The Dirichlet Problem for the Minimal Surface Equation

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Abstract:

The minimal surface equation is an elliptic equation but it is nonlinear and is not uniformly elliptic. It is the Euler-Lagrange equation for variational problems which involve minimising the area of the graphs of functions. For the most part we will solve the variational problem with Dirichlet boundary values, that is, when the values of the function are prescribed on the boundary of some given set. We will present some existence results using the Direct Method from the Calculus of Variations and also some interior gradient estimates. All of the techniques can be generalised to include more difficult equations but the essence of the ideas is much clearer when dealing with this particular equation especially as it has such strong geometrical meaning. The material presented closely follows Chapters 12 and 13 from the book "Minimal Surfaces and Functions of Bounded Variation" by E.Giusti.

Chapter 1: Existence of Solutions

1. Problem Description

Suppose Ω is a given bounded open subset of $\mathbb{R}^n$ and ϕ is a given real valued function defined on $\partial\Omega$, the boundary of Ω. Let

$$\mathcal{C} = \{u \ : \ u \in C^1(\Omega) \cap C^0(\overline{\Omega}),\ u(x) = \phi(x) \text{ for } x \in \partial\Omega\}$$

For $u \in \mathcal{C}$ define

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx$$

then the problem we will consider is:

(P) *Find $u \in \mathcal{C}$ such that $\mathcal{A}(u) \le \mathcal{A}(v)$ for all $v \in \mathcal{C}$.*

We will assume that ϕ and $\partial\Omega$ are smooth although for the most part it is sufficient to assume $\partial\Omega$ is locally Lipschitz and ϕ is continuous.

Note that $\mathcal{A}(u)$ is the area of the graph of u so that the problem could be rephrased as finding the graph of least area amongst all graphs having a prescribed boundary given by the graph of ϕ over $\partial\Omega$. Of course this is *not* the same as finding the *surface* of least area amongst those having the prescribed boundary.

Also note that although we have restricted ourselves to C^1 functions in the definition for $\mathcal{C}$ we would achieve the same infimum for $\mathcal{A}$ over $\mathcal{C}$ if we only allowed C^∞ functions or alternatively if we allowed all functions from the Sobolev Space $W^{1,1}(\Omega)$. This follows from the standard approximation results of $W^{1,1}(\Omega)$ functions by C^∞ functions.

The material we present is mostly from Chapters 12 and 13 of the book by Giusti[G]. It can also be found in the books [MM] and (in a more general setting) [GT]. The large volume [N] by Nitsche contains much information about minimal surfaces and includes a particular section of interest on non-parametric minimal surfaces which gives several useful references. Problems dealing with minimal surfaces rather than just minimal graphs are treated in the lectures on Geometric Measure Theory and Classical Minimal Surfaces.

2. Euler-Lagrange Equation

Suppose u is a solution to problem (P) and $\eta \in C^1(\Omega)$ is 0 on $\partial\Omega$. Then the function

$$F(t) = \mathcal{A}(u + t\eta)$$

must have a minimum at $t = 0$. Thus $F'(0) = 0$. Differentiating under the integral sign shows

$$\int_\Omega \frac{\nabla u \cdot \nabla \eta}{\sqrt{1 + |\nabla u|^2}}\, dx = 0 \tag{2.1}$$

If u is twice differentiable then integration by parts yields

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\frac{\partial u}{\partial x_i}}{\sqrt{1+|\nabla u|^2}} \right) = 0 \tag{2.2}$$

or, equivalently,

$$div\,(a(\nabla u)) = 0 \tag{2.3}$$

where $a(p) = (a_1(p), a_2(p), \ldots, a_n(p))$ and $a_i(p) = \dfrac{p_i}{\sqrt{1+|p|^2}}$.

This partial differential equation is known as the *minimal surface equation.*

Exercise:

(i) Verify the above derivation of the minimal surface equation.

(ii) Show that the equation can also be written in the form

$$\sum_{i,j=1}^{n} a_{ij}(\nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0$$

and for each p, $a_{ij}(p)$ is a symmetric positive definite matrix. Show that the ratio of the maximum and minimum eigenvalues for this matrix is unbounded as $|p| \longrightarrow \infty$.

This shows that the minimal surface equation is elliptic but not uniformly elliptic. It is also quasilinear (i.e. linear in the highest derivatives) and can be written in divergence form (i.e. form (2.3)).

Even if u is not a solution to problem (P) we can still form the function $F(t)$ as above. If we now apply Taylor's Theorem to F we find

$$F(t) = \mathcal{A}(u) + tF'(0) + t^2 F''(s) \tag{2.4}$$

where s is some point between 0 and t. Calculations as in the exercise above show that $F''(s) \geq 0$ and so u provides the minimum if and only if $F'(0) = 0$. Later we will see that solutions to problem (P) are automatically C^∞ and so

Theorem 1. *A function $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ solves problem (P) if and only if it is in $C^\infty(\Omega) \cap C^0(\overline{\Omega})$ and satisfies*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\frac{\partial u}{\partial x_i}}{\sqrt{1 + |\nabla u|^2}} \right) = 0$$

and

$$u(x) = \phi(x), \quad \text{for } x \in \partial\Omega.$$

Exercise:

(i) Show that $\mathcal{A}$ is convex on $\mathcal{C}$.

(ii) Show that $\mathcal{A}$ is, in fact, strictly convex. That is

$$\mathcal{A}(\lambda u + (1 - \lambda)v) \leq \lambda\mathcal{A}(u) + (1 - \lambda)\mathcal{A}(v)$$

with strict inequality unless $u = v$.

(iii) Using the strict convexity show that there is at most one solution to Problem (P).

