LECTURES ON SECOND ORDER ELLIPTIC AND PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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1 INTRODUCTION

The aim of these lectures is to give an introduction to the theory of linear second order elliptic and parabolic partial differential equations. A *partial differential equation of order k* is an equation involving an unknown function u of two or more variables and its derivatives up to order k:

$$F(x, u, Du, \cdots, D^k u) = 0\,. \tag{1.1}$$

Here x denotes the independent variables which typically vary over some domain in a Euclidean space $\mathbb{R}^n$ with $n \geq 2$. Equation (1.1) is said to be *linear* if the left hand side of (1.1) is an affine function of u and its derivatives. Thus a general linear second order partial differential equation can be written in the form

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u + c(x) u = f(x)\,. \tag{1.2}$$

Here are some important examples of second order linear equations.

Laplace's equation

$$\Delta u = \sum_{i=1}^{n} D_{ii} u = 0\,. \tag{1.3}$$

Poisson's equation

$$\Delta u = f(x)\,. \tag{1.4}$$

Heat equation

$$\frac{\partial u}{\partial t} = \Delta u\,. \tag{1.5}$$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u\,. \tag{1.6}$$

In the last two examples the independent variables are the spatial variable x and the time t; the Laplacian is taken with respect to the spatial variables only. We can also consider inhomogeneous heat and wave equations which are obtained by adding a function of x and t to the right hand sides of (1.5) and (1.6). We may also wish to consider these equations with lower order terms, or with Δ replaced by a more general operator of the form (1.2).

These are the most fundamental second order linear partial differential equations. They occur in mathematical physics and in various branches of mathematics. Equations (1.3) and (1.4) typically describe an equilibrium situation, while equations (1.5) and (1.6) describe diffusion and oscillatory phenomena respectively, as their names suggest. Notice also that any solution of (1.3) is automatically a time independent solution of (1.5) and (1.6). This suggests that that the theory for the heat and wave equations should be a generalization of the theory for Laplace's equation.

It turns out that the behaviour of equation (1.2) is determined primarily by the highest order or *principal part* of the equation. We make the following definitions. Equation (1.2) is said to be *elliptic* at a point x if the matrix $[a^{ij}(x)]$ is positive definite, i.e.,

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j > 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n\,,\ \xi \neq 0\,. \tag{1.7}$$

Equation (1.2) is said to be *elliptic* in a region $\Omega \subset \mathbb{R}^n$ if it is elliptic at each point of Ω.

The terminology comes from the two variable theory, in which a linear second order equation such as (1.2) is classified as being *elliptic*, *hyperbolic* or *parabolic* at a point x according to whether the matrix $[a^{ij}(x)]$ has two nonzero eigenvalues of the same sign, of opposite signs, or one zero and one nonzero eigenvalue. Of course, in higher dimensions there are more possibilities.

Equations (1.3) and (1.4) are clearly elliptic everywhere, while equations (1.5) and (1.6) are not elliptic. However, they are elliptic in the spatial directions (the meaning of this should be clear). In this sense equation (1.5) is closer to being elliptic than (1.6), since it is only first order in the time direction, and can in fact be regarded as a *degenerate elliptic* equation, while (1.6) is second order with respect to the time variable, and is fundamentally different. An equation of the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a^{ij}(x,t)D_{ij}u + \sum_{i=1}^{n} b^i(x,t)D_iu + c(x,t)u + f(x,t) \tag{1.8}$$

is said to be *parabolic* at (x,t) if

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j > 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n\,,\ \xi \neq 0\,. \tag{1.9}$$

It is said to be *parabolic* on a region $Q \subset \mathbb{R}^n \times \mathbb{R}$ if it is parabolic at each $(x,t) \in Q$. Thus the heat equation (1.5) is an example of a parabolic equation. The wave equation is an example of another class of equations, *hyperbolic equations*, which can be defined analogously to parabolic equations.

To conclude this introduction we give a brief outline of the topics we will cover. In Sections 2 and 3 we will present the classical theory for the Laplace and Poisson equations. The basic problem we will discuss for equations (1.3) and (1.4) is the *Dirichlet problem*: given a bounded (or possibly unbounded) domain Ω in $\mathbb{R}^n$, that is, a connected open set in $\mathbb{R}^n$, find a function u which satisfies the differential equation in Ω and is equal to a given function ϕ on $\partial\Omega$. The kinds of questions we might ask are the following.

(i) Does there exist a solution to the Dirichlet problem?

(ii) If so, is it unique?

(iii) How does the solution depend on the given functions f and ϕ? If these functions and the boundary $\partial\Omega$ have a certain degree of smoothness, does the solution u inherit some smoothness? If so, how much?

(iv) Are there explicit formulae for the solution in terms of the data?

We will answer these questions using only very simple tools—essentially only calculus.

In the next two sections we will discuss general linear second order elliptic equations from two points of view. In Section 4 we will describe the more classical Schauder theory, which is essentially a generalization of the the results of Sections 2 and 3. In Section 5 we will describe a more modern functional analytic approach. In the final section we will describe how the elliptic theory can be extended to the more complicated parabolic setting.

Finally, some remarks about references. The standard reference for the theory of second order elliptic equations is [GT]; in particular, Chapters 2 to 8 contain all the material we will describe in the first four lectures (and much more as well). Expositions of various parts of the elliptic theory can also be found in [E] (Sections 2.2 and 6), [F] (Chapters 2 and 6), [J] (Chapters 4 and 6) and [S]. Some references for the theory of parabolic equations are [E] (Sections 2.3 and 7.1), [F] (Chapter 4), [Fr], [J] (Chapter 7), [LSU] and [S] (Lecture 10).

2 LAPLACE'S EQUATION

In this section we will develop some of the theory for Laplace's equation

$$\Delta u = 0\,, \tag{2.1}$$

and its inhomogeneous counterpart Poisson's equation

$$\Delta u = f\,. \tag{2.2}$$

Results for these special cases are fundamental for the development of the theory of more general elliptic equations.

Let Ω be a domain in $\mathbb{R}^n$. We denote the set of real valued continuous functions on Ω by $C^0(\Omega)$. The set of functions in $C^0(\Omega)$ having a continuous extension to the closure of Ω, $\bar{\Omega}$, is denoted by $C^0(\bar{\Omega})$. For any positive integer k we denote by $C^k(\Omega)$ the set of functions which are k times continuously differentiable on Ω. We denote by $C^k(\bar{\Omega})$ the set of functions in $C^k(\Omega)$ all of whose derivatives of order $\leq k$ have continuous extensions to $\bar{\Omega}$. The spaces $C^\infty(\Omega)$ and $C^\infty(\bar{\Omega})$ are defined in the obvious way.

The spaces $C^k(\bar{\Omega})$, $k < \infty$, are Banach spaces with norm given by

$$\|u\|_{C^k(\bar{\Omega})} = |u|_{k;\Omega} = \sum_{j=0}^{k} \sup_{|\beta|=j} \sup_{\Omega} |D^\beta u|\,. \tag{2.3}$$

Here $\beta = (\beta_1, \cdots, \beta_n)$ denotes a multi-index with each β_i a nonnegative integer, $|\beta| = \sum_{i=1}^n \beta_i$, and $D^\beta u = \frac{\partial^{|\beta|} u}{\partial^{\beta_1} x_1 \cdots \partial^{\beta_n} x_n}$.

A function $u \in C^2(\Omega)$ is said to be *harmonic* (*subharmonic*, *superharmonic*) in Ω if at each point of Ω we have

$$\Delta u = 0 \quad (\geq 0\,,\ \leq 0)\,. \tag{2.4}$$

We now want to derive some properties of harmonic functions. First we recall the divergence theorem in $\mathbb{R}^n$. If Ω is a bounded domain in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$ (weaker conditions on $\partial\Omega$ suffice) and outer unit normal ν, then for any vectorfield $w \in [C^1(\bar{\Omega})]^n$ we have

$$\int_\Omega \operatorname{div} w\, dx = \int_{\partial\Omega} w \cdot \nu\, ds \tag{2.5}$$

where ds denotes the $(n-1)$-dimensional area element on $\partial\Omega$. In particular, if $u \in C^2(\bar{\Omega})$, we may take $w = Du$ in (2.5) to obtain

$$\int_\Omega \Delta u\, dx = \int_{\partial\Omega} Du \cdot \nu\, ds = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, ds\,. \tag{2.6}$$

If in addition $v \in C^2(\bar{\Omega})$, then we may take $w = vDu$ in (2.5) to obtain *Green's first identity*

$$\int_\Omega v\Delta u\,dx + \int_\Omega Du \cdot Dv\,dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\,ds\,. \tag{2.7}$$

Interchanging u and v in (2.7) and subtracting, we obtain *Green's second identity*

$$\int_\Omega (v\Delta u - u\Delta v)\,dx = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu}\right)ds\,. \tag{2.8}$$

We can now prove the following *mean value inequalities.*

Theorem 2.1 *Let $u \in C^2(\Omega)$ satisfy $\Delta u = 0$ $(\geq 0\,, \leq 0)$ in Ω. Then for any ball $B = B_R(y) \subset\subset \Omega$ we have*

$$u(y) = (\leq, \geq)\frac{1}{n\omega_n R^{n-1}}\int_{\partial B} u\,ds\,, \tag{2.9}$$

$$u(y) = (\leq, \geq)\frac{1}{\omega_n R^n}\int_B u\,dx\,. \tag{2.10}$$

Remark Here ω_n denotes the measure of the unit ball in $\mathbb{R}^n$ and $B \subset\subset \Omega$ means that $\bar{B} \subset \Omega$.

Proof of Theorem 2.1 Let $\rho \in (0, R)$ and apply the identity (2.6) to the ball $B_\rho = B_\rho(y)$ to obtain

$$\int_{\partial B_\rho}\frac{\partial u}{\partial\nu}\,ds = \int_{B_\rho}\Delta u\,dx = (\geq, \leq)\,0\,.$$

Introducing radial and angular coordinates $r = |x-y|$ and $\omega = (x-y)/r$, and writing $u(x) = u(y + r\omega)$, we see that

$$\begin{aligned}
\int_{\partial B_\rho}\frac{\partial u}{\partial\nu}\,ds &= \int_{\partial B_\rho}\frac{\partial u}{\partial r}(y+\rho\omega)\,ds \\
&= \rho^{n-1}\int_{|\omega|=1}\frac{\partial u}{\partial r}(y+\rho\omega)\,d\omega \\
&= \rho^{n-1}\frac{\partial}{\partial\rho}\int_{|\omega|=1}u(y+\rho\omega)\,d\omega \\
&= \rho^{n-1}\frac{\partial}{\partial\rho}\left(\rho^{1-n}\int_{\partial B_\rho}u\,ds\right).
\end{aligned}$$

It follows that for any $\rho \in (0, R)$,

$$\rho^{1-n}\int_{\partial B_\rho}u\,ds = (\leq, \geq)R^{1-n}\int_{\partial B_R}u\,ds\,.$$

The relation (2.9) follows from this since

$$\lim_{\rho\to 0} \rho^{1-n} \int_{\partial B_\rho} u\, ds = n\omega_n u(y)\,.$$

The solid mean value inequalities (2.10) follow from (2.9) by integrating with respect to ρ from 0 to R.

The mean value inequalities have a number of useful consequences. The first of these is the *strong maximum principle* for subharmonic functions and the *strong minimum principle* for superharmonic functions.

Theorem 2.2 *Let $u \in C^2(\Omega)$ satisfy $\Delta u \geq 0$ (≤ 0) in Ω, and suppose there is a point $y \in \Omega$ such that $u(y) = \sup_\Omega u$ ($\inf_\Omega u$). Then u is constant. Consequently, a harmonic function cannot assume an interior maximum or minimum unless it is constant.*

Proof Suppose u is subharmonic in Ω, and let $M = \sup_\Omega u$ and $\Omega_M = \{x \in \Omega : u(x) = M\}$. By assumption Ω_M is not empty, and since u is continuous, Ω_M is closed relative to Ω. Now let z be any point of Ω_M and apply the mean value inequality (2.10) to the subharmonic function $u - M$ in a ball $B = B_R(z) \subset\subset \Omega$. We obtain

$$0 = u(z) - M \leq \frac{1}{\omega_n R^n} \int_B (u - M)\, dx \leq 0\,,$$

which implies that $u = M$ in B. Thus Ω_M is also open relative to Ω, and therefore $\Omega_M = \Omega$. The result for superharmonic functions follows by replacing u by $-u$.

From the strong maximum and minimum principles we immediately obtain the following *weak maximum* and *minimum principles.*

Theorem 2.3 *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $\Delta u \geq 0$ (≤ 0) in a bounded domain Ω. Then*

$$\sup_\Omega u = \sup_{\partial\Omega} u \quad (\inf_\Omega u = \inf_{\partial\Omega} u)\,. \tag{2.11}$$

Consequently, if u is harmonic, then

$$\inf_{\partial\Omega} u \leq u(x) \leq \sup_{\partial\Omega} u \quad \textit{for} \quad x \in \Omega\,. \tag{2.12}$$

A consequence of this is the following uniqueness result for the Dirichlet problem for Poisson's equation on bounded domains.

Theorem 2.4 *Let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $\Delta u = \Delta v$ in Ω, $u = v$ on $\partial\Omega$, where Ω is a bounded domain. Then $u = v$ in Ω.*

Proof Let $w = u - v$. Then $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$, so by the previous theorem $w = 0$ in Ω.

Remarks (i) The conclusions of Theorems 2.3 and 2.4 are generally false if Ω is unbounded. It is easy to construct counterexamples on a halfspace.

(ii) By Theorem 2.3 we also see that if u, v, w are harmonic, subharmonic and superharmonic functions in a bounded domain Ω which agree on $\partial\Omega$, then

$$v \le u \le w \quad \text{in} \quad \Omega .$$

This explains the terms "subharmonic" and "superharmonic".

We will see later that harmonic functions are smooth, in fact, analytic. Assuming this for the moment, we can obtain estimates for their derivatives from the mean value equality. If u is harmonic on Ω, then so is each component of the gradient Du, so for any ball $B = B_R(y) \subset\subset \Omega$ we have

$$Du(y) = \frac{1}{\omega_n R^n} \int_B Du\, dx = \frac{1}{\omega_n R^n} \int_{\partial B} u\nu\, ds ,$$

and hence, letting $R \to \operatorname{dist}(y, \partial\Omega)$,

$$|Du(y)| \le \frac{n}{\operatorname{dist}(y, \partial\Omega)} \sup_\Omega |u| .$$

By successive application of this result we obtain the following.

Theorem 2.5 *Let u be harmonic in Ω. Then for any $\Omega' \subset\subset \Omega$ and any multi-index α we have*

$$\text{(2.13)} \qquad \sup_{\Omega'} |D^\alpha u| \le \left(\frac{n|\alpha|}{\operatorname{dist}(\Omega', \partial\Omega)} \right)^{|\alpha|} \sup_\Omega |u| .$$

From Theorem 2.5 and the Arzela-Ascoli theorem we see that harmonic functions have a strong compactness property.

Theorem 2.6 *Any bounded sequence of harmonic functions on a domain Ω contains a subsequence which converges uniformly on compact subsets of Ω to a harmonic function.*

Another important consequence of the mean value property of harmonic functions is the *Harnack inequality.* It tells us that the values of a nonnegative harmonic function u are comparable on any compact subset of the domain on which u is defined. The strong maximum principle is a special case of this.

Theorem 2.7 *Let u be a nonnegative harmonic function in a domain Ω. Then for any bounded subdomain $\Omega' \subset\subset \Omega$ there is a constant C depending only on n, Ω and Ω' such that*

$$\text{(2.14)} \qquad \sup_{\Omega'} u \le C \inf_{\Omega'} u .$$

Proof Let $y \in \Omega$ and choose $R > 0$ so that $B_{4R}(y) \subset \Omega$. Then for any two points $x_1, x_2 \in B_R(y)$ we have, by the mean value inequality,

$$\begin{aligned} u(x_1) &= \frac{1}{\omega_n R^n} \int_{B_R(x_1)} u \le \frac{1}{\omega_n R^n} \int_{B_{2R}(y)} u\,, \\ u(x_2) &= \frac{1}{\omega_n (3R)^n} \int_{B_{3R}(x_2)} u \ge \frac{1}{\omega_n (3R)^n} \int_{B_{2R}(y)} u\,. \end{aligned}$$

Consequently

$$\sup_{B_R(y)} u \le 3^n \inf_{B_R(y)} u\,. \tag{2.15}$$

Now let $\Omega' \subset\subset \Omega$ and choose $x_1, x_2 \in \overline{\Omega'}$ so that $u(x_1) = \sup_{\Omega'} u$, $u(x_2) = \inf_{\Omega'} u$. We can join x_1 and x_2 by an arc Γ such that $\operatorname{dist}(\Gamma, \partial\Omega) > 4R$ for some positive R. By the Heine-Borel Theorem Γ can be covered by a finite number of balls $\{B_j\}_{j=1}^N$ (with N depending only on n, Ω and Ω') of radius R, such that $B_1 = B_R(x_1)$, $B_N = B_R(x_2)$, and $B_j \cap B_{j+1} \ne \emptyset$ for $j = 1, \cdots, N-1$. Using the estimate (2.15) in each ball B_j, we obtain

$$\begin{aligned} u(x_1) \le \sup_{B_1} u &\le 3^n \inf_{B_1} u \\ &\le 3^n \inf_{B_1 \cap B_2} u \\ &\le 3^n \sup_{B_1 \cap B_2} u \\ &\le 3^n \sup_{B_2} u \\ &\le 3^{2n} \inf_{B_2} u\,. \end{aligned}$$

Continuing in the obvious way we finally obtain

$$u(x_1) \le 3^{nN} u(x_2)\,.$$

Hence the estimate (2.14) holds with $C = 3^{nN}$.

We now turn to the existence of harmonic functions. A simple computation shows that the only spherically symmetric harmonic function, up to additive and multiplicative constants, is given by

$$\Gamma(x) = \Gamma(|x|) = \begin{cases} \dfrac{1}{n(2-n)\omega_n} |x|^{2-n} & \text{if } n > 2 \\ \dfrac{1}{2\pi} \log |x| & \text{if } n = 2. \end{cases} \tag{2.16}$$

Γ is called the *fundamental solution* of Laplace's equation.

Now let $u \in C^2(\Omega)$ and $y \in \Omega$. We want to use Green's second identity (2.8) with $v(x) = \Gamma(x - y)$. Since $\Gamma(x - y)$ has a singularity at $x = y$ we cannot use (2.8) directly. Instead we replace Ω by $\Omega - B_\rho$ where $B_\rho = B_\rho(y)$ for sufficiently small ρ. From (2.8) we then obtain, since Γ is harmonic in $\Omega - B_\rho$,

$$\int_{\Omega - B_\rho} \Gamma \Delta u \, dx = \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) ds + \int_{\partial B_\rho} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) ds . \tag{2.17}$$

We have

$$\begin{aligned} \left| \int_{\partial B_\rho} \Gamma \frac{\partial u}{\partial \nu} \, ds \right| &= \left| \Gamma(\rho) \int_{\partial B_\rho} \frac{\partial u}{\partial \nu} \, ds \right| \\ &\leq n\omega_n \rho^{n-1} \Gamma(\rho) \sup_{B_\rho} |Du| \to 0 \text{ as } \rho \to 0 \end{aligned}$$

and

$$\begin{aligned} \int_{\partial B_\rho} u \frac{\partial \Gamma}{\partial \nu} \, ds &= -\Gamma'(\rho) \int_{\partial B_\rho} u \, ds \\ &= \frac{-1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho} u \, ds \to -u(y) \text{ as } \rho \to 0 . \end{aligned}$$

Consequently, letting $\rho \to 0$ in (2.17) we obtain *Green's representation formula*

$$\begin{aligned} u(y) &= \int_{\partial\Omega} \left(u \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y) \frac{\partial u}{\partial \nu} \right) ds \\ &\quad + \int_\Omega \Gamma(x-y) \Delta u \, dx , \quad y \in \Omega . \end{aligned} \tag{2.18}$$

For harmonic u we obtain the representation

$$u(y) = \int_{\partial\Omega} \left(u \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y) \frac{\partial u}{\partial \nu} \right) ds , \quad y \in \Omega . \tag{2.19}$$

Since the integrand is analytic with respect to y, it follows that harmonic functions are analytic.

For any function $f \in L^1(\Omega)$, the integral $\int_\Omega \Gamma(x-y) f(x) \, dx$ is called the *Newtonian potential* of f.

We can also obtain a slightly more general representation formula. Suppose $h \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is harmonic. Then, by Green's second identity (2.8)

$$- \int_{\partial\Omega} \left(u \frac{\partial h}{\partial \nu} - h \frac{\partial u}{\partial \nu} \right) ds = \int_\Omega h \Delta u \, dx . \tag{2.20}$$

Writing $G = \Gamma + h$ and adding (2.18) and (2.20) we obtain the formula

$$u(y) = \int_{\partial\Omega} \left(u \frac{\partial G}{\partial \nu}(x-y) - G(x-y) \frac{\partial u}{\partial \nu} \right) ds + \int_\Omega G(x-y) \Delta u \, dx . \tag{2.21}$$

If in addition we can choose h so that $G = 0$ on $\partial\Omega$, we have

$$u(y) = \int_{\partial\Omega} u \frac{\partial G}{\partial \nu}(x-y)\, ds + \int_\Omega G(x-y)\Delta u\, dx\,. \tag{2.22}$$

The function G is called the *Green's function of the first kind* for Ω. It is unique, by Theorem 2.4. Its existence is equivalent, by the above, to the solvability of the Dirichlet problem for Laplace's equation. It is not possible to construct the Green's function explicitly except for special domains such as the ball. Nevertheless, the representation (2.22) is very useful for studying the Dirichlet problem for Poisson's equation. We will return to it later.

Let $B_R = B_R(0)$, and for $x \in B_R$, $x \neq 0$, let

$$\bar{x} = \frac{R^2}{|x|^2}x$$

denote its inverse with respect to B_R. If $x = 0$ we take $\bar{x} = \infty$. It can then be verified that the Green's function for B_R is given by

$$\begin{aligned} G(x,y) &= \begin{cases} \Gamma(|x-y|) - \Gamma\left(|y||x-\bar{y}|/R\right), & y \neq 0 \\ \Gamma(|x|) - \Gamma(R), & y = 0 \end{cases} \\ &= \Gamma\left(\sqrt{|x|^2+|y|^2-2x\cdot y}\right) - \Gamma\left(\sqrt{(|x||y|/R)^2 + R^2 - 2x\cdot y}\right) \qquad (2.23) \\ &\qquad \text{for all } x,y \in B_R,\ x \neq y. \end{aligned}$$

Hence if $u \in C^2(B_R) \cap C^1(\bar{B}_R)$ is harmonic we have the *Poisson integral formula*

$$u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R} \frac{u(x)\, ds_x}{|x-y|^n}\,. \tag{2.24}$$

Notice that for $y = 0$ this reduces to the mean value equality (2.9). It can be shown by an approximation argument that this formula remains valid for $u \in C^2(B_R) \cap C^0(\bar{B}_R)$ ([GT], Theorem 2.6). An immediate consequence of this is the following result on the solvability of the Dirichlet problem for Laplace's equation on balls.

Theorem 2.8 *Let ϕ be a continuous function on ∂B_R. Then the function u defined by*

$$u(x) = \begin{cases} \dfrac{R^2 - |x|^2}{n\omega_n R} \displaystyle\int_{\partial B_R} \frac{\phi(y)\, ds_y}{|x-y|^n} & \text{if } x \in B_R \\ \phi(x) & \text{if } x \in \partial B_R \end{cases} \tag{2.25}$$

is the unique solution in $C^2(B_R) \cap C^0(\bar{B}_R)$ of the Dirichlet problem $\Delta u = 0$ in B_R, $u = \phi$ on ∂B_R.

As a consequence of this we see that the mean value property in fact characterizes harmonic functions.

Theorem 2.9 *A function $u \in C^0(\Omega)$ is harmonic if and only if it satisfies the mean value equality*

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u\, ds \tag{2.26}$$

for every ball $B = B_R(y) \subset\subset \Omega$.

Proof By Theorem 2.8, for any ball $B \subset\subset \Omega$ there exists a harmonic function h on B such that $h = u$ on ∂B. The difference $w = u - h$ satisfies the mean value property on any ball contained in B, so w satisfies the maximum principle, since this was the only property that was used in its proof. Hence $w = 0$ in B, and u is harmonic.

To prove the existence of solutions of the Dirichlet problem on more general domains we need to do more work. There are several approaches which can be used. The one which follows most directly from the theory we have developed so far is the *Perron method.* We will describe the main ideas here; details can be found in [GT], Section 2.8.

First we extend the definition of subharmonic and superharmonic functions in the following way. A function $u \in C^0(\Omega)$ is *subharmonic* in Ω if for every ball $B \subset\subset \Omega$ and every function h which is harmonic in B and satisfies $u \leq h$ on ∂B we also have $u \leq h$ in B. This is a natural definition in view of Theorem 2.9. The definition of superharmonic function can be extended similarly.

Now let Ω be bounded and let ϕ be a bounded function on $\partial\Omega$. A subharmonic function $u \in C^0(\bar{\Omega})$ is called a *subfunction* relative to ϕ if it satisfies $u \leq \phi$ on $\partial\Omega$. The set of subfunctions is denoted by S_ϕ. It is not empty since any constant function $\leq \inf \phi$ belongs to S_ϕ. The set of superfunctions may be defined similarly.

We have the following result ([GT], Theorem 2.12).

Theorem 2.10 *The function $u(x) = \sup_{v \in S_\phi} v(x)$ is harmonic in Ω.*

The function u is called the *Perron solution* of the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega\,, \qquad u = \phi \quad \text{on} \quad \partial\Omega\,. \tag{2.27}$$

It is a prospective solution of (2.27) in the sense that if (2.27) has a solution $w \in C^2(\Omega) \cap C^0(\bar{\Omega})$, then $u = w$, since $w \in S_\phi$ and $w \geq v$ for all $v \in S_\phi$, by the maximum principle.

The definition of the Perron solution gives us no information about its boundary behaviour. However, if ϕ is continuous at $\xi \in \partial\Omega$, and if there is a superharmonic function $w \in C^0(\bar{\Omega})$ such that $w > 0$ in $\bar{\Omega} - \{\xi\}$ and $w(\xi) = 0$, then u is also continuous at ξ. For then, for any $\epsilon > 0$ we can find a positive number k such that $\phi(\xi) + \epsilon + kw$ and $\phi(\xi) - \epsilon - kw$ are superfunction and subfunction respectively relative to ϕ, and the assertion follows. w is called a *barrier* at ξ. We say that a boundary point of a domain is *regular* if there exists a barrier at that point. The existence of barriers is connected to the geometry of the domain. For example, in two dimensions a point $\xi \in \partial\Omega$ is regular if it is the endpoint of a simple arc lying in the exterior of Ω; the function

$$w = -\mathrm{Re}\frac{1}{\log z} \tag{2.28}$$

is a (local) barrier. This covers most reasonable two dimensional domains. On the other hand, it can be shown that in higher dimensions a domain bounded by a surface with a sufficiently sharp inward pointing cusp has a nonregular point at the tip of the cusp. Finally, if there is a ball $B = B_R(y)$ such that $\bar{B} \cap \bar{\Omega} = \{\xi\}$ (this is called the *exterior sphere condition*), then ξ is a regular point. A barrier can be constructed from the fundamental solution, namely

$$w(x) = \begin{cases} R^{2-n} - |x-y|^{2-n} & \text{for } n \geq 3 \\ \log \dfrac{|x-y|}{R} & \text{for } n = 2\,. \end{cases} \tag{2.29}$$

We therefore conclude the following result for the Dirichlet problem.

Theorem 2.11 *Let Ω be a bounded domain in $\mathbb{R}^n$. The Dirichlet problem (2.27) has a unique solution belonging to $C^\infty(\Omega) \cap C^0(\bar{\Omega})$ for arbitrary continuous boundary data ϕ if and only if every boundary point is regular.*

Remark We will see in Sections 4 and 5 that if $\partial\Omega$ and ϕ are sufficiently smooth, then the solution u has more smoothness up to the boundary. In particular, if $\partial\Omega$ and ϕ are C^∞, then $u \in C^\infty(\bar{\Omega})$.

3 THE NEWTONIAN POTENTIAL

In this section we investigate Poisson's equation

$$\Delta u = f \tag{3.1}$$

by studying the Newtonian potential of f. Recall from Section 2 that any $C^2(\Omega) \cap C^1(\bar{\Omega})$ solution of (3.1) has the representation

$$u(x) = \int_{\partial\Omega} \left(u \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y)\frac{\partial u}{\partial \nu} \right) ds_y + \int_{\Omega} \Gamma(x-y) f(y)\, dy\,. \tag{3.2}$$

The first integrand is smooth as a function of x, so the smoothness of u is determined by the last integral. In particular, if u has compact support in Ω, then u is given by the Newtonian potential of f.

If $f \in C_0^\infty(\Omega)$ (i.e., f is smooth and has compact support in Ω), then its Newtonian potential

$$w(x) = \int_{\Omega} \Gamma(x-y) f(y)\, dy\,. \tag{3.3}$$

belongs to $C^\infty(\bar{\Omega})$, as can be seen by writing

$$\begin{aligned} w(x) &= \int_{\Omega} \Gamma(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy \\ &= \int_{\mathbb{R}^n} \Gamma(z) f(x-z)\, dz\,. \end{aligned}$$

More generally, it is reasonable to expect the second derivatives of w to have the same smoothness as f, since, roughly speaking, solving (3.1) essentially amounts to integrating f twice. However, Δu is only the trace of the second derivative matrix and some cancellation can occur. In fact, there are examples showing that u need not have any C^2 solution if f is only continuous. In this sense the spaces $C^k(\Omega)$ are not well suited to the study of partial differential equations. As it turns out, we do indeed gain two derivatives if we measure differentiability in the right kinds of spaces. There are two main kinds of spaces which are appropriate for this, Hölder spaces and Sobolev spaces. In this section we will use Hölder spaces; Sobolev spaces will be introduced in Section 5.

Let Ω be a bounded domain in $\mathbb{R}^n$, let k be a nonnegative integer and $\alpha \in (0,1]$. The *global Hölder space* $C^{k,\alpha}(\bar{\Omega})$ is defined to be the set of all functions in $C^k(\bar{\Omega})$ for which the quantity

$$[D^k u]_{\alpha;\Omega} = \sup_{\substack{x,y\in\Omega \\ |\beta|=k}} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \tag{3.4}$$

is finite. In this case we say that the k-th order derivatives of u are Hölder continuous on $\bar{\Omega}$ with exponent α. We define a norm on $C^{k,\alpha}(\bar{\Omega})$ by

$$|u|_{k,\alpha;\Omega} = |u|_{k;\Omega} + [D^k u]_{\alpha;\Omega}\,, \tag{3.5}$$

where

$$|u|_{k;\Omega} = \sum_{j=0}^{k} \sup_{\Omega} |D^j u|\,. \tag{3.6}$$

$C^{k,\alpha}(\bar{\Omega})$ is a nonreflexive, nonseparable Banach space. The inclusion $C^{k',\alpha'}(\bar{\Omega}) \subset C^{k,\alpha}(\bar{\Omega})$, $k'+\alpha' > k+\alpha$, is true for domains with sufficiently smooth boundary, but is not true in general.

The *local Hölder space* $C^{k,\alpha}(\Omega)$ is defined to be the set of functions in $C^k(\Omega)$ whose derivatives of order k are Hölder continuous with exponent α on any compact subset of Ω. We denote by $C_0^{k,\alpha}(\Omega)$ the set of functions in $C^{k,\alpha}(\Omega)$ having compact support in Ω.

It is convenient to introduce nondimensional norms on the spaces $C^k(\bar{\Omega})$ and $C^{k,\alpha}(\bar{\Omega})$. Setting $d = \operatorname{diam}\Omega$, we define

$$|u|'_{k;\Omega} = \sum_{j=0}^{k} d^j \sup_{\Omega} |D^j u| \tag{3.7}$$

and

$$|u|'_{k,\alpha;\Omega} = |u|'_{k;\Omega} + d^{k+\alpha}[D^k u]_{\alpha;\Omega}\,. \tag{3.8}$$

The main results of this section are the following interior and boundary estimates ([GT], Theorems 4.6 and 4.11).

Theorem 3.1 *Let $u \in C^2(\Omega)$, $f \in C^{0,\alpha}(\Omega)$, $\alpha \in (0,1)$, satisfy $\Delta u = f$ in a domain Ω in $\mathbb{R}^n$. Then $u \in C^{2,\alpha}(\Omega)$ and for any two concentric balls $B_1 = B_R(x_0)$, $B_2 = B_{2R}(x_0) \subset\subset \Omega$ we have*

$$|u|'_{2,\alpha;B_1} \le C(n,\alpha)(|u|_{0;B_2} + R^2|f|'_{0,\alpha;B_2})\,. \tag{3.9}$$

In the following theorem let B_1 and B_2 be concentric balls as above centred at a point $x_0 \in \{x_n = 0\}$, let $B_j^+ = B_j \cap \{x_n > 0\}$ for $j = 1,2$, and let $T = B_2 \cap \{x_n = 0\}$.

Theorem 3.2 *Let $u \in C^2(B_2^+) \cap C^0(\bar{B}_2^+)$, $f \in C^{0,\alpha}(\bar{B}_2^+)$, $\alpha \in (0,1)$, satisfy $\Delta u = f$ in B_2^+, $u = 0$ on T. Then $u \in C^{2,\alpha}(\bar{B}_1^+)$ and*

$$|u|'_{2,\alpha;B_1^+} \le C(n,\alpha)(|u|_{0;B_2^+} + R^2|f|'_{0,\alpha;B_2^+})\,. \tag{3.10}$$

To prove these results it suffices to establish the corresponding assertions for the Newtonian potential of f, since u differs from w by a harmonic function h, for which the required estimate (in the interior case) follows from Theorem 2.5. In the case of Theorem 3.2 we can reduce to a similar situation as in Theorem 3.1 by some extension and reflection procedures. Proving the result for w is simply a matter of computation. The singularity of Γ prevents us from differentiating directly under the integal, so we proceed by considering a modified function w_ϵ obtained by replacing $\Gamma(x-y)$ in equation (3.3) by $\eta(|x-y|/\epsilon)\Gamma(x-y)$ for a smooth function η such that $0 \le \eta \le 1$, $\eta = 0$ for $t \le 1$ and $\eta = 1$ for $t \ge 2$, and letting $\epsilon \to 0$. Assuming without loss of generality that $\partial\Omega$ is smooth enough for the divergence theorem to hold, we find that

$$D_i w(x) = \int_\Omega D_i\Gamma(x-y)f(y)\,dy\,, \quad i = 1,\cdots,n\,, \tag{3.11}$$

and

$$\begin{aligned} D_{ij}w(x) &= \int_\Omega D_{ij}\Gamma(x-y)(f(y)-f(x))\,dy \\ &\quad -f(x)\int_{\partial\Omega} D_i\Gamma(x-y)\nu_j(y)\,ds_y\,, \quad i,j = 1,\cdots,n\,. \end{aligned} \tag{3.12}$$

The Hölder continuity of D^2w follows from (3.12) after some messy but straightforward computation.

To prove the boundary estimate for w we observe that the representation (3.12) holds with Ω replaced by B_2^+. In addition, the portion of the boundary integral

$$\int_{\partial B_2^+} D_i\Gamma(x-y)\nu_j(y)\,ds_y \tag{3.13}$$

over T is zero if either i or $j \ne n$, so we can proceed exactly as in the interior case if i or $j \ne n$. Finally, $D_{nn}w$ can be estimated directly from the equation $\Delta w = f$ once we have estimated $D_{kk}w$ for $k = 1,\cdots,n-1$.

We can extend Theorems 3.1 and 3.2 to obtain the following global Hölder estimate in balls.

Theorem 3.3 *Let $B = B_R(x_0)$ be a ball in $\mathbb{R}^n$. Let $u \in C^2(B)\cap C^0(\bar{B})$, $f \in C^{0,\alpha}(\bar{B})$, $\alpha \in (0,1)$, satisfy $\Delta u = f$ in B, $u = 0$ on ∂B. Then $u \in C^{2,\alpha}(\bar{B})$ and we have*

$$|u|_{2,\alpha;B} \le C(n,\alpha)(|u|_{0;B} + R^2|f|'_{0,\alpha;B})\,. \tag{3.14}$$

Proof We may assume that ∂B passes through the origin. The inversion mapping $x \mapsto x^* = x/|x|^2$ is a smooth, bicontinuous mapping of $\mathbb{R}^n - \{0\}$ onto itself which maps B onto a halfspace B^*. Furthermore, the *Kelvin transform* of u, defined by

$$v(x) = |x|^{2-n}u\left(\frac{x}{|x|^2}\right) \tag{3.15}$$

belongs to $C^2(B^*)\cap C^0(\bar{B}^*)$ and satisfies

$$\begin{aligned}(3.16)\qquad \Delta_{x^*}v(x^*) &= |x^*|^{-n-2}\Delta_x u(x)\,, \quad x^*\in B^*\,,\ x\in B\,,\\ &= |x^*|^{-n-2} f\left(\frac{x^*}{|x^*|^2}\right)\,, \quad x^*\in B^*\,.\end{aligned}$$

We now apply Theorem 3.2 to the Kelvin transform v, noting that any point of ∂B may be taken as the origin.

Remark The term $|u|_{0;B}$ can be dropped from the right hand side of (3.14), since the functions $v^{\pm}(x) = \pm K(|x-x_0|^2 - R^2)$ are sub- and supersolution respectively of the Dirichlet problem $\Delta u = f$ in B, $u = 0$ on ∂B, if $K \geq \frac{1}{2n}\sup_B |f|$. The maximum principle then implies

$$(3.17)\qquad \sup_B |u| \leq \frac{R^2}{2n}\sup_B |f|\,.$$

We can now obtain the following extension of Theorem 2.9.

Theorem 3.4 *Let $f\in C^{0,\alpha}(\bar{B})$, $\alpha\in(0,1)$, and $\phi\in C^0(\bar{B})$. Then the Dirichlet problem*

$$(3.18)\qquad \Delta u = f \ \text{in } B, \quad u = \phi \ \text{on } \partial B\,,$$

has a unique solution u belonging to $C^{2,\alpha}(B)\cap C^0(\bar{B})$. If $\phi\in C^{2,\alpha}(\bar{B})$, then u belongs to $C^{2,\alpha}(\bar{B})$.

Proof Set $v = u - w$ where w is the Newtonian potential of f. Then problem (3.18) is equivalent to the problem $\Delta v = 0$ in B, $v = \phi - w$ on ∂B, which has a unique solution belonging to $C^2(B)\cap C^0(\bar{B})$, by Theorem 2.9. Hence (3.18) has a unique solution u belonging to $C^2(B)\cap C^0(\bar{B})$. The regularity assertions follow from Theorems 3.1 and 3.3, the second of these applied to $\tilde{u} = u - \phi$ rather than to u itself.

Remarks (i) All the results above require $\alpha\in(0,1)$; they are false in the cases $\alpha = 0$ and $\alpha = 1$.

(ii) The conclusion of Theorem 3.4 is true not just for balls but for any bounded domain with sufficiently smooth boundary ($\partial\Omega\in C^{2,\alpha}$ is sufficient). The main goal of the next two sections is to prove this result and an analogous result for more general second order linear elliptic equations.

4 SCHAUDER THEORY

The aim of this section is to extend the results of the previous sections on Poisson's equation to general second order linear elliptic equations. We will obtain a result analogous to Theorem 3.4 for general linear equations on arbitrary bounded domains subject to certain smoothness assumptions.

We will consider equations of the form

$$(4.1) \qquad Lu = \sum_{i,j=1}^{n} a^{ij}(x) D_{ij}u + \sum_{i=1}^{n} b^i(x) D_i u + c(x)u = f(x)\,,$$

with $a^{ij} = a^{ji}$. Usually we will write (4.1) without the summation signs; summation over repeated indices is understood. We will assume that the equation is *elliptic*, i.e., the coefficient matrix $[a^{ij}(x)]$ is positive definite at each point x. We denote the maximum and minimum eigenvalues of $[a^{ij}(x)]$ by $\Lambda(x)$ and $\lambda(x)$ respectively, so that

$$(4.2) \qquad 0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n - \{0\}$. We say (4.1) is *strictly elliptic* in Ω if $\lambda \geq \lambda_0 > 0$ for some constant λ_0, and *uniformly elliptic* in Ω if Λ/λ is bounded in Ω.

Results for elliptic equations of the form (4.1) usually require additional assumptions on the coefficients. We shall assume throughout this section that

$$(4.3) \qquad \frac{|b^i(x)|}{\lambda(x)} \leq b_0 \quad \text{for } i = 1, \cdots, n,\ x \in \Omega\,,$$

for some constant $b_0 < \infty$. Conditions on c will also be necessary, but these will be stated as needed.

Our first aim is to prove the *weak maximum principle* for solutions of (4.1).

Theorem 4.1 *Let L be an elliptic operator of the form (4.1), and suppose $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies*

$$(4.4) \qquad Lu \geq 0 \ (\leq 0) \ \textit{in } \Omega\,, \quad c = 0 \ \textit{in } \Omega\,,$$

in a bounded domain Ω. Then u achieves its maximum (minimum) on $\partial\Omega$, i.e.,

$$(4.5) \qquad \sup_{\Omega} u = \sup_{\partial\Omega} u \quad \left(\inf_{\Omega} u = \inf_{\partial\Omega} u\right).$$

Proof At an interior maximum point x_0 we have $Du = 0$ and $D^2u \leq 0$ in the sense of matrices, so that $a^{ij}(x_0)D_{ij}u(x_0) \leq 0$. Consequently, if we have the strict inequality $Lu > 0$ in Ω, we immediately obtain a contradiction. Using

(4.3) we may now fix a constant $k > 0$ such that $Le^{kx_1} > 0$. Then for any $\epsilon > 0$ we have $L(u + \epsilon e^{kx_1}) > 0$ in Ω, so that

$$\sup_{\Omega}(u + \epsilon e^{kx_1}) = \sup_{\partial\Omega}(u + \epsilon e^{kx_1}).$$

We now obtain (4.5) by letting $\epsilon \to 0$.

We can generalize this result to the case $c \leq 0$ in Ω. We have $L_0 u = a^{ij}D_{ij}u + b^i D_i u \geq -cu \geq 0$ in $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ if $Lu \geq 0$ in Ω, so the maximum of u on Ω^+ must be achieved on $\partial\Omega^+$. Thus we have

Theorem 4.2 *Let L be elliptic in a bounded domain Ω, and suppose that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies*

$$Lu \geq 0 \ (\leq 0) \text{ in } \Omega, \quad c \leq 0 \text{ in } \Omega. \tag{4.6}$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ \quad \left(\inf_{\Omega} u \geq \inf_{\partial\Omega} u^-\right). \tag{4.7}$$

If $Lu = 0$ in Ω, then

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|. \tag{4.8}$$

As with Poisson's equation this leads to a uniqueness result for the Dirichlet problem

$$Lu = f \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega. \tag{4.9}$$

Theorem 4.3 *Let L be elliptic in a bounded domain Ω with $c \leq 0$. Suppose that $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $Lu = Lv$ in Ω, $u = v$ on $\partial\Omega$. Then $u = v$ in Ω. If $Lu \geq Lv$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

The following *strong maximum principle* generalizes Theorem 2.2.

Theorem 4.4 *Let L be uniformly elliptic, $c = 0$ and $Lu \geq 0$ (≤ 0) in a (possibly unbounded) domain Ω. If u attains its maximum (minimum) at an interior point, then u is constant in Ω. If $c \leq 0$ and c/λ is bounded, then u cannot achieve a nonnegative maximum (nonpositive minimum) in the interior unless it is constant.*

We omit the proof since we will not need this result. See [GT], Chapter 4, for a proof.

In the remainder of this section we want to explain how to prove the existence of solutions for the Dirichlet problem (4.9). First we give a notion of smoothness for boundaries of domains.

We will say that a domain Ω in $\mathbb{R}^n$ has boundary of class $C^{k,\alpha}$ for a nonnegative integer k and $\alpha \in [0, 1]$ if for each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and a one to one mapping ψ of B onto a domain $D \subset \mathbb{R}^n$ such that

(i) $\psi(B \cap \Omega) \subset \mathbb{R}^n{}_+ = \{x \in \mathbb{R}^n : x_n > 0\}$;

(ii) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n{}_+$;

(iii) $\psi \in C^{k,\alpha}(B), \quad \psi^{-1} \in C^{k,\alpha}(D)$.

We will say that a domain Ω has a boundary portion $T \subset \partial\Omega$ of class $C^{k,\alpha}$ if for each $x_0 \in T$ there is a ball $B = B(x_0)$ in which the above conditions are satisfied and such that $B \cap \partial\Omega \subset T$.

We can now state the fundamental results for linear elliptic equations of the form (4.1).

Theorem 4.5 *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{2,\alpha}$ for some $\alpha \in (0,1)$, and let L be as above with*

$$(4.10) \qquad a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for all} \quad x \in \Omega\,,\ \xi \in \mathbb{R}^n\,,$$

and

$$(4.11) \qquad |a^{ij}|_{0,\alpha;\Omega}\,, |b^i|_{0,\alpha;\Omega}\,, |c|_{0,\alpha;\Omega} \leq \Lambda\,,$$

where λ and Λ are positive constants.

(i) *If $c \leq 0$, then for any $f \in C^{0,\alpha}(\bar{\Omega})$ and any $\phi \in C^{2,\alpha}(\bar{\Omega})$ the Dirichlet problem*

$$(4.12) \qquad Lu = f \quad \text{in} \quad \Omega\,, \quad u = \phi \quad \text{on} \quad \partial\Omega$$

has a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$.

(ii) *If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is any solution of* (4.12) *(not necessarily with $c \leq 0$), then $u \in C^{2,\alpha}(\bar{\Omega})$ and*

$$(4.13) \qquad |u|_{2,\alpha;\Omega} \leq C\left(|u|_{0;\Omega} + |\phi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}\right)$$

where C depends only on $n, \lambda, \Lambda, \alpha$ and Ω. If in addition $\partial\Omega\,, \phi \in C^{k+2,\alpha}$ and

$$(4.14) \qquad |a^{ij}|_{k,\alpha;\Omega}\,, |b^i|_{k,\alpha;\Omega}\,, |c|_{k,\alpha;\Omega} \leq \Lambda\,,$$

then $u \in C^{k+2,\alpha}(\bar{\Omega})$ and

$$(4.15) \qquad |u|_{k+2,\alpha;\Omega} \leq C\left(|u|_{0;\Omega} + |\phi|_{k+2,\alpha;\Omega} + |f|_{k,\alpha;\Omega}\right)$$

where C depends only on $n, k, \lambda, \Lambda, \alpha$ and Ω.

Remarks (i) It is essential that $\alpha \in (0,1)$ in the above theorem—the conclusions are false in the cases $\alpha = 0$ and $\alpha = 1$.

(ii) As well as the above global regularity assertions and estimates there are completely interior and partially interior (i.e., in a neighbourhood of a boundary point) versions of these results. The global result is obtained by patching together these local results.

(iii) If $c \leq 0$, then $|u|_{0;\Omega}$ can be estimated in terms of f and ϕ. In fact,

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |\phi| + C \sup_{\Omega} |f|/\lambda \tag{4.16}$$

where C depends only on diam Ω and $\sup_{\Omega} |b|/\lambda$ (see [GT], Theorem 3.7, and also the remark following Theorem 3.3).

(iv) The existence assertion of Theorem 4.5 is generally false if we do not assume $c \leq 0$. However, the following is true regardless of the sign of c: the problem (4.12) has a unique solution in $C^{2,\alpha}(\bar{\Omega})$ for any $f \in C^{0,\alpha}(\bar{\Omega})$ and any $\phi \in C^{2,\alpha}(\bar{\Omega})$ if and only if the homogeneous problem

$$Lu = 0 \quad \text{in} \quad \Omega\,, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{4.17}$$

has only the trivial solution $u = 0$ (see [GT], Theorem 6.15).

We will sketch the main ideas in the proof of Theorem 4.5. First, Theorem 4.5 has two main parts—an assertion about the existence of a solution and an assertion that any solution satisfies certain estimates. The estimates play a key role in the proof of existence, so let's assume that we have already proved these and concentrate on the existence problem for the moment.

First, we can simplify things by assuming zero boundary values—this can be achieved by replacing u by $u - \phi$. Next, in place of the problem (4.12) let's consider a family of problems

$$L_t u \equiv tLu + (1-t)\Delta u = f \quad \text{in} \quad \Omega\,, \quad u = 0 \quad \text{on} \quad \partial\Omega\,, \tag{4.18}$$

where $t \in [0,1]$. Notice that $L_0 = \Delta$ and $L_1 = L$, and that the coefficients of L_t satisfy the conditions (4.10) and (4.11) with

$$\lambda_t = \min\{1, \lambda\}\,, \quad \Lambda_t = \max\{1, \Lambda\}\,.$$

The operator L_t is a bounded linear operator from the Banach space $\mathcal{B}_1 = \{u \in C^{2,\alpha}(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ into the Banach space $\mathcal{B}_2 = C^{0,\alpha}(\bar{\Omega})$. The solvability of the Dirichlet problem (4.18) is equivalent to the invertibility of the mapping L_t. Let u_t denote the solution of (4.18), assuming that there is one. Then, by the estimates (4.13) and (4.16), we have

$$|u_t|_{2,\alpha;\Omega} \leq C|f|_{0,\alpha;\Omega}\,, \tag{4.19}$$

for a constant C depending only on n, λ, Λ and diam Ω, but not on t. In other words,

$$\|u\|_{\mathcal{B}_1} \leq C\|L_t u\|_{\mathcal{B}_2}\,. \tag{4.20}$$

Now suppose that (4.18) is solvable for some $s \in [0,1]$, i.e., L_s is onto. By (4.20) L_s is one to one and hence the inverse L_s^{-1} exists. Furthermore, L_s^{-1} is a bounded linear operator with $\|L_s^{-1}\| \leq C$, by (4.19). The equation $L_t u = f$ is equivalent to the equation

$$\begin{aligned} L_s u &= f + (L_s - L_t)u \\ &= f + (t-s)L_0 - (t-s)L_1 \,, \end{aligned}$$

which in turn is equivalent to

$$u = L_s^{-1} f + (t-s) L_s^{-1}(L_0 - L_1)u \,.$$

The mapping T from $\mathcal{B}_1$ to itself given by

$$Tu = L_s^{-1} f + (t-s) L_s^{-1}(L_0 - L_1)u$$

is is clearly a contraction mapping (and hence has a fixed point) if

$$|s-t| < \delta = [C(\|L_0\| + \|L_1\|)]^{-1} \,.$$

It follows that L_t is onto for all $t \in [0,1]$ with $|s-t| < \delta$. By repeatedly applying this argument on subintervals of $[0,1]$ of length less than δ we conclude that L_t is onto for all $t \in [0,1]$ if it is onto for any fixed $t \in [0,1]$, in particular, for $t = 0$.

To summarize, to prove the solvability in $C^{2,\alpha}(\bar{\Omega})$ of the Dirichlet problem (4.12) it is sufficent to prove the estimate (4.19) for any solution of (4.18) for any $t \in [0,1]$, and to prove the solvability in $C^{2,\alpha}(\bar{\Omega})$ of the Dirichlet problem for the special case of Poisson's equation. In particular, by Theorem 3.4 we can solve the Dirichlet problem (4.12) in $C^{2,\alpha}(\bar{\Omega})$ in the special case that Ω is a ball, assuming of course that the coefficients and the data satisfy the hypotheses of Theorem 4.5. But once we have this we can solve the Dirichlet problem (4.12) on a ball for continuous boundary data ϕ, obtaining a solution $u \in C^{2,\alpha}(B) \cap C^0(\bar{B})$. We do this by approximating ