method as a major tool in the geometric calculus of variations.

**Theorem.** *Let $M$ and $N$ be compact Riemannian manifolds and let $N$ have non-positive sectional curvature. Then (2) has a unique smooth solution $F$ for all time. For any reasonable initial data $F_0$ the solution converges, as $t \to \infty$, to a harmonic map which belongs to the same homotopy class as $F_0$.*

## Curve Shortening Flow

Consider a planar curve $\Gamma \subset \mathbf{R}^2$ with position vector $x$ and normal $\nu$. We consider the evolution equation

$$\frac{\partial x}{\partial t} = k\nu \tag{3}$$

where $k$ denotes the curvature of $\Gamma$ at the point at which we want to move $\Gamma$. For a closed curve we consider more specifically $x = x(u,t)$ with $u \in [0, 2\pi), t \in (0,T)$. Let

$$s(u,t) = \int_0^u \left| \frac{\partial x}{\partial u'}(u',t) \right| du'$$

be the arclength function of the curve $\Gamma_t$ parametrized by $x(t,\cdot)$ and $\tau = \frac{\partial x}{\partial s}$ the unit tangent vector. Then $k\nu = \frac{\partial \tau}{\partial s} = \frac{\partial^2 x}{\partial s^2}$ whence (3) can be written as

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2}.$$

This looks like a linear heat equation for $x$. The nonlinearity of the problem is hidden in the definition of arclength which depends on $\frac{\partial x}{\partial u}(u,t)$.

If our curve is given as the graph of a function $y = f(u,t)$, equation (3) reduces to the single equation

$$\frac{\partial f}{\partial t} = \frac{1}{1 + (\frac{\partial f}{\partial u})^2} \frac{\partial^2 f}{\partial u^2}. \tag{3'}$$

A solution of (3') is equivalent to a solution of (3) up to tangential diffeomorphisms. The evolution process described by (3) is called curve shortening flow since it arises as the steepest descent flow for the length functional

$$L(\Gamma) = \int_0^{2\pi} \left| \frac{\partial x}{\partial u} \right| du.$$

The following theorems give a description of the global behaviour of solutions of (3).

**Theorem.** *(Gage & Hamilton, [GH]). Any closed convex curve in the plane remains convex and shrinks to a point in finite time. If we rescale the solution $\Gamma_t$ such as to keep the enclosed area constant we obtain a solution which converges smoothly to a round circle in the plane.*

**Theorem.** *(Grayson, [Gr1]) Any closed embedded curve in the plane remains embedded and becomes convex in finite time (after which the previous theorem takes over).*

We can also consider curve shortening flow of curves on two dimensional surfaces. The main result here is due to Grayson:

**Theorem.** *(Grayson, [Gr2]) Any closed embedded curve on a compact 2-surface has a unique evolution $(\Gamma_t)$ which remains embedded and either shrinks to a point in finite time or converges to a geodesic for $t \to \infty$.*

This result can be used to give a new proof of an old theorem of Ljusternik - Schnirelman which states that for any metric on $S^2$ there exist at least 3 simple closed geodesic loops.

### Mean Curvature Flow

This flow includes the curve shortening flow as a special case for $n = 1$. We consider the evolution equation

$$\frac{\partial x}{\partial t} = -H\nu \tag{4}$$

for hypersurfaces $M \subset \mathbf{R}^{n+1}$. Here $H$ denotes the mean curvature of $M_t = x(t,\cdot)(M)$ where $x(t,\cdot)$ is an immersion of $M$ into $\mathbf{R}^{n+1}$ at time $t$.
In view of the identity

$$\Delta_{M_t} x = -H\nu$$

where

$$\Delta_{M_t} x^k = \frac{1}{\sqrt{g(t)}} \partial_i \left( \sqrt{g(t)} g^{ij}(t) \partial_j x^k(t) \right)$$

for $k = 1, \ldots, n+1$ we obtain the nonlinear system of PDE's

$$\frac{\partial x}{\partial t} = \Delta_{M_t} x.$$

In the special case where $M_t = \operatorname{graph} u(t,\cdot)$ equation (4) is equivalent (up to tangential diffeomorphisms) to the single parabolic PDE

$$\frac{\partial u}{\partial t} = (\delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2})\ D_i D_j u. \tag{4'}$$

The first result about global behaviour of solutions of (4) was proved by G. Huisken in 1984.

**Theorem.** *(Huisken, [Hu1]) Let $M_0 \subset \mathbf{R}^{n+1}$ be compact and convex. Then the surfaces $M_t$ are convex and contract smoothly to a point in finite time. If we rescale $M_t$ such as to preserve the area, then they converge to an $n$-sphere (with the prescribed area) in infinite time.*

Without the convexity assumption, $M_t$ may develop singularities in finite time as for example if $M_0$ is a dumbbell shaped surface obtained by attaching two large spheres to the ends of a sufficiently long and thin cylinder.

However, there are some initial situations which guarantee the existence of a smooth longterm solution. For instance, in the case where $M_0 = \operatorname{graph} u_0$ with $u_0 : \mathbf{R}^n \to \mathbf{R}$, the following result holds.

**Theorem.** *(E. - Huisken, [EH1,2]). The equation (4) (or equivalently (4')) admits a smooth solution $M_t = \operatorname{graph} u(t,\cdot)$ for all $t > 0$ if we merely require that $u_0$ is locally Lipschitz continuous.*

Note, that in contrast to the linear heat equation $\frac{\partial u}{\partial t} = \Delta u$ on $\mathbf{R}^n$, no assumption about growth behaviour of $u_0$ at infinity has to be made. In the linear case,

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbf{R}^n} u_0(y) e^{\frac{-|x-y|^2}{4t}} dy$$

which blows up in finite time unless $u_0$ has controlled (e.g. polynomial) growth for $|y| \to \infty$.

Stationary solutions of (4) are minimal surfaces ($H_M \equiv 0$). In fact, one hopes to employ mean curvature flow in Riemannian manifolds to find minimal surfaces (analogously to Grayson's result about geodesic loops in 2 - surfaces).

In $\mathbf{R}^{n+1}$, closed compact minimal surfaces do not exist which can be seen by integrating the equation $\Delta_M x = 0$ by parts on $M$.
However, in $\mathbf{R}^{n+1}$ equation (4) can be used to solve the Dirichlet problem for the minimal surface equation.

**Theorem.** *(Huisken, [Hu4]) Let $\Omega \subset \mathbf{R}^n$ have smooth boundary with nonnegative mean curvature. Let $\varphi : \partial\Omega \to \mathbf{R}$ and $u_0 : \Omega \to \mathbf{R}$ be smooth. Then (4') admits a smooth solution $u(t,\cdot)$ for all $t > 0$ which converges to a minimal graph (solution of the minimal surface equation) as $t \to \infty$.*

## 5. Ricciflow of Metrics

We would like to evolve a metric on a given manifold to a "better" metric, such as for example a metric of constant sectional curvature. The main interest in considering such an evolution process stems from the fact that the topology of manifolds which admit metrics of constant sectional curvature is well-understood.

More specifically, we would like to find a family of metrics $(g_{ij}(t))$ on a manifold $M$ satisfying the equation

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} \tag{5}$$

with initial metric $g_{ij}(0)$. This flow was proposed and studied by R. Hamilton in 1982 [H2], see in particular the theorem below.

If for example the initial metric is invariant under $SO(3)$ with spheres $S^2$ as orbits then this is preserved under (5). In this case, the metric $g_{ij}(t)$ looks like

$$ds^2 = b(r,t)^2 dr^2 + a(r,t)^2 d\omega^2$$

where $\omega$ denotes coordinates on $S^2$ and $r$ is the radius function. The equation (5) reduces to the parabolic system of equations

$$\begin{aligned}\frac{\partial a}{\partial t} &= \frac{1}{b^2}\frac{\partial^2 a}{\partial r^2} - \frac{1}{b^2}\frac{\partial a}{\partial r}\frac{\partial b}{\partial r} + \frac{1}{ab^2}(\frac{\partial a}{\partial r})^2 - \frac{1}{a}\\ \frac{\partial b}{\partial t} &= \frac{2}{ab}\frac{\partial^2 b}{\partial r^2} - \frac{2}{ab^2}\frac{\partial a}{\partial r}\frac{\partial b}{\partial r}\end{aligned}$$

in this situation.

The recent interest in geometric evolution equations was initiated mainly by the following theorem due to R. Hamilton:

**Theorem.** *(Hamilton, [H2]) For any initial metric $(g_{ij}(0))$ on a compact 3-manifold $M$ with $\mathrm{Ric}_{ij}(0) > 0$, there exists a unique solution $(g_{ij}(t))_{t\in(0,T)}$ of (5) such that $\mathrm{Ric}_{ij}(t) > 0$ for all $t > 0$ and $\mathrm{Vol}(t) = \int_M \sqrt{\det g_{ij}(t)} \to 0$ as $t \to T$. If we rescale the metrics as to keep the volume fixed, they converge to a metric of constant positive sectional curvature.*

The rescaled flow mentioned in the theorem is given by the following:

$$\frac{\partial}{\partial t}\tilde{g}_{ij} = \frac{2}{3}\tilde{r}\,\tilde{g}_{ij} - 2\widetilde{\mathrm{Ric}}_{ij}$$

where $\tilde{R}$ denotes the scalar curvature of $\tilde{g}$ and $\tilde{r} = \frac{1}{\widetilde{\mathrm{vol}}(M)}\int_M \tilde{R}$. One readily checks that for such metrics

$$\frac{d}{dt}\widetilde{\mathrm{vol}}\,(M) = 0.$$

Combining the above theorem with a classification result for constant curvature manifolds (see Wolf [W]) we obtain an important classification result for compact 3-manifolds with positive Ricci curvature.

**Corollary.** *Any compact 3-manifold which admits a metric with positive Ricci curvature is diffeomorphic to the sphere $S^3$ or a quotient of $S^3$ by a finite linear group.*

### Gradient flows

All of the above flows except for the Ricci flow arise as so-called *gradient flows* for some geometric energy functional. Here, we will only sketch the underlying idea. For more detailed information especially about suitable function spaces for particular geometric problems we refer to the book by Struwe [S].

We would like to minimize energy functions $E : X \to \mathbf{R}$ on some space $X$, typically a Sobolev space of functions, mappings or tensorfields endowed with some inner product $\langle \cdot, \cdot \rangle_X$. Let $x = x(t)$ be a differentiable path in $X$, along which $E$ decreases fastest. Such a path will have its tangent direction at every point determined by some generalization of the gradient of $E$ at that point.
More specifically, we have the following: In many applications we are dealing with spaces $X$ where the directional derivative (Fréchet derivative) of $E$ along $x(t)$ can be represented in terms of a vectorfield (gradient) on $X$ by

$$\frac{d}{dt}E(x(t)) = \langle \nabla E(x(t)), \frac{dx}{dt}(t) \rangle_X .$$

We then consider the so-called *gradient flow* for $E$ defined by

$$\frac{dx}{dt} = -\nabla E(x).$$

In particular,

$$\frac{dE}{dt}(x) = -\|\nabla E(x)\|_X^2$$

where $\| \cdot \|_X$ denotes the norm on $X$ induced by $\langle \cdot, \cdot \rangle_X$.

**Examples:**

(i) Euclidean space: $X = \mathbf{R}^n$, $E(x) = \frac{1}{2}|x|^2$, $\nabla E(x) = x$, $\frac{dx}{dt} = -x$.
Steepest descent paths are rays into the origin.

(ii) Heat equation on a compact Riemannian manifold $M$: Let $X = H^{1,2}(M)$ and the energy be given by

$$E(f) = \frac{1}{2} \int_M |\nabla f|^2.$$

Let $t \to (f(t))$ be a path in $H^{1,2}(M)$. Assume for simplicity that all functions $f(t)$ along this path are smooth. We then calculate

$$\begin{aligned}
\frac{d}{dt}E(f(t)) &= \frac{1}{2} \int_M \frac{\partial}{\partial t}|\nabla f|^2 \\
&= \int_M \langle \nabla f, \frac{\partial}{\partial t} \nabla f \rangle \\
&= \int_M \langle \nabla f, \nabla \frac{\partial f}{\partial t} \rangle \\
&= -\int_M \Delta f \frac{\partial f}{\partial t} .
\end{aligned}$$

If we define an inner product on $X$ by $\langle f, g \rangle_X = \int_M fg$ the above reads as

$$\frac{d}{dt}E(f) = \langle -\Delta f, \frac{\partial f}{\partial t} \rangle_X$$

whence $\nabla E(f) = -\Delta f$. This leads to the heat equation $\frac{\partial f}{\partial t} = \Delta f$.

(iii) Mean curvature flow: Consider immersions $x : M \to \mathbf{R}^{n+1}$ of an $n$-dimensional hypersurface. As energy we choose the area of the image

$$E(x) = \text{area}\,(x(M))$$

If $x = x(t)$ denotes a path of immersions we calculate using the first variation formula (see lectures by Marty Ross in these proceedings)

$$\frac{d}{dt}E(x(t)) = \int_M H\langle \nu, \frac{dx}{dt}\rangle.$$

Regarding the integral on the right hand side as an inner product on an appropriate space of immersions we arrive at

$$\frac{dx}{dt} = -\nabla E(x) = -H\nu.$$

### Remarks on Short Time Existence of Solutions

The equations (1) - (5) above are all of parabolic type and, except for (1), are nonlinear. Short time existence of a solution is usually proved by first considering the so called linearized equation. (For the existence theory for linear parabolic equations we refer to [F], [LSU].) If we consider a nonlinear evolution equation (this could also be a system of equations) of the form

$$\frac{\partial u}{\partial t} = F(u, \nabla u, \nabla^2 u)$$

then the linearized equation is given by

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} F(u+\epsilon w, \nabla(u+\epsilon w), \nabla^2(u+\epsilon w)) \\ &\equiv a_{ij}\nabla_i\nabla_j w + \text{lower order terms}\end{aligned}$$

where $(a_{ij})$ is a positive definite matrix with entries depending on $u$. The linear existence and regularity theory (Schauder theory) in combination with an implicit function theorem (or a fixed point theorem) argument in an appropriate Banach space can then be employed to solve the nonlinear equation for a short time interval . One difficulty with the above equations (3)-(5) is that they are invariant under diffeomorphisms so the standard short time existence arguments do not apply immediately. One can however formulate and solve a modified but equivalent equation in each case (see for example [H1]. For instance, in the case of the curve-shortening flow (3), one can write $\Gamma_t$ as a normal graph over $\Gamma_0$ which leads to an equation of the type given by (3').

### Longterm Behaviour for Solutions of the Linear Heat Equation

Here we give an outline of the proof of the theorem about longterm behaviour of the solution of the linear heat equation discussed earlier. The techniques used are most easily presented for linear equations but are also applicable to a variety of nonlinear evolution equations.

We would like to show that a solution $f$ of

$$\frac{\partial f}{\partial t} = \Delta f$$

with smooth initial function $f(0) = f_0$ on a smooth compact Riemannian manifold $M$ converges smoothly to its average value. We first observe that the average

$$\bar{f}(t) = \frac{1}{\mathrm{vol}(M)} \int_M f(t)$$

is time independent. Indeed,

$$\begin{aligned} \frac{d}{dt}\bar{f}(t) &= \frac{1}{\mathrm{vol}(M)} \int_M \frac{\partial}{\partial t} f(t) \\ &= \frac{1}{\mathrm{vol}(M)} \int_M \Delta f = 0. \end{aligned}$$

The last identity is a consequence of the divergence theorem. In order to prove that $f$ converges to its average in $L^2(M)$ we use the

**Theorem.** *(Poincaré inequality). There exists a constant $c_0 > 0$ depending only on the geometry of $M$ s.t.*

$$\int_M |f - \bar{f}|^2 \leq c_0 \int_M |\nabla f|^2$$

*for all $f \in C^1(M)$.*

The constant $c_0$ is an isoperimetric constant which is related to the quantity

$$\sup_{\Omega \subset M,\ \Omega \text{ open}} \frac{\min\{\mathrm{Vol}\ (\Omega),\ \mathrm{Vol}\ (M \sim \Omega)\}}{\mathrm{Area}\ (\partial \Omega)}.$$

For instance, on manifolds with thin necks (think of a dumbbell) $c_0$ can be very large.

**Lemma.** *If $f$ satisfies $\frac{\partial}{\partial t} f = \Delta f$ on a compact manifold $M$ then*

$$\int_M |f - \bar{f}|^2(t) \leq e^{-2c_0^{-1}t} \cdot \int_M |f - \bar{f}|^2(0)$$

*for every $t \geq 0$. This implies in particular that as $t \to \infty$*

$$|f - \bar{f}| \to 0$$

*in the $L^2-$ sense.*

**Proof.** Using $\frac{\partial \bar{f}}{\partial t} = 0$ we calculate

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_M |f-\bar{f}|^2)\right) &= \int_M (f-\bar{f})\frac{\partial}{\partial t}(f-\bar{f}) \\ &= \int_M (f-\bar{f})\frac{\partial f}{\partial t} \\ &= \int_M (f-\bar{f})\Delta f \\ &= -\int_M |\nabla f|^2 \\ &\leq -c_0^{-1}\int_M |f-\bar{f}|^2\end{aligned}$$

whence

$$\frac{d}{dt}\int_M |f-\bar{f}|^2 \leq -2c_0^{-1}\int_M |f-\bar{f}|^2.$$

Integrating this differential inequality yields the result.

Next we would like to show that $|f-\bar{f}| \to 0$ uniformly as $t \to \infty$ and that all derivatives of $f$ converge to 0 uniformly as well. The main tool we are going to employ is the weak maximum principle which holds for more general classes of parabolic operators. We state it here for the heat operator.

**Proposition.** *Let $u : M \times [0,T] \to \mathbf{R}$ be a sufficiently regular function satisfying the inequality*

$$\frac{\partial u}{\partial t} \leq \Delta u$$

*on a compact manifold $M$ for $t \in (0,T]$, then*

$$\max_M u(t) \leq \max_M u(0)$$

*for all $t \in [0,T]$.*

*Remark.* More generally we may consider inequalities of the form

$$\frac{\partial u}{\partial t} \leq \Delta u + \langle \mathbf{a}, \nabla u\rangle.$$

Then $\max_M u(t) \leq \max_M u(0)$ as long as $\sup_{M\times(0,T]} |\mathbf{a}| < \infty$.

**Proof.** Heuristically we proceed as follows. We have

$$\Delta u = g^{ij}(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k}).$$

At a point where $u$ attains a maximum at time $t$ the first partials of $u$ vanish and the matrix $(\frac{\partial^2 u}{\partial x^i \partial x^j})$ is negative semidefinite. Hence $\Delta u \leq 0$ and therefore $\frac{\partial u}{\partial t} \leq 0$ at such a point. Hence $u$ cannot increase here. This argument is almost rigorous.

In more detail, one checks that the function $t \to M(t) \equiv \max_{x \in M} u(x,t)$ is Lipschitz continuous and hence differentiable a.e.. Furthermore, for a.e. $t \in (0, T]$

$$\begin{aligned} M'(t) &\leq \max\{\frac{\partial u}{\partial t}(x,t),\ x \in M \,\text{s.t.}\, u(x,t) = M(t)\} \\ &\leq \max\{\Delta u(x,t),\ x \in M \,\text{s.t.}\, u(x,t) = M(t)\} \\ &\leq 0. \end{aligned}$$

Hence $M(t) \leq M(0)$ for all $t \in [0, T]$.

An immediate consequence of the maximum principle is the following

**Proposition.** *A solution $f$ of the heat equation satisfies the inequality*

$$\max_M |f(t)| \leq \max_M |f(0)|.$$

Next we would like to use the maximum principle to obtain a time independent bound on $|\nabla f|$. In fact, we would even like to show that $|\nabla f| \to 0$ uniformly as $t \to \infty$.

Let us first calculate

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla f|^2.$$

In the process of finding a simple expression for this we will have to interchange 3rd derivatives of $f$ at some stage. This will introduce the Riemann curvature tensor of the metric on $M$ into the calculations.

**Proposition.** (Bochner formula) *For any smooth function $f$ on $M$ there holds the formula*

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + \mathrm{Ric}\,(\nabla f, \nabla f)$$

*where $\langle \cdot, \cdot \rangle$ denotes the metric on $M$ and* Ric(. , .) *its Ricci curvature.*

**Proof.** Recall (see R. Bartnik's notes) that the Riemann tensor on $M$ was defined in terms of interchanging second covariant derivatives of vector fields on $M$ (careful; my sign convention for the Riemann tensor may be the opposite of the one used elsewhere). Let $X$ be a vectorfield on $M$. We use the shorthand notation

$$X_{ijk} = \nabla_k \nabla_j X_i.$$

We also work in a local orthonormal frame on $M$ such that we can conveniently write all tensor identities using only lower indices. Then

$$X_{ijk} - X_{ikj} = X_l R_{lijk}$$

where we sum over repeated indices. The Ricci curvature is given by

$$R_{ij} = R_{ikjk}.$$

For a function $f$ on $M$ we have in particular (with $f_i$ denoting $\nabla_i f$)

$$\begin{aligned} f_{kik} &= f_{kki} + f_l R_{lkik} \\ &= (\Delta f)_i + f_l R_{li}. \end{aligned}$$

We now calculate

$$\Delta f_i = f_{ikk} = f_{kik} = (\Delta f)_i + f_l R_{li}$$

and therefore

$$\begin{aligned} \frac{1}{2}\Delta|\nabla f|^2 &= (f_i f_{ij})_j = f_{ij} f_{ij} + f_i f_{ijj} \\ &= f_{ij} f_{ij} + f_i((\Delta f)_i + R_{ij} f_j) \\ &= |\nabla^2 f|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + \mathrm{Ric}\,(\nabla f, \nabla f). \end{aligned}$$

**Lemma.** *If $f$ satisfies the heat equation $\frac{\partial f}{\partial t} = \Delta f$ then*

$$(\frac{\partial}{\partial t} - \Delta)|\nabla f|^2 = -2|\nabla^2 f|^2 - 2\,\mathrm{Ric}\,(\nabla f, \nabla f).$$

**Proof.** Calculate

$$\begin{aligned} \frac{\partial}{\partial t}|\nabla f|^2 &= 2\langle \nabla f, \frac{\partial}{\partial t}\nabla f\rangle \\ &= 2\langle \nabla f, \nabla \frac{\partial}{\partial t} f\rangle \\ &= 2\langle \nabla f, \nabla(\Delta f)\rangle \end{aligned}$$

and apply the Bochner formula.

There are a variety of estimates we can obtain by applying the weak maximum principle:

**Proposition.** *Let $f$ satisfy $\frac{\partial f}{\partial t} = \Delta f$ on a compact Riemannian manifold $M$. Then there exists a constant $c = c(\mathrm{Ric}) > 0$ s.t. for all $t > 0$*

$$\max_M |\nabla f|^2(t) \leq \frac{c}{t} \max_M f^2(0).$$

**Proof.** Since $M$ is compact there exists $K > 0$ such that

$$\mathrm{Ric}_{ij} \geq -K g_{ij}$$

holds on $M$. Using this in the above lemma we estimate

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla f|^2 \leq 2K|\nabla f|^2.$$

We also calculate

$$\left(\frac{\partial}{\partial t} - \Delta\right) f^2 = -2|\nabla f|^2.$$

Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right) (t\,|\nabla f|^2 + (K + \frac{1}{2})f^2) \le 0.$$

The maximum principle yields

$$\max_M (t\,|\nabla f|^2 + (K + \frac{1}{2})f^2) \le (K + \frac{1}{2}) \max_M f^2(0)$$

*Remark.* By applying the maximum principle to the functions $e^{2Kt}|\nabla f|^2$ and $e^{-2Kt}|\nabla f|^2$, respectively, one obtains the following gradient estimates:

If $R_{ij} \ge K g_{ij}$ for $K > 0$ then

$$\max_M |\nabla f|^2(t) \le e^{-2Kt} \max_M |\nabla f|^2(0).$$

If $R_{ij} \ge -K g_{ij}$ for $K > 0$ then

$$\max_M |\nabla f|^2(t) \le e^{2Kt} \max_M |\nabla f|^2(0).$$

The above proposition establishes that $|\nabla f| \to 0$ uniformly on $M$ as $t \to \infty$. We now proceed to bound higher derivatives of $f$. We will only discuss estimates for $\nabla^2 f$ in detail. The methods carry over readily to higher derivatives.
In order to estimate

$$\left(\frac{\partial}{\partial t} - \Delta\right) \nabla^2 f$$

we calculate

$$\frac{\partial}{\partial t}\nabla^2 f = \nabla^2 \frac{\partial}{\partial t} f = \nabla^2 \Delta f$$

and

$$\begin{aligned} \Delta|\nabla^2 f|^2 &= (f_{ij} f_{ij})_{kk} = (2 f_{ij} f_{ijk})_k \\ &= 2 f_{ijk} f_{ijk} + 2 f_{ij} f_{ijkk} \\ &= 2|\nabla^3 f|^2 + 2 f_{ij} f_{ijkk}. \end{aligned}$$

Using $R_{lijkk}$ to denote $\nabla_k R_{lijk}$ we compute

$$\begin{aligned} f_{ijkk} &= f_{ikjk} + (f_l R_{lijk})_k \\ &= f_{kijk} + f_{lk} R_{lijk} + f_l\, R_{lijkk} \\ &= f_{kikj} + f_{li} R_{lkkj} + f_{km} R_{mikj} + f_{lk} R_{lijk} + f_l R_{lijkk} \\ &= f_{kkij} + (f_n R_{nkik})_j + f_{li} R_{ikkj} + f_{km} R_{mikj} + f_{lk} R_{lijk} + f_l R_{lijkk} \\ &= (\Delta f)_{ij} + f_{nj} R_{nkik} + f_{li} R_{lkkj} + f_{km} R_{mikj} + f_{lk} R_{lijk} + f_n R_{nkikj} + f_l R_{lijkk}. \end{aligned}$$

Let $S * T$ denote a tensor obtained by contractions of tensor products of $S$ and $T$. Then the above formula can be written as

$$\Delta \nabla^2 f = \nabla^2 \Delta f + \nabla^2 f * R + \nabla f * \nabla R.$$

We therefore obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla^2 f|^2 &= -2|\nabla^3 f|^2 + \nabla^2 f * \nabla^2 f * R + \nabla^2 f * \nabla f * \nabla R \\ &\leq -2|\nabla^3 f|^2 + c(n, |R|)\, |\nabla^2 f|^2 + c(n, |\nabla R|)\, |\nabla^2 f||\nabla f| \\ &\leq -2|\nabla^3 f|^2 + c(n, |R|)\, |\nabla^2 f|^2 + c(n, |\nabla R|)\, |\nabla f|^2\end{aligned}$$

where we used the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and denoted all constants depending only on $n$, $|R|$ and $|\nabla R|$ by $c(n, |R|)$ and $c(n, |\nabla R|)$ respectively. Let now

$$g = |\nabla^2 f|^2 + L|\nabla f|^2$$

where $L > 0$ is a large constant. Then

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) g &\leq c|\nabla^2 f|^2 + c|\nabla f|^2 - 2L|\nabla^2 f|^2 \\ &\leq -|\nabla^2 f|^2 + c|\nabla f|^2 \\ &= -g + (c + L)|\nabla f|^2 \\ &\leq -g + C.\end{aligned}$$

Here we have chosen $L$ large depending on $n$, $|R|$ and $|\nabla R|$ and denoted by $C$ any constant depending on $n$, $|R|$, $|\nabla R|$ and $\max_{M\times(0,\infty)} |\nabla f|^2$. We want to show that $g$ satisfies the inequality

$$\max_{M\times[0,\infty)} g \leq C + \max_M g(0).$$

To this end let $\epsilon > 0$. Suppose there exists a first time $t_0 > 0$ such that at $x_0 \in M$

$$g(x_0, t_0) = C + \max_M g(0) + \epsilon.$$

Then at $(x_0, t_0)$ we have $\frac{\partial g}{\partial t} \geq 0$ and $\Delta g \leq 0$. Hence

$$0 \leq -\left(C + \max_M g(0) + \epsilon\right) + C$$

at $(x_0, t_0)$ which yields a contradiction. This argument establishes the following

**Proposition.** *We have the estimate*

$$\max_{M\times[0,\infty)} |\nabla^2 f|^2 \leq C$$

*where $C$ depends on $n, |R|, |\nabla R|, \max_{M\times[0,\infty)} |\nabla f|$ and $\max_M |\nabla^2 f|^2(0)$.*

*Remark.* One can even establish a bound for $t^2|\nabla^2 f|^2$ and therefore prove decay of $|\nabla^2 f|$ in time. In fact, for all $m \geq 1$

$$t^m |\nabla^m f|^2$$

can be bounded in terms of $n, m$, derivatives of Riem up to order $m-1$ and $\max_M |f(0)|$. To prove such an estimate for $|\nabla^m f|^2$ one shows that

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla^m f|^2 \leq -2|\nabla^{m+1} f|^2 + c(1 + |\nabla^m f|^2)$$

where $c$ depends on $n, m, |R|, |\nabla R|, \ldots, |\nabla^{m-1} R|$ and $|\nabla^{m-1} f|$ and then proceeds by induction on $m$.

We are now able to prove that $f - \bar{f}$ converges to 0 as $t \to \infty$ uniformly. This is achieved by combining the above higher order estimates with the $L^2$- decay estimate

$$\left(\int_M |f - \bar{f}|^2(t)\right)^{\frac{1}{2}} \leq c e^{-\delta t}$$

proved earlier.

**Lemma.** *There exist positive constants $C$ and $\delta$ such that*

$$\int_M |\nabla f|^2(t) \leq C e^{-\delta t}$$

*for all $t \geq 0$.*

**Proof.** Using $\Delta \bar{f} = 0$, the inequality $\left(\sum_{i=1}^n a_{ij}\right)^2 \leq n \sum_{i,j=1}^n a_{ij}^2$ and the above decay estimate we calculate at time $t$

$$\begin{aligned}
\int_M |\nabla f|^2 &= -\int_M (f - \bar{f})\Delta f \\
&\leq \int_M |f - \bar{f}|\,|\Delta f| \\
&\leq \sqrt{n} \int_M |f - \bar{f}|\,|\nabla^2 f| \\
&\leq \sqrt{n} \left(\int_M |f - \bar{f}|^2\right)^{\frac{1}{2}} \mathrm{Vol}(M)^{\frac{1}{2}} \max_{M\times[0,\infty)} |\nabla^2 f| \\
&\leq c\sqrt{n}\,\mathrm{Vol}(M)^{\frac{1}{2}} \max_{M\times[0,\infty)} |\nabla^2 f| e^{-\delta t}.
\end{aligned}$$

In view of the bound on $|\nabla^2 f|$ the results follows. To complete the proof of convergence we need the following Sobolev inequality on $M$:

There exists a constant $C = C(n, M, p)$ such that for all $C^1$- functions $f$ on $M$ and for every $p > n$

$$\max_M |f - \bar{f}| \leq C \left(\int_M |\nabla f|^p\right)^{\frac{1}{p}}.$$

Combining this with the previous lemma and the global bound on $|\nabla f|$ we estimate

$$\begin{aligned}\left(\int_M |\nabla f|^p\,(t)\right)^{\frac{1}{p}} &\leq \max_{M\times[0,\infty)} |\nabla f|^{1-\frac{2}{p}} \left(\int_M |\nabla f|^2(t)\right)^{\frac{1}{2}\cdot\frac{2}{p}} \\ &\leq \max_{M\times[0,\infty)} |\nabla f|^{1-\frac{2}{p}} C^{\frac{2}{p}} e^{-\delta\frac{2}{p}t} \\ &\leq C\, e^{-\delta t}.\end{aligned}$$

Applying the Sobolev inequality to $(f-\bar{f})(t)$ we therefore conclude for all $t \geq 0$

$$\max_M |f-\bar{f}|(t) \leq Ce^{-\gamma t}$$

with constants $C, \gamma > 0$. This establishes uniform convergence.

## Selected Topics from Mean Curvature Flow

### *1. Evolution equations of geometric quantities*

We would like to calculate how geometric quantities of a given hypersurface $M \subset \mathbf{R}^{n+1}$ change as this surface is deformed locally in the direction of an arbitrary normal vector field. The evolution equations for mean curvature flow are then obtained by substituting mean curvature as the normal speed.

More specifically, we consider vectorfields $X$ defined in a neighbourhood of $M$ which generate a 1-parameter family of diffeomorphisms of $\mathbf{R}^{n+1}$. We may assume without loss of generality that our vectorfields are of the form $X = \eta\nu$ where $\nu$ is a choice of normal of $M$ suitably extended and $\eta$ is a smooth function defined in a neighbourhood of $M$. In case of mean curvature flow we have $\eta = -H$.

We shall introduce a convenient set of coordinates in a normal neighbourhood about $M$:

Let $p \in M$ and $\Omega \subset \mathbf{R}^{n+1}$ be an open neighbourhood of $p$. Let $X \in C_c^\infty(\Omega; \mathbf{R}^{n+1})$ be a vectorfield which generates a 1-parameter family of diffeomorphisms

$$\varphi : \Omega \times (-\epsilon, \epsilon) \to \Omega$$

such that $\varphi_t(x) = \varphi(x,t) = x$ for all $x \in \mathbf{R}^n \sim \Omega$ and $t \in (-\epsilon, \epsilon)$, $\varphi_0(x) = x$ for all $x \in \Omega$ and $\frac{\partial\varphi}{\partial t}_{|t=0}(x) = X(x)$. Let $M_t = \varphi_t(M)$.

We introduce a local orthonormal frame $\tau_1 \ldots \tau_n, \nu$ near $p \in M$ with the property

$$\langle \nabla_{\tau_i}\tau_j, \tau_k\rangle(p) = 0, \qquad \langle \tau_i, \tau_j\rangle(p) = \delta_{ij}$$

for $1 \leq i,j,k \leq n$ where $\nabla$ denotes the standard connection on $\mathbf{R}^{n+1}$. The vectorfields $\tau_i(t) = \varphi_{t*}(\tau_i)$ then yield a local (not necessarily orthonormal) frame for $M_t$. By definition of $\tau_i(t)$ we have in particular

$$\nabla_{\tau_i}X - \nabla_X\tau_i = [X, \tau_i] = 0$$

in a neighbourhood of $M_0$. Assuming for simplicity that $X = \eta\nu$ where $\eta \in C_c^\infty(\Omega)$ and denoting $X = \frac{d}{dt}_{|t=0}$ we have the following

**Proposition.** *The metric satisfies*

$$\frac{d}{dt}\Big|_{t=0} g_{ij} = 2\eta h_{ij}$$

*where $h_{ij} = \langle \nabla_{\tau_i} \nu, \tau_j \rangle$ is the second fundamental form of $M$.*

**Proof.** We calculate

$$\begin{aligned}
X(g_{ij}) &= X\langle \tau_i, \tau_j \rangle \\
&= 2\langle \nabla_X \tau_i, \tau_j \rangle = 2\langle \nabla_{\tau_i} X, \tau_j \rangle \\
&= 2\langle \nabla_{\tau_i}(\eta\nu), \tau_j \rangle \\
&= 2\tau_i\ (\eta)\langle \nu, \tau_j \rangle + 2\eta\langle \nabla_{\tau_i}\nu, \tau_j \rangle \\
&= 2\eta h_{ij}
\end{aligned}$$

where we have used $[X, \tau_i] = 0$ in the second line.

**Corollary.** *The inverse metric $g^{ij}$ and the volume element $\sqrt{g} = \sqrt{\det g_{ij}}$ satisfy the equations*

*(i)*
$$\frac{d}{dt}\Big|_{t=0} g^{ij} = -2\eta h^{ij}$$

*and*

*(ii)*
$$\frac{d}{dt}\Big|_{t=0} \sqrt{g} = \eta H \sqrt{g}.$$

**Proof.** (i) is easy. To establish (ii) we compute

$$\frac{d}{dt}\Big|_{t=0} \sqrt{g} = \frac{1}{2}\sqrt{g}\, g^{ij} \frac{d}{dt}\Big|_{t=0} g_{ij} = \eta H \sqrt{g}.$$

**Proposition.** *The second fundamental form satisfies the equation*

$$\frac{d}{dt}\Big|_{t=0} h_{ij} = -\nabla_i^M \nabla_j^M \eta + \eta h_{ik} h_{kj}.$$

**Proof.** We calculate

$$
\begin{aligned}
X(h_{ij}) &= -X\langle \nabla_{\tau_i}\tau_j, \nu\rangle \\
&= -\langle \nabla_X \nabla_{\tau_i}\tau_j, \nu\rangle - \langle \nabla_{\tau_i}\tau_j, \nabla_X \nu\rangle \\
&= -\langle \nabla_{\tau_i} \nabla_X\tau_j, \nu\rangle - \langle \nabla_{\tau_i}\tau_j, \nabla_X \nu\rangle \\
&= -\langle \nabla_{\tau_i} \nabla_X\tau_j, \nu\rangle \qquad\qquad (\text{since } \nabla^T_{\tau_i}\tau_j(p) = 0 \text{ and } \nabla_X\nu \in T_pM) \\
&= \langle \nabla_{\tau_i} \nabla_{\tau_j} X, \nu\rangle \qquad\qquad\ (\text{since } [X, \tau_j] = 0) \\
&= -\langle \nabla_{\tau_i} \nabla_{\tau_j} (\eta\ \nu), \nu\rangle \\
&= -\langle \nabla_{\tau_i} (\tau_j(\eta)\nu + \eta\nabla_{\tau_j}\nu), \nu\rangle \\
&= -\tau_i\tau_j(\eta) - \tau_j(\eta)\langle \nabla_{\tau_i}\nu, \nu\rangle - \tau_i\ (\eta)\langle \nabla_{\tau_j}\nu, \nu\rangle - \eta\langle \nabla_{\tau_i}\nabla_{\tau_j}\nu, \nu\rangle \\
&= -\tau_i\tau_j(\eta) + \eta\langle \nabla_{\tau_j}\nu, \nabla_{\tau_i}\nu\rangle \qquad (\text{using } \nabla_{\tau_j}\nu \in T_pM) \\
&= -\tau_i\tau_j(\eta) + \eta h_{ik}h_{kj} \\
&= -\nabla^M_i\nabla^M_j\eta + \eta h_{ik}h_{kj} \qquad (\text{normal coordinates}).
\end{aligned}
$$

**Corollary.** *The mean curvature satisfies*

$$\frac{d}{dt}_{|t=0} H = -\Delta_M\eta - \eta|A|^2$$

*where* $|A|^2 = h_{ij}h_{ij}$.

**Proof.**

$$
\begin{aligned}
X(H) &= X(g^{ij}h_{ij}) \\
&= X(g^{ij})h_{ij} + g^{ij}X(h_{ij}) \\
&= -2\eta h_{ij}h_{ij} + X(h_{ii}) \\
&= -2\eta|A|^2 - \Delta_M\eta + \eta|A|^2.
\end{aligned}
$$

**Corollary.** (First variation formula) *The surface area (n-dimensional Hausdorff measure) satisfies*

$$\frac{d}{dt}_{|t=0} \mathcal{H}^n(M_t) = \int_M \eta H\sqrt{g}.$$

**Corollary.** (Second variation formula) *The second derivative of area is given by*

$$
\begin{aligned}
\frac{d^2}{dt^2}_{|t=0} \mathcal{H}^n(M_t) = &\int_M \langle\nabla\eta, \nu\rangle\eta H\sqrt{g} - \int_M \eta\Delta_M\eta\sqrt{g} \\
&- \int_M \eta^2|A|^2\sqrt{g} + \int_M \eta^2H^2\sqrt{g}.
\end{aligned}
$$

*Remark.* If $M$ is minimal, i.e. $H \equiv 0$ and $\eta \in C_c^\infty(M)$ then the second variation formula reduces to

$$\frac{d^2}{dt^2}_{|t=0} \mathcal{H}^n(M_t) = \int_M |\nabla^M\eta|^2\sqrt{g} - \int_M \eta^2|A|^2\sqrt{g}.$$

A minimal surface is called *stable* if

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{H}^n(M_t) \geq 0.$$

This yields the inequality

$$\int_M |A|^2 \eta^2 \sqrt{g} \leq \int_M |\nabla^M \eta|^2 \sqrt{g}$$

for all $\eta \in C_c^\infty(M)$.

To derive evolution equations for mean curvature flow we consider variations of $M = M_t$ for each $t$.

**Theorem.** (Evolution equations for mean curvature flow)

(i) $$\frac{d}{dt} g_{ij} = -2Hh_{ij}$$

(ii) $$\frac{d}{dt} g^{ij} = 2Hh^{ij}$$

(iii) $$\frac{d}{dt} \sqrt{g} = -H^2 \sqrt{g}$$

(iv) $$\left(\frac{d}{dt} - \Delta_{M_t}\right) H = H|A|^2$$

(v) $$\left(\frac{d}{dt} - \Delta_{M_t}\right) h_{ij} = -2H - h_{ik}h_{kj} + |A|^2 h_{ij}$$

(vi) $$\left(\frac{d}{dt} - \Delta_{M_t}\right) |A|^2 = 2|A|^4 - 2|\nabla A|^2$$

(vii) $$\frac{d}{dt} \nu = \nabla^{M_t} H$$

**Proof.** (i) - (iii) follow by replacing $\eta$ by $-H$ in the rate of change formulas for general deformations. To derive (iv) and (v) we need to express $\nabla_i^{M_t} \nabla_j^{M_t} H$ in terms of $\Delta_{M_t} h_{ij}$.

**Proposition.** (Simons' identity)

(+) $$\nabla_i^M \nabla_j^M H = \Delta_M h_{ij} - Hh_{ik}h_{kj} + |A|^2 h_{ij}$$

(++) $$\Delta_M |A|^2 = -2|A|^4 + 2|\nabla^M A|^2 + 2Hh_{ik}h_{kj}h_{ij} + 2h_{ij}\nabla_i^M \nabla_j^M H$$

**Proof.** (++) follows easily from (+) by multiplication with $h_{ij}$. To prove (+) we employ the Gauss equations

$$R_{ijkl}^M = h_{ik}h_{jl} - h_{il}h_{jk}$$

and the Codazzi equations

$$h_{ijk} = h_{ikj}$$

the latter being shorthand for $\nabla_k^M h_{ij} = \nabla_j^M h_{ik}$. We then calculate

$$\begin{aligned}
\Delta_M h_{ij} &= h_{ijkk} \\
&= h_{ikjk} \qquad \text{(Codazzi)} \\
&= h_{kijk} \qquad (\text{since } h_{ik} = h_{ki}) \\
&= h_{kikj} + h_{mi} R^M_{mkjk} + h_{km} R^M_{mijk} \qquad (\text{from definition of } R^M) \\
&= h_{kkij} + h_{mi}(h_{mj}h_{kk} - h_{mk}h_{kj}) + h_{km}(h_{mj}h_{ik} - h_{mk}h_{ij}) \quad \text{(Codazzi and Gauss)} \\
&= H_{ij} + H h_{ik} h_{kj} - |A|^2 h_{ij} \qquad \text{(relabelling indices)}
\end{aligned}$$

Equations (iv) - (vi) of the theorem now follow immediately. To prove (vii) we proceed as follows:

We assume w.l.o.g. that $\eta \neq 0$ in a neighbourhood of $p$ for the purpose of our calculations. Otherwise we interpret $\frac{d}{dt}\nu$ as $\frac{d}{dt}\nu(\varphi_t(x))$ using the chain rule (the result of the calculation will still be the same). If $X = \eta\nu$ is transverse to $M$ then $\frac{d}{dt}\nu = \nabla_X \nu$ and we calculate

$$\begin{aligned}
\nabla_X \nu &= \langle \nabla_X \nu, \tau_i \rangle \tau_i \qquad (\text{since } \nabla_X \nu \in T_p M) \\
&= -\langle \nu, \nabla_X \tau_i \rangle \tau_i \\
&= -\langle \nu, \nabla_{\tau_i} X \rangle \tau_i \qquad (\text{since } [X, \tau_i] = 0) \\
&= -\langle \nu, \nabla_{\tau_i}(-H\nu) \rangle \tau_i \\
&= \tau_i(H) \langle \nu, \nu \rangle \tau_i = \tau_i(H) \tau_i \\
&= \nabla^M H.
\end{aligned}$$

*2. Formation and structure of singularities for mean curvature flow*

Without any special assumptions on $M_0$ such as convexity or graph property, the solution $(M_t)$ will in general develop singularities in finite time. Here we discuss some techniques which are relevant for dealing with isolated singularities. We would first like to give a short proof of the fact that singularities for "dumb-bell" like hypersurfaces do indeed form. The argument is entirely based on the weak maximum principle. This was stated earlier for the ordinary heat operator. The proof, however, carries over unchanged if we consider $\Delta_{M_t}$ instead. We begin by comparing the flowing hypersurfaces with spheres moving by contraction.

**Proposition.** (Sphere comparison) *Let $(M_t)$ be a family of hypersurfaces moving by mean curvature. If*

$$M_0 \subset B_R(0)$$

*then*

$$M_t \subset B_{\sqrt{R^2 - 2nt}}(0).$$

*If*

$$M_0 \subset \mathbf{R}^{n+1} \sim B_R(0)$$

*then*

$$M_t \subset \mathbf{R}^{n+1} \sim B_{\sqrt{R^2 - 2nt}}(0).$$

**Proof.** *The position vector of $M_t$ satisfies*

$$\frac{d}{dt}|x|^2 = -2H\langle \nu, x\rangle$$

and

$$\Delta_{M_t}|x|^2 = 2n - 2H\langle \nu, x\rangle.$$

Hence

$$(\frac{d}{dt} - \Delta_{M_t})(|x|^2 + 2nt) = 0.$$

We can therefore employ the weak maximum principle with $f = |x|^2 + 2nt$ to obtain the result.

**Proposition.** *Let $(M_t)$ be a family of hypersurfaces moving by mean curvature. If for $\beta < n-1$*

$$M_0 \subset \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}, x_{n+1}^2 \geq \frac{x_1^2 + \ldots + x_n^2 - \epsilon}{n-1-\beta}\}$$

*then*

$$M_t \subset \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}, x_{n+1}^2 \geq \frac{x_1^2 + \ldots + x_n^2 + \epsilon - 2\beta t}{n-1-\beta}\}$$

*for $t \leq \frac{\epsilon}{2\beta}$. In particular, $M_{\frac{\epsilon}{2\beta}}$ is contained inside a cone.*

**Proof.** Using $(\frac{d}{dt} - \Delta_{M_t})x = 0$ we check that $(\frac{d}{dt} - \Delta_{M_t})x_{n+1}^2 = -2|\nabla^M x_{n+1}|^2$ and hence that

$$(\frac{d}{dt} - \Delta_{M_t})(|x|^2 - (n-\beta)x_{n+1}^2 + 2\beta t) \leq 0.$$

Applying the weak maximum principle yields the result.

We now consider a "dumbbell" - shaped initial hypersurface $M_0$. More specifically, we assume that $M_0$ consists of a long thin cylinder of radius $\sqrt{\epsilon}$ with a large sphere of radius $R$ attached at either end of the cylinder. Such a surface will be contained inside a hyperboloid with minimum diameter $\sqrt{\epsilon}$ and will contain two spheres of radius $R$ inside the volume it encloses. By the proposition, $M_{\frac{\epsilon}{2\beta}}$ will be contained inside a cone with vertex at 0 while if $R$ is large enough compared to $\epsilon$ will still contain two spheres and can therefore not have vanished completely. Therefore a singularity must have formed at 0 (possibly also at other points).

One can show that if a singularity forms at time $T$ we have

$$\limsup_{t \to T} \max_{M_t} |A|^2 = \infty.$$

In fact, if for compact $M_t$ we had a bound on $|A|^2$ up to time $T$, we could also obtain bounds on all derivatives of $A$ (we will return to this later) and would therefore be able to extend the solution $(M_t)$ a little beyond time $T$.

There is a certain minimum "blow-up" rate for $|A|^2$ at the first singular time $T$:

**Lemma.** *If $M_t$ is singular at $t = T$, i.e.*

$$\limsup_{t\to T} \max_{M_t} |A|^2 = \infty$$

*then*

$$\max_{M_t} |A|^2 \geq \frac{1}{2(T-t)}$$

*for all $t \in (0,T)$.*

**Proof.** The quantity $|A|^2$ satisfies the evolution equation

$$(\frac{d}{dt} - \Delta_{M_t})|A|^2 = 2|A|^4 - 2|\nabla^{M_t} A|^2$$

and hence

$$\frac{d}{dt} \max_{M_t} |A|^2 \leq 2(\max_{M_t} |A|^2)^2.$$

Integrating this differential inequality and using that $\limsup_{t\to T} \max_{M_t} |A|^2 = \infty$ yields the estimate.

If we also had an upper blow-up rate for $|A|^2$ of this kind we would be able to give a complete description of the structure of isolated singularities. This is done by means of a monotonicity formula first proved by G. Huisken ([Hu2]):

**Proposition.** *(*Monotonicity formula*) For $t < T$ the hypersurfaces $M_t$ satisfy*

$$\frac{d}{dt}\int_{M_t} \rho_{x_0,T}(x,t) = -\int_{M_t} \rho_{x_0,T}(x,t)\left|H - \frac{\langle x - x_0, \nu\rangle}{2(T-t)}\right|^2$$

*where*

$$\rho_{x_0,T}(x,t) = \frac{1}{(4\pi(T-t))^{\frac{n}{2}}} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right).$$

*Remark.* The proof uses the fact that $\frac{1}{(4\pi(T-t))^{\frac{1}{2}}}\rho_{x_0,T}$ satisfies the backward heat equation on $\mathbf{R}^{n+1}$,

$$\left(\frac{\partial}{\partial t} + \Delta_{\mathbf{R}^{n+1}}\right) \frac{1}{(4\pi(T-t))^{\frac{1}{2}}}\rho_{x_0,T} = 0$$

In fact, a monotonicity formula like the one above holds for any such backward heat kernel ([Hu3], [H3]):

Let $k$ solve the equation

$$\frac{d}{dt}k = -\Delta_{\mathbf{R}^{n+1}} k.$$

Then

$$\frac{d}{dt}\left((2(T-t))^{\frac{1}{2}}\int_{M_t} k\right) = -(2(T-t))^{\frac{1}{2}}\int_{M_t} |H - \frac{\langle \nabla k, \nu\rangle}{k}|^2 k$$
$$- (2(T-t))^{\frac{1}{2}}\int_{M_t}\left(\nabla_\nu\nabla_\nu k - \frac{\nabla_\nu k\,\nabla_\nu k}{k} + \frac{k}{2(T-t)}\right).$$

for all $t < T$. It turns out that the second integral on the right hand side is nonnegative for any backward solution of the heat equation and therefore

$$\frac{d}{dt}\left((2(T-t))^{\frac{1}{2}}\int_{M_t} k\right) \le -(2(T-t))^{\frac{1}{2}}\int_{M_t} |H - \frac{\langle \nabla k, \nu\rangle}{k}|^2 k.$$

To prove this more general version of the monotonicity formula we first calculate

$$\frac{d}{dt}k = \frac{\partial}{\partial t}k + \langle \nabla k, \frac{dx}{dt}\rangle = \frac{\partial}{\partial t}k - H\langle \nu, \nabla k\rangle$$

as well as

$$\begin{aligned}\Delta_{M_t} k &= \mathrm{div}_{M_t}\nabla^{M_t} k = \mathrm{div}_{M_t}(\nabla k - \langle \nabla k, \nu\rangle \nu)\\ &= \mathrm{div}_{\mathbf{R}^{n+1}}\nabla k - \langle \nabla_\nu \nabla k, \nu\rangle - \langle \nabla k, \nu\rangle \mathrm{div}_{M_t}\nu\\ &= \Delta_{\mathbf{R}^{n+1}} k - \nabla_\nu\nabla_\nu k - H\langle \nabla k, \nu\rangle.\end{aligned}$$

Hence using $\frac{d}{dt}\sqrt{g_t} = -H^2\sqrt{g_t}$ we compute

$$\begin{aligned}\frac{d}{dt}\left((2(T-t))^{\frac{1}{2}}\int_{M_t} k\right) &= (T-t)^{-\frac{1}{2}}\int_{M_t} k + (2(T-t))^{\frac{1}{2}}\int_{M_t}\frac{dk}{dt} - (2(T-t))^{\frac{1}{2}}\int_{M_t} kH^2\\ &= (T-t)^{-\frac{1}{2}}\int_{M_t} k - (2(T-t))^{\frac{1}{2}}\int_{M_t}\Delta_{\mathbf{R}^{n+1}} k\\ &\quad - (2(T-t))^{\frac{1}{2}}\int_{M_t} kH^2 - (2(T-t))^{\frac{1}{2}}\int_{M_t} kH\langle \nu, \nabla k\rangle\\ &= (T-t)^{-\frac{1}{2}}\int_{M_t} k - (2(T-t))^{\frac{1}{2}}\int_{M_t}\Delta_{M_t} k\\ &\quad - (2(T-t))^{\frac{1}{2}}\int_{M_t} 2H\langle \nu, \nabla k\rangle - (2(T-t))^{\frac{1}{2}}\int_{M_t}\nabla_\nu\nabla_\nu k\end{aligned}$$

Completing the square

$$\left|H - \frac{\nabla k\cdot \nu}{k}\right|^2$$

yields the monotonicity formula.

Let us now *assume* that $(M_t)$ satisfies the so-called *type I* blow-up rate

$$(*) \qquad \max_{M_t}|A|^2 \le \frac{C}{T-t}$$

for some $C > 0$. Note that there exist solutions $(M_t)$ which do not satisfy (*). Assumption (*) guarantees that we can rescale the surface about the isolated singularity and pass to a *smooth* limit which moves by homothety:

Suppose $0 \in \mathbf{R}^{n+1}$ is a singular point of the flow at the time $T$. We define rescaled immersions

$$\tilde{x}(p,s) = (2(T-t))^{-\frac{1}{2}} x(p,t)$$

for $p \in M^n$ and $s = s(t) = -\frac{1}{2}\log(T-t)$. The hypersurfaces

$$\tilde{M}_s = \tilde{x}(\cdot, s)(M^n)$$

are defined for $-\frac{1}{2}\log T \leq s < \infty$ and satisfy

$$\frac{d}{ds}\tilde{x} = \tilde{H}\tilde{\nu} + \tilde{x}.$$

In view of (*) the rescaled curvature satisfies

$$\max_{\tilde{M}_s} |\tilde{A}|^2 \leq C$$

for all $s$ with constant independent of $s$. This implies also an estimate for the position of $\tilde{M}_s$ by estimating

$$|x\ (p,t)| \leq \int_t^T |H(p,\tau)| d\tau \leq C \int_t^T \frac{1}{(T-t)^{\frac{1}{2}}} d\tau \leq C(T-t)^{\frac{1}{2}}$$

whence

$$|\tilde{x}(p,s)| \leq C$$

for all $p \in M^n$ and all $s$. One can now establish global bounds on $|\tilde{\nabla}^m\ \tilde{A}|$ for any $m \geq 0$ and then use standard convergence results to infer that for a subsequence $(s_j) \to \infty$ we have

$$\tilde{M}_{s_j} \to \tilde{M}_\infty$$

in $C^\infty$ where $\tilde{M}_\infty$ is a smooth hypersurface. Rescaling the monotonicity formula (with $k = \rho_{0,T}$) yields

$$\frac{d}{ds}\int_{\tilde{M}_s} \tilde{\rho} = -\int_{\tilde{M}_s} \tilde{\rho}\,|\tilde{H} - \langle \tilde{x}, \tilde{\nu}\rangle|^2$$

where $\tilde{\rho}(\tilde{x}) = e^{-\frac{1}{2}|\tilde{x}|^2}$.
Hence

$$\int_0^\infty \int_{\tilde{M}_s} \tilde{\rho}\,|\tilde{H} - \langle \tilde{x}, \tilde{\nu}\rangle|^2 \leq \int_{\tilde{M}_0} \tilde{\rho} < \infty$$

which implies (after selecting another subsequence) that

$$\lim_{k\to\infty} \int_{\tilde{M}_{s_k}} \tilde{\rho}|\tilde{H} - \langle \tilde{x}, \tilde{\nu}\rangle|^2 = 0.$$

Therefore $\tilde{M}_\infty$ satisfies the equation

$$(\#) \qquad H = \langle x, \nu \rangle.$$

The geometric significance of this equation is given by the fact that an initial surface $M_0$ satisfying (#) moves homothetically under mean curvature flow. Indeed, let

$$x(p,t) = (2(T-t))^{\frac{1}{2}} x(p,0)$$

where $M_0 = x(\cdot, 0)\ (M^n)$ satisfies (#). Then for $p \in M^n$,

$$\left(\frac{d}{dt} x(p,t)\right)^{\perp} = \frac{1}{(2(T-t))^{\frac{1}{2}}} \langle H(p,0), \nu \rangle = -H(p,t)\nu(p,t).$$

It turns out that the only embedded solutions of (#) with positive mean curvature are the round sphere (of radius $\sqrt{n}$) in the compact case and the cylinder in the non-compact case. This is a result due to G. Huisken ([Hu3]). There are many more interesting facts known about the structure of singularities but presenting these would exceed the scope of these lectures.

*3. Mean curvature flow of graphs.*

I would like to conclude with a few results about the flow of graphs and show that in this case mean curvature flow does not develop singularities:

We consider hypersurface $M_t = \operatorname{graph} u(t, \cdot)$ which move by mean curvature.The functions $u(t, \cdot) = \mathbf{R}^n \to \mathbf{R}$ then satisfy the single PDE

$$\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2} \operatorname{div} \left( \frac{Du}{\sqrt{1+|Du|^2}} \right).$$

The mail goal is to establish an estimate on $\sqrt{1+|Du|^2}$ such that the equation becomes uniformly parabolic which would then allow us to apply methods for linear equations. However, rather than working with the quantity $\sqrt{1+|Du|^2}$ directly we consider with the geometrically more natural quantity

$$v = \langle \nu, e_{n+1} \rangle^{-1}.$$

which agrees with up to tangential diffeomorphisms. Here, graphs are characterized by the condition $\langle \nu, e_{n+1} \rangle > 0$ for some choice of normal field.

From the evolution equation for $\nu$ we obtain

$$\frac{d}{dt} \nu_{n+1} = \langle \nabla^{M_t} H, e_{n+1} \rangle.$$

Using a local orthonormal frame as before we can also calculate $\Delta_{M_t}\nu_{n+1}$ by

$$\begin{aligned}\Delta_{M_t}\nu_{n+1} &= \tau_i\tau_i\nu_{n+1} = \tau_i\langle\nabla_{\tau_i}\nu, e_{n+1}\rangle\\ &= \tau_i(h_{ik}\langle\tau_k, e_{n+1}\rangle) = \tau_i h_{ik}\langle\tau_k, e_{n+1}\rangle + h_{ik}\langle\nabla_{\tau_i}\tau_k, e_{n+1}\rangle\\ &= \tau_k h_{ii}\langle\tau_k, e_{n+1}\rangle - h_{ik}h_{ik}\langle\nu, e_{n+1}\rangle \qquad \text{(using the Codazzi equations)}\\ &= \langle\nabla^{M_t}H, e_{n+1}\rangle - |A|^2\nu_{n+1}\end{aligned}$$

Therefore,

$$\left(\frac{d}{dt} - \Delta_{M_t}\right)\nu_{n+1} = |A|^2\nu_{n+1}$$

whence

$$\left(\frac{d}{dt} - \Delta_{M_t}\right)v = -|A|^2 v - 2v^{-1}|\nabla^{M_t}v|^2.$$

We now would like to employ the weak maximum principle. However, since we are on a non-compact hypersurface $M_t$ we cannot argue as before in a pointwise fashion to prove it.

**Proposition.** *Suppose $f = f(x,t)$ satisfies the inequality*

$$\left(\frac{d}{dt} - \Delta_{M_t}\right)f \le \langle \mathbf{a}, \nabla^{M_t} f\rangle$$

*where $\mathbf{a}$ is a vectorfield satisfying $\sup_{(0,T)}\sup_{M_t}|\mathbf{a}| < \infty$. Then*

$$\sup_{M_t} f \le \sup_{M_0} f$$

*for all $t \in (0,T)$.*

**Proof.** Similarly to the proof of the monotonicity formula for $\rho_{x_0,T}$ one can show ([EH1]) that the function $f_k^2 = (\max(f-k,0))^2, k > 0$ satisfies the inequality

$$\frac{d}{dt}\int_{M_t} f_k^2\rho_{x_0,T} \le \frac{1}{2}\sup_{(0,T)}\sup_{M_t}|a|^2\int_{M_t} f_k^2\rho_{x_0,T}.$$

Hence, if $k \ge \sup_{M_0} f$ we have $\int_{M_0} f_k^2\rho_{x_0,T} = 0$ which implies

$$\int_{M_t} f_k^2\rho_{x_0,T} = 0$$

for all $t \in (0,T)$ and therefore $\sup_{M_t} f \le k$ for all $t \in (0,T)$.

From the evolution equation for $v$ (since all the terms on the RHS are nonpositive) we immediately obtain a gradient estimate:

**Proposition.** *The gradient function satisfies the estimate*

$$\sup_{M_t} v \leq \sup_{M_0} v$$

*for all* $t \in (0, \infty)$.

This establishes that if $M_0$ grows linearly then this is preserved for all $t > 0$. In particular, this proves that the above equation for $u$ is uniformly parabolic (since we now also have a bound on $\sqrt{1 + |Du|^2}$). Standard theory for parabolic equations (see [F], [LSU]) then yields estimates on all higher derivatives of $u$. Let me indicate how (at least in the case of the curvatures) such estimates can be obtained in a more geometric fashion:

Combining the equations for $|A|^2$ and $v$ one calculates using also the inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$ that

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) |A|^2 v^2 \leq -2v^{-1} \langle \nabla^{M_t} v, \nabla^{M_t} (|A|^2 \ v^2) \rangle$$

and furthermore

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) (t|A|^2 v^2 + v^2) \leq -2v^{-1} \langle \nabla^{M_t} v, \nabla^{M_t} (t|A|^2 v^2 + tv^2) \rangle.$$

Since $v^{-1} |\nabla^{M_t} v| \leq |A| v$ we can then apply the weak maximum principle to obtain

**Theorem.** *For all* $t > 0$ *we have*

$$\sup_{M_t} (t|A|^2 v^2 + v^2) \leq \sup_{M_0} v^2.$$

This yields in particular that

$$\sup_{M_t} |A|^2 \to 0$$

as $t \to \infty$, i.e. the hypersurfaces $M_t$ become flat as $t \to \infty$. There are also local versions of all these estimates which can be used in conjunction with a spherical barrier argument to prove that mean curvature flow admits a smooth solution $M_t = \mathrm{graph} u(t, \cdot)$ even if the initial data $M_0$ is merely locally Lipschitz continuous (see [EH2]).

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