3. Examples

It is not hard to find examples of functions satisfying (2.2). Trivially any linear function satisfies the equation but many more complicated, and often beautiful, examples can be found in the literature (see [N]). By Theorem 1, if we use appropriate boundary conditions, we will then have examples where Problem (P) has a solution.

We can also construct examples where there is no solution.

Let $\Omega \subseteq \mathbb{R}^2$ be the annulus

$$\Omega = \{x \in \mathbb{R}^2 \ : \ \rho < |x| < R\}$$

and ϕ be the function

$$\phi(x) = \begin{cases} M, & \text{if } |x| = \rho, \\ 0, & \text{if } |x| = R. \end{cases}$$

where $0 < \rho < R$ and M are given constants.

Suppose that Problem (P) has a solution u.

Now define a new function on $[\rho, R]$ by

$$v(r) = \frac{1}{2\pi} \int_{|y|=1} u(ry)\, ds, \qquad \text{for } \rho < r < R.$$

That is, $v(r)$ is defined as the average of u around the circle of radius r. Now define

$$\tilde{u}(x) = v(|x|), \qquad \text{for } x \in \Omega.$$

Exercise:

Show that $\mathcal{A}(\tilde{u}) \le \mathcal{A}(u)$ and that $\tilde{u}$ also satisfies the same boundary conditions as u.

The exercise shows that $\tilde{u}$ is also a solution to Problem (P) and in fact uniqueness theorems (see the exercise in Section 2) would then show that $\tilde{u} = u$. From Theorem 1, $\tilde{u}$ is a radially symmetric solution to (2.2) and so we must have

$$v''(r) = -\frac{1}{r}v'(r)\left(1 + (v'(r))^2\right)$$

as well as $v(\rho) = M$ and $v(R) = 0$. Solving this ordinary differential equation gives

$$v(r) = c \log \frac{R + \sqrt{R^2 - c^2}}{r + \sqrt{r^2 - c^2}}$$

where c is a constant to be determined such that $v(\rho) = M$ and $0 \le c \le \rho$. However

$$v(\rho) = c \log \frac{R + \sqrt{R^2 - c^2}}{\rho + \sqrt{\rho^2 - c^2}} \le \rho \log\left(R + \sqrt{R^2 - \rho^2}\right) \tag{3.1}$$

So if ρ and R are fixed and then M is chosen larger than the right hand side of (3.1) the condition for $v(\rho)$ cannot be satisfied and so u cannot have been a solution to Problem (P).

4. First Existence Results

The last example shows that some conditions must be imposed on Ω and ϕ if we are to expect a solution (and smoothness is not enough). Later we shall give a precise condition, but for the moment we impose a condition and present a proof used by Miranda[M]. The condition is

Bounded Slope Condition (BSC): There is a constant K such that for each $x_0 \in \partial\Omega$ there are linear functions π_+ and π_- satisfying

(i) $\pi_-(x_0) = \phi(x_0) = \pi_+(x_0)$,

(ii) $\pi_-(x) \le \phi(x) \le \pi_+(x), \quad$ for all $x \in \partial\Omega$,

(iii) $|\nabla \pi_-| \le K$ and $|\nabla \pi_+| \le K$.

Some work by Hartman[H] shows that this condition is more or less equivalent to assuming that Ω is uniformly convex and ϕ and $\partial\Omega$ are $C^{1,1}$.

Theorem 2. *Suppose Ω is a bounded open subset of $\mathbb{R}^n$ and ϕ is a function defined on $\partial\Omega$. If Ω and ϕ satisfy a Bounded Slope Condition then Problem (P) has a solution. Furthermore the solution is unique.*

Rather than solve the problem directly we will first look at the easier problem of minimising over the set $\mathcal{C}_k$ rather than $\mathcal{C}$ where

$$\mathcal{C}_k = \{u \in \mathcal{C} \ : \ |u(x) - u(y)| \leq k|x - y|, \text{ for all } x, y \in \Omega\}.$$

The condition introduced here is known as a Lipschitz condition with constant k.

The plan is to look at a minimising sequence of functions in $\mathcal{C}_k$, show that it has a convergent subsequence and that the limit is the solution for the easy problem. Next we show, using the Bounded Slope Condition, that if k is taken sufficiently large then the solution to the $\mathcal{C}_k$ problem actually satisfies $|u(x)-u(y)| \leq k'|x-y|$ for some $k' < k$ and that this means that u in fact solves Problem (P). To make all the steps we need some preliminary lemmas. The first of these is a lower semicontinuity result. Such results are of fundamental importance in the Calculus of Variations and there are many more powerful ones than the simple result we present here.

Lemma 1. (Lower Semicontinuity) *Suppose $\{u_i\}$ is a sequence of functions all with Lipschitz constant k on Ω. Suppose the sequence converges uniformly to u. Then u also has Lipschitz constant k and*

$$\mathcal{A}(u) \leq \liminf_{i \longrightarrow \infty} \mathcal{A}(u_i).$$

Proof: Suppose $g = (g_1, g_2, \dots, g_{n+1})$ is a differentiable vector valued function with compact support in Ω, i.e. $g \in C_0^1(\Omega; \mathbb{R}^{n+1})$. Suppose also that $|g(x)| \leq 1$. Then integration by parts shows

$$\int_\Omega g_{n+1} + u \sum_{j=1}^n \frac{\partial g_j}{\partial x_j}\, dx = \int_\Omega g_{n+1} - u \sum_{j=1}^n g_j \frac{\partial u}{\partial x_j}\, dx \leq \int_\Omega |g|\sqrt{1 + |\nabla u|^2}\, dx \leq \mathcal{A}(u).$$

On the other hand if we take $g = (1, -\nabla u)/\sqrt{1 + |\nabla u|^2}$ then equality will actually be achieved. However this choice for g may not be allowed since it does not have compact support in Ω. We multiply g by a sequence of smooth cut-off functions which are 0 near $\partial\Omega$ but 1 in the rest of Ω. We choose the sequence so that it converges to 1 everywhere in Ω and then it is not hard to see that the integrals above converge to $\mathcal{A}(u)$. Thus

$$(4.1). \qquad \mathcal{A}(u) = \sup\left\{ \int_\Omega g_{n+1} + u \sum_{j=1}^n \frac{\partial g_j}{\partial x_j}\, dx \ : \ g \in C_0^1(\Omega; \mathbb{R}^{n+1}), |g| \leq 1 \right\}$$

Suppose $g \in C_0^1(\Omega; \mathbb{R}^{n+1})$ and $|g| \leq 1$ then using this idea for each of the functions together with uniform convergence

$$\int_\Omega g_{n+1} + u \sum_{j=1}^n \frac{\partial g_j}{\partial x_j}\, dx = \lim_{i \to \infty} \int_\Omega g_{n+1} + u_i \sum_{j=1}^n \frac{\partial g_j}{\partial x_j}\, dx \leq \liminf_{i \longrightarrow \infty} \mathcal{A}(u_i).$$

Taking the supremum over all such g gives the result.

Equation (4.1) is an important one in the theory of the minimal surface equation and it is the basis for the theory based in the space of functions of Bounded Variation. (See [G].)

Lemma 2. (Existence of Minimiser in $\mathcal{C}_k$) *If $\mathcal{C}_k$ is nonempty then there is a function $u_k \in \mathcal{C}_k$ such that $\mathcal{A}(u_k) \leq \mathcal{A}(v)$ for all $v \in \mathcal{C}_k$.*

Proof: Let $\{u_j\}$ be a sequence in $\mathcal{C}_k$ such that $\mathcal{A}(u_j) \longrightarrow \inf\{\mathcal{A}(v) \ : \ v \in \mathcal{C}_k\}$. By the Lipschitz condition this sequence is equicontinuous. Further, again using the Lipschitz condition, it is easy to show that $|u| \leq \sup|\phi| + k$ diameter of Ω. Thus the sequence is bounded and so by the Ascoli-Arzela theorem we can select a uniformly convergent subsequence. The result now follows by using Lemma 1.

Lemma 3. (Test for Minimiser in C) *Suppose u is a minimiser in $\mathcal{C}_k$ and there is $k' < k$ such that $|u(x) - u(y)| \leq k'|x - y|$ for $x, y \in \Omega$. Then u is a minimiser in $\mathcal{C}$.*

Proof: Suppose $\eta \in C_0^1(\Omega)$ and form $F(t)$ as in Section 2. Since, for t sufficiently small $u + t\eta \in \mathcal{C}_k$ and u minimises in $\mathcal{C}_k$ we must have $F'(0) = 0$. Then the result follows by Theorem 1.

Lemma 4. (Comparison Principle) *Suppose u and w are Lispschitz continuous functions with Lipschitz constant at most k and $u \leq w$ on $\partial\Omega$. Suppose also that u and w both minimise $\mathcal{A}(v)$ amongst functions with Lipschitz constant at most k and equaling u and w respectively on $\partial\Omega$. Then $u \leq w$ in Ω.*

Proof: Let $u_1(x) = \min\{u(x), w(x)\}$ and $E = \{x \in \Omega \ : \ w(x) < u(x)\}$. Suppose that the set E is nonempty (otherwise the Lemma is true). Note that $u_1(x) = w(x)$ if $x \in E$ and $u_1(x) = u(x)$ otherwise. Also $u_1 = u$ on $\partial\Omega$ and the Lipschitz constant of u_1 is at most k. Thus $\mathcal{A}(u) \leq \mathcal{A}(u_1)$ and so

$$\int_E \sqrt{1 + |\nabla u|^2}\, dx \leq \int_E \sqrt{1 + |\nabla w|^2}\, dx.$$

Similarly taking $u_1(x) = \max\{u(x), w(x)\}$ and comparing it with w we find the opposite inequality holds so that, in fact

$$\int_E \sqrt{1 + |\nabla u|^2}\, dx = \int_E \sqrt{1 + |\nabla w|^2}\, dx.$$

Obviously we have $u = w$ on ∂E and both u and w must minimise

$$\int_E \sqrt{1 + |\nabla u|^2}\, dx$$

amongst functions in

$$\mathcal{E} = \{v \ : \ v = u = w \text{ on } \partial E, \ |v(x) - v(y)| \leq k|x - y|, x, y \in E\}.$$

Now consider $\tilde{u} = (u + w)/2$ which belongs to $\mathcal{E}$. Since the function $\sqrt{1 + |p|^2}$ is strictly convex on $\mathbb{R}^n$ we have $\sqrt{1 + |\nabla \tilde{u}|^2} < \frac{1}{2}\left(\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla w|^2}\right)$ whenever $\nabla u \neq \nabla w$. Thus if $\nabla u \neq \nabla w$ in a set of positive measure we have

$$\int_E \sqrt{1 + |\nabla \tilde{u}|^2}\, dx < \int_E \sqrt{1 + |\nabla u|^2}\, dx$$

which is a contradiction. Thus $\nabla u = \nabla w$ almost everywhere and so $u = w$ on E which contradicts the definition of E.

Similar arguments apply without the restriction on the Lipschitz constant and so they will apply to Problem (P).

Theorem 3. *There is at most one solution to Problem (P).*

There are several useful consequences of Lemma 4 and we give two. The first is a maximum principle giving global bounds on the difference of solutions in terms of their difference on the boundary and the second gives a bound for the Lipschitz constant of solutions in terms of the Lipschitz constant on the boundary.

Lemma 5. (Maximum Principle) *Suppose u and w are Lispschitz continuous functions with Lipschitz constant at most k. Suppose also that u and w both minimise $\mathcal{A}(v)$ amongst functions with Lipschitz constant at most k and equaling u and w respectively on $\partial\Omega$. Then*

(i) $\sup\{u(x) - w(x) \ : \ x \in \Omega\} = \sup\{u(x) - w(x) \ : \ x \in \partial\Omega\}$

(ii) $u(x) \leq \sup_{\partial\Omega} \phi,$ for all $x \in \Omega$

Proof: If w is a minimiser and α is a real number then $w + \alpha$ is also a minimiser but for boundary values increased by α. Choose $\alpha = \sup\{u(x) - w(x) \ : \ x \in \partial\Omega\}$ and then apply Lemma 4 to u and $w + \alpha$.

Lemma 6. (Maximum Principle for Difference Ratios) *Let u minimise $\mathcal{A}(v)$ in $\mathcal{C}_k$. Then*

$$|u(x) - u(y)| \leq k'|x - y|, \qquad \text{for } x, y \in \Omega,$$

where

$$k' = sup\left\{ \frac{|u(x) - u(z)|}{|x - z|} \; : x \in \Omega, \; z \in \partial\Omega \right\}.$$

Proof: Let x_1 and x_2 belong to Ω and set $\tau = x_2 - x_1$. Now consider the function $u_\tau(x) = u(x + \tau)$ and the set $\Omega_\tau = \{x \; : \; x + \tau \in \Omega\}$. Clearly u_τ minimises an integral like $\mathcal{A}(v)$ (but over Ω_τ instead of Ω) amongst functions equal to u_τ on $\partial\Omega_\tau$ which have Lipschitz constant at most k. Now consider u and u_τ on the set $\Omega \cap \Omega_\tau$ which contains x_1. Both functions are minimisers in appropriate classes and so we can apply Lemma 5 to obtain

$$\sup\{u(x) - u_\tau(x) \; : \; x \in \Omega \cap \Omega_\tau\} \leq \sup\{u(x) - u_\tau(x) \; : \; x \in \partial(\Omega \cap \Omega_\tau)\}$$

and so, in particular

$$u(x_1) - u(x_2) = u(x_1) - u_\tau(x_1) \leq \sup\{u(x) - u(x + \tau) \; : \; x \in \partial(\Omega \cap \Omega_\tau)\}.$$

However, if $x \in \partial(\Omega \cap \Omega_\tau)$ then either x or $x + \tau$ is in $\partial\Omega$ and so by the definition of k' the right hand side of the inequality above is bounded by $k'|\tau| = k'|x_1 - x_2|$. Thus $u(x_1) - u(x_2) \leq k'|x_1 - x_2|$. Reversing the roles of x_1 and x_2 gives the required result.

We are now ready to give the proof of Theorem 2. A crucial step in the application of the Bounded Slope Condition is to note that the linear functions π_+ and π_- satisfy the minimal surface equation and so, by Theorem 1, are minimisers with respect to their own boundary values.

Proof of Theorem 2:

Choose $k > K$ where K is the constant of the Bounded Slope Condition. Apply Lemma 2 to find a function u which minimises in $\mathcal{C}_k$. Fix a point x_0 in $\partial\Omega$ and note that the functions π_+ and π_- given in the BSC both minimise $\mathcal{A}(v)$ with respect to their own boundary values. Further

$$\pi_-(x) \leq \phi(x) = u(x) \leq \pi_+(x), \qquad \text{for } x \in \partial\Omega \tag{4.3}$$

By Lemma 4, inequality (4.3) holds throughout Ω and not just on the boundary. But $\pi_-(x_0) = \phi(x_0) = u(x_0) = \pi_+(x_0)$ and both linear functions have slope at most K, so

$$-K|x - x_0| \leq \pi_-(x) - \pi_-(x_0) \leq u(x) - u(x_0) \leq \pi_+(x) - \pi_+(x_0) \leq K|x - x_0|.$$

Since $x_0 \in \partial\Omega$ was arbitrary we have shown that k' in Lemma 6 is at most K. Lemma 3 shows that we have the required function.

5. More General Existence Results

The only place we used the Bounded Slope Condition in the last Theorem was to obtain a bound for $|u(x) - u(z)|/|x - z|$ when $x \in \partial\Omega$ and $z \in \Omega$. The linearity of π_- and π_+ was used to show that each satisfied the minimal surface equation and so we could use the Comparison Principle to say that u had to lie between the two linear functions. We will generalise the Comparison Principle to allow functions other than just solutions of the minimal surface equation and then we will introduce the concept of *barriers* which generalise the ideas behind π_- and π_+.

A function $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a **subsolution** for the minimal surface equation if

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\frac{\partial v}{\partial x_i}}{\sqrt{1 + |\nabla v|^2}} \right) \geq 0$$

or equivalently

$$\left(1 + |\nabla v|^2\right) \triangle\, v - \sum_{i,j=1}^{n} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \geq 0$$

Similarly, reversing the inequalities we can define **supersolutions**.

Note that by Theorem 1 solutions to Problem (P) are both sub- and supersolutions.

Lemma 1. (Weak Maximum Principle) *Suppose u minimises $\mathcal{A}(v)$ over $\mathcal{C}_k$. Suppose u_1 and u_2 are sub- and supersolutions respectively with Lipschitz constants at most k. Suppose also that $u_1(x) \leq u(x) \leq u_2(x)$ for $x \in \partial\Omega$. Then*

$$u_1(x) \leq u(x) \leq u_2(x) \qquad \text{for all } x \in \Omega.$$

Proof: Suppose $\eta \in C_0^1(\Omega)$ and $\eta \geq 0$. Consider $F(t) = \mathcal{A}(u_1 + t\eta)$. Then

$$\begin{aligned} F'(t) &= \int_\Omega \frac{(\nabla u_1 + t\nabla\eta) \cdot \nabla\eta}{\sqrt{1 + |\nabla u_1 + t\nabla\eta|^2}}\, dx \\ &= \int_\Omega \frac{(\nabla u_1) \cdot \nabla\eta}{\sqrt{1 + |\nabla u_1 + t\nabla\eta|^2}} + \frac{t|\nabla\eta|^2}{\sqrt{1 + |\nabla u_1 + t\nabla\eta|^2}}\, dx \end{aligned}$$

Using integration by parts on the first term and the definition of subsolutions shows that $F'(t) \geq 0$ for $t \geq 0$. Thus, from (2.4)

$$\mathcal{A}(u_1) \leq \mathcal{A}(u_1 + \eta) \qquad \text{for } \eta \in C_0^1(\Omega),\ \eta \geq 0 \tag{5.1}$$

Similarly $\mathcal{A}(u_2) \leq \mathcal{A}(u_2 - \eta)$.

We can now repeat the proof of Lemma 4 in Section 4 except we use (5.1) (over E rather than over Ω) in place of u being a minimiser over E.

We can use this Comparison Principle to define suitable replacements for π_- and π_+.

Given $x_0 \in \partial\Omega$ and $\mathcal{N}$ a neighbourhood of x_0 we say the function v^+ is an **upper barrier** at x_0 on $\mathcal{N}$ if

(i) v^+ is a supersolution on $\Omega \cap \mathcal{N}$,

(ii) $v^+(x_0) = \phi(x_0)$,

(iii) $v^+(x) \geq \phi(x)$ if $x \in \partial\Omega \cap \mathcal{N}$,

(iv) $v^+(x) \geq \sup_{\partial\Omega} \phi(x)$ if $x \in \Omega \cap \partial\mathcal{N}$.

Similarly v^- is a **lower barrier** if it is a subsolution on $\Omega \cap \mathcal{N}$, $v^-(x_0) = \phi(x_0)$, $v^-(x) \leq \phi(x)$ if $x \in \partial\Omega \cap \mathcal{N}$ and $v^-(x) \leq \inf_{\partial\Omega} \phi(x)$ if $x \in \Omega \cap \partial\mathcal{N}$.

Example: π_- and π_+ are lower and upper barriers with $\mathcal{N}$ taken as $\mathbb{R}^n$.

Theorem 2. *Suppose there are constants $\epsilon > 0$ and K such that for every $x_0 \in \partial\Omega$ there is a neighbourhood $\mathcal{N}$ containing $\{x : |x - x_0| < \epsilon\}$ and upper and lower barriers v^+ and v^- at x_0 on $\mathcal{N}$ with Lipschitz constants at most K. Then Problem (P) has a solution.*

Proof: Choose $k > \max\{K, 2\sup|\phi(x)|/\epsilon\}$ and let u be the minimiser in C_k. If $x_0 \in \partial\Omega$ then by the previous Lemma, $|u(x) - u(x_0)| < K|x - x_0|$ if $|x - x_0| < \epsilon$. On the other hand, if $|x - x_0| \geq \epsilon$ then $|u(x) - u(x_0)| \leq 2\sup|\phi(x)| \leq |x - x_0| 2\sup|\phi(x)|/\epsilon$.

Now Lemma 3 of Section 4 applies.

The idea of upper and lower barriers was used by Jenkins and Serrin [JS] to obtain optimal existence results for Problem (P). It is very clearly explained in a general setting in Chapter 13 of the book by Gilbarg and Trudinger [GT].

The barriers are usually constructed in terms of the function $d(x)$ which is the distance of x from $\partial\Omega$. This function obviously has properties related to the smoothness and geometry of $\partial\Omega$. In particular if $\partial\Omega$ is C^2 then so is d, at least in a neighbourhood of $\partial\Omega$. Even more importantly the second derivatives of d are related to the curvatures of $\partial\Omega$ and in particular, on $\partial\Omega$, $-\triangle d$ is just n times the mean curvature of $\partial\Omega$. For precise statements and proofs of these results see appendix B of [G] or appendix A of [GT].

An upper barrier at x_0 for $\mathcal{N} = \{x : d(x) < a\}$ can be constructed with the form

$$v^+(x) = \phi(x) + \psi(d(x))$$

with $\psi(d) = c\log(1 + \beta d)$ for suitable choice of constants a, c and β. The choices can be made and the barriers constructed provided $\partial\Omega$ has nonnegative mean curvature.

Theorem 3. *Let Ω be a bounded open set in $\mathbb{R}^n$ with a C^2 boundary having nonnegative mean curvature. Let ϕ be a C^2 function defined on $\partial\Omega$. Then Problem (P) has a solution.*

Remarks:

(i) In $\mathbb{R}^2$ the condition of nonnegative mean curvature is equivalent to the convexity of Ω but this is not the case in higher dimensions.

(ii) Jenkins and Serrin showed that the result is optimal in the sense that if $\partial\Omega$ has negative mean curvature at any point then there is smooth data ϕ with $\sup|\phi|$ arbitrarily small such that Problem (P) has no solution. See [GT].

(iii) Despite (ii), given any set Ω there is always some data for which the problem has a solution (e.g. $\phi \equiv 0$). Results showing that smallness in the Lipschitz norm will guarantee solutions together with counterexamples to show the results are optimal can be found in [W].

6. Interior Regularity of solutions

We have so far proved the existence of solutions which are Lipschitz continuous. In this section we show that such solutions are actually C^∞ inside Ω. (Indeed they are in fact analytic.) After some initial regularity results the higher order regularity follows from the theory of linear differential equations.

Lemma 1. *If u is a Lipschitz continuous solution of Problem (P) then u has Hölder continuous first derivatives in Ω.*

Proof: We consider the difference quotients

$$u^h(x) = [u(x + h\alpha) - u(x)]/h$$

for a fixed unit vector α and show that they satisfy a linear differential equation. We then use linear theory to obtain bounds on the Hölder norms which are independent of h. Taking the limit as $h \to 0$ then gives the result.

Suppose Ω' has closure contained in Ω and $v \in C_0^1(\Omega')$. If h is sufficiently small then $v_h(x) = v(x - h\alpha) \in C_0^1(\Omega)$ and so from equation (2.1)

$$\int_\Omega a(\nabla u) \cdot \nabla v_h \, dx = 0 \tag{6.1}$$

with $a(p)$ as in (2.3). By the change of variables $x \to x + h\alpha$

$$\int_\Omega a(\nabla u(x + h\alpha)) \cdot \nabla v \, dx = 0 \tag{6.2}$$

Subtract (6.2) from the equation like (6.1) that has v rather than v_h and we have

$$\int_\Omega [a(\nabla u(x+h\alpha)) - a(\nabla u(x))] \cdot \nabla v \, dx = 0$$

and hence

$$\int_\Omega \int_0^1 \frac{\partial}{\partial p_j} a_i(\nabla u(x) + t(\nabla u(x+h\alpha) - \nabla u(x))) \cdot \frac{\partial}{\partial x_j}[u(x+h\alpha) - u(x)] \frac{\partial v}{\partial x_i} \, dx = 0$$

where we have summed i and j from 1 to n. Now by dividing by h we obtain an equation

$$\text{(6.3)} \qquad \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u^h}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = 0 \qquad \text{for all } v \in C_0^1(\Omega')$$

where a_{ij} are determined from the derivatives of a_i as in the equation before. Now we know u is Lipschitz continuous and so $|\nabla u|$ is bounded and consequently, using previous calculations about the ellipticity of the minimal surface equation,

$$\text{(6.4)} \qquad \lambda |\zeta|^2 \le \sum_{i,j=1}^n a_{ij}(x) \zeta_i \zeta_j \le \Lambda |\zeta|^2 \qquad \text{for all } \zeta \in I\!R^n, x \in \Omega$$

where $\lambda > 0$ and Λ are constants depending on the bounds for $|\nabla u|$. By the De Giorgi-Nash theory (see[GT] Theorem 8.22) we then have the required bounds.

Theorem 1. *Solution for Problem (P) are in $C^\infty(\Omega)$.*

Proof: From equation (2.1) we have

$$\int_\Omega \sum_{i=1}^n a_i(\nabla u) \frac{\partial v}{\partial x_i} \, dx = 0 \qquad \text{for } v \in C_0^1(\Omega).$$

If we take $v = \frac{\partial \eta}{\partial x_s}$ we can integrate by parts to obtain

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial w}{\partial x_j} \frac{\partial \eta}{\partial x_i} \, dx = 0$$

where $a_{ij}(x) = \frac{\partial a_i}{\partial p_j}(\nabla u(x))$ and $w = \frac{\partial u}{\partial x_s}$. Actually for this to be valid we need to know that $u \in W^{2,2}(\Omega)$. This can be proved by using a technique very similar to the last Lemma. We use (6.3) and (6.4) and then apply Theorem 8.8 of [GT] (also see Theorem 5.5 in the notes by John Urbas in these volumes) to get uniform estimates on the difference quotients. Since ∇u is Hölder continuous, by the previous Lemma , so are $a_{ij}(x)$. Linear theory now shows that w has Hölder continuous first derivatives and so u has Hölder continuous second derivatives. By further differentiating the equation we can step up to any derivative we like.

As mentioned before solutions are actually analytic and a proof of this can be found in [MC].

7. Other Existence Methods

We have presented just one of the many methods available to prove the existence of solutions. Other possible techniques are viscosity solutions, the Perron process and the methods of continuity, fixed point theorems and topological degree which are used extensively in the theory of nonlinear partial differential equations (see [GT]). A further rather interesting method is the direct method in the Calculus of Variations. This method can be easily illustrated for the Dirichlet Integral

$$\mathcal{D}(u) = \int_\Omega |\nabla u|^2 \, dx.$$

We can attempt to minimise this integral amongst functions in $W^{1,2}(\Omega)$ which satisfy some prescribed boundary values. If we look at a minimising sequence, then using the Poincaré inequality it is a simple matter to show that the sequence is bounded in $W^{1,2}(\Omega)$. It must then have a weakly convergent subsequence and using lower semicontinuity for the norm under weak convergence we can show the limit function is the required solution. If we try to apply the same technique to $\mathcal{A}(u)$ then the natural space to use would be $W^{1,1}(\Omega)$. Unfortunately this is not a reflexive space and so we cannot obtain a weakly convergent subsequence. We can obtain a subsequence which converges in $L^1(\Omega)$ but the limit may not be in $W^{1,1}(\Omega)$. To try to overcome this it is necessary to use a different space of functions. Using equation (4.1) as a starting point the area of the graph of functions of Bounded Variation is defined. Then it is easy to prove that minimising sequences converge to a function of Bounded Variation (or at least a subsequence does). The limit is the solution we have found before, when it exists. However the limit always exists even when we have shown that Problem (P) has no solution. This leads to the notion of a generalised solution which doesn't necessarily satisfy the boundary values. There are many very interesting results about such solutions (see [G]).

8. Non-graphical Minimal Surfaces

The problem we have considered is to find the least area amongst all surfaces which can be written as graphs and which have a prescribed boundary. There may possibly be surfaces which are not graphs and which have a smaller area. If Ω is convex then it is known that this does not happen and our solution provides the least area amongst all surfaces. However if Ω is not convex (nonnegative mean curvature is not enough) then there are examples where our solution does not provide the least area amongst all surfaces. For such examples and further information see [N p367] and [HLL].

Chapter 2: Interior Gradient Estimates

1. Introduction

We have seen previously that boundary gradient estimates (or Lipschitz constant bounds) transfer to give the same estimate in the interior. It is possible to have solutions of the minimal surface equation whose gradient becomes unbounded on approach to $\partial\Omega$. In this Chapter we show that even in this case we can still bound the gradient of the solution u provided we keep away from $\partial\Omega$. Specifically we show that if u satisfies the minimal surface equation in Ω and $x_0 \in \Omega$ then

$$(1.1) \qquad |\nabla u(x_0)| \leq c_1 \exp\left\{c_2 \sup_{\Omega} \left(u(x) - u(x_0)\right)/d\right\}$$

where c_1 and c_2 are constants depending only on n and d is the distance of x_0 from $\partial\Omega$.

This result is one of the cornerstones of the theory about solutions of the minimal surface equation. It has numerous applications particularly to the convergence of sequences of solutions. We shall present just one application, namely the one to Problem (P) when we assume that ϕ is only continuous and so cannot obtain uniform bounds on the Lipschitz constants. The result was originally proved by Bombieri, De Giorgi and Miranda [BDM] and has since been extended and improved in many directions. There is a good account in [G] and a more comprehensive and general one in [GT].

A different (and simpler) approach to proving the result can be found in [K].

2. Sketch of Proof of Estimate

The presentation here is only a sketch of the main ideas but it closely follows [G] and further details and explanation can be found there.

The main idea of the proof is to look at a function of the gradient, namely

$$w = \log \sqrt{1 + |\nabla u|^2}$$

and show that it satisfies a differential inequality. However it is important to look at the differential inequality as one on the surface $(x, u(x))$ rather than on Ω. This means that we should take derivatives tangential to the surface rather than in the usual coordinate directions. Once this is done it turns out that w is subharmonic and so we can attempt to mimic the theory of subharmonic functions to find a bound for w. In order to carry out the calculations on the surface it is convenient to introduce some notation.

We assume the point of interest is the origin and that $u(0) = 0$ .Let B_R be a ball of radius R in $\mathbb{R}^{n+1}$ and $\mathcal{B}_R$ be a ball of radius R in $\mathbb{R}^n$. S will the surface determined by the graph of u and $S_R = S \cap B_R$. ν is the upward unit normal to the surface S and so

$$\nu_{n+1} = 1/\sqrt{1+|\nabla u|^2}, \qquad \nu_i = -\nu_{n+1}\frac{\partial u}{\partial x_i}, \quad i = 1, \dots, n$$

The tangential differential operators are

$$\delta_i = \frac{\partial}{\partial x_i} - \nu_i \sum_{j=1}^{n} \nu_j \frac{\partial}{\partial x_j},$$

the Laplacian on the surface is

$$\mathcal{D} = \sum_{i=1}^{n} \delta_i \delta_i$$

and some useful identities, which are readily checked are

$$\delta_i \nu_j = \delta_j \nu_i, \quad \sum_{i=1}^{n} \nu_i \delta_i = 0, \quad \sum_{i=1}^{n} \nu_i \delta_j \nu_i = 0 \tag{2.1}$$

$$\mathcal{D}\nu_i = -c^2 \nu_i \qquad \text{where } c^2 = \sum_{i,j=1}^{n} (\delta_i \nu_j)^2 \tag{2.2}$$

Furthermore, integration by parts is valid

$$\int_S u \delta_i v \, dS = -\int_S v \delta_i u \, dS \tag{2.3}$$

provided uv has compact support in S. From (2.2) and the definition of w

$$\mathcal{D}w = |\delta w|^2 + c^2 \geq 0 \tag{2.4}$$

and so w is subharmonic on the surface S.

Exercise: Check the above identities.

The next result corresponds exactly to the mean value property for subharmonic functions on $\mathbb{R}^n$

Lemma 1. (Theorem 13.2 of [G]) *Suppose u solves the minimal surface equation and $u(0) = 0$. Then*

(2.5).
$$w(0) \leq \frac{1}{|\mathcal{B}_R|} \int_{S_R} w \, dS$$

In the proof we take inequality (2.4) and multiply by a suitable radially symmetric test function whose Laplacian approximates the Dirac delta function plus other cleverly chosen terms. Then apply integration by parts twice and take limits.

We are trying to find a bound for $w(0)$ in terms of R which corresponds to d the distance from $\partial\Omega$ and $\sup|u|$. According to the Lemma we only have to bound the right hand side of (2.5).

Lemma 2. (Theorem 13.4 of [G]) *Suppose u is a solution to the minimal surface equation in the ball $\mathcal{B}_{3R}$ and $u(0) = 0$. Then*

$$\frac{1}{|\mathcal{B}_R|}\int_{S_R} w\, dS \le c_1 \left\{1 + \sup_{\mathcal{B}_{3R}} u/R\right\} \tag{2.6}$$

The proof starts with (2.3) when $i = n+1$, $u = 1$ and v is a suitably chosen function with compact support involving the solution u. Then follow several estimates which make use of the differential inequality for w.

The required estimate (1.1) follows easily from the two Lemmas and the definition of w.

3. Application to Continuous Boundary Data

The existence theorem of the first Chapter can only work if we can expect the required solution to be Lipschitz continuous on $\overline{\Omega}$. If the boundary values ϕ are merely continuous and not Lipschitz continuous then the methods cannot possibly work although it still makes sense to look for solutions in $C^1(\Omega) \cap C^0(\overline{\Omega})$. We use the interior gradient estimate to show that solutions do exist if we assume the usual condition on the mean curvature of $\partial\Omega$.

Theorem 1. *Let Ω be a bounded open set in $\mathbb{R}^n$ with a C^2 boundary having nonnegative mean curvature. Let ϕ be a continuous function defined on $\partial\Omega$. Then there is a unique solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the boundary value problem:*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\frac{\partial u}{\partial x_i}}{\sqrt{1+|\nabla u|^2}}\right) = 0 \qquad \text{in } \Omega$$

$$u = \phi \qquad \text{on } \partial\Omega.$$

Moreover u is also the unique solution to Problem (P).

Proof: Let $\{\phi_j\}$ be a sequence of C^2 functions which converge uniformly to ϕ on $\partial\Omega$. For each j we can find a solution u_j to Problem (P) with boundary data ϕ_j by the work in Chapter 1. We also know that each u_j is C^∞ and satisfies the minimal surface equation. If we apply the Maximum Principle (Lemma 5 of Section 4) to the sequence we see

$$\sup_{\Omega}|u_j - u_i| \le \sup_{\partial\Omega}|\phi_j - \phi_i|$$

and so the sequence $\{u_j\}$ must converge uniformly to some function $u \in C^0(\overline{\Omega})$. Now suppose Ω' is any set whose closure is contained in Ω. Because $\sup_\Omega |u_j|$ is bounded we can use the interior gradient estimate for each j to conclude there is a constant L not depending on j such that

$$\sup_{\Omega'} |\nabla u_j| \leq L. \tag{3.1}$$

On Ω' each u_j satisfies a linear equation in divergence form, namely

$$\sum_{i=1}^{n} \frac{\partial}{x_i}\left(a(x)\frac{\partial u}{\partial x_i}\right) = 0$$

where $a(x) = 1/\sqrt{1+|\nabla u_j|^2}$ is uniformly bounded above and below away from zero. By the theory of linear equations we then have bounds like (3.1) on the derivatives of any order (See the talks of John Urbas in this volume). Using the bounds on the third derivatives we can pick a subsequence so that u_j converges together with their first and second derivatives to a function which must be the u found above. Thus $u \in C^2(\Omega')$ and since each u_j satisfies the minimal surface equation in Ω' so does u.

To prove the last result about Problem (P) it is only necessary to show that $\mathcal{A}(u)$ is finite since then our previous calculations in Section 2 of Chapter 1 show that u provides the minimum. This is done in [G].

4. Application to the Bernstein Problem

A famous result of Bernstein states that, in 2 dimensions, the only solutions of the minimal surface equation defined everywhere are linear i.e their graphs are planes. This was later extended to all dimensions smaller than 8 but a counterexample is available in dimension 8. However if we add some additional conditions it is possible to obtain results even in these higher dimensions. We present one such result below. For additional details and references consult Chapter 17 of [G].

Theorem 1. *Suppose u is a solution of the minimal surface equation defined everywhere in $\mathbb{R}^n$. Suppose there is a constant K such that for all x*

$$u(x) \leq K\,(1+|x|)\,. \tag{4.1}$$

Then u is a linear function.

Proof: Suppose $\xi \in \mathbb{R}^n$ and $R > |\xi|$. From (1.1), applied in the ball $B_R(\xi)$, centre

ξ and radius R,

$$
\begin{aligned}
|\nabla u\,(\xi)| &\le c_1 \exp\left\{ c_2 \sup_{B_R(\xi)} \frac{u(x)-u(\xi)}{R} \right\} \\
&\le c_1 \exp\left\{ c_2 \sup_{B_{2R}(0)} \frac{u(x)-u(\xi)}{R} \right\} \\
&\le c_1 \exp\left\{ c_2 \frac{K\,(1+2R)-u(\xi)}{R} \right\}
\end{aligned}
$$

Letting $R \to \infty$ gives

$$
|\nabla u\,(\xi)| \le c_1 \exp\left(2c_2 K\right) = c_3 \tag{4.2}
$$

Now let $w = \partial u/\partial x_s$ then by differentiating the minimal surface equation with respect to x_s we have

$$
\frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial w}{\partial x_j} \right) = 0
$$

where a_{ij} is defined in terms of u and ∇u. Further because of (4.2) this equation is uniformly elliptic. We can then apply the De Giorgi–Nash estimate ([GT] Theorem 8.22) to show that if $x \in B_R(0)$ then

$$
\begin{aligned}
|w(x) - w(\xi)| &\le C \left| \frac{x-\xi}{R} \right|^{\alpha} \sup_{B_{2R}(0)} w \\
&\le c_4 \left| \frac{x-\xi}{R} \right|^{\alpha}
\end{aligned}
$$

Keeping x fixed and letting $r \to \infty$ shows $w(x) = w(\xi)$ and so $w = \partial u/\partial x_s$ is constant. Doing this for each $s = 1, 2, \ldots, n$ shows that ∇u is constant and so u must be a linear function.

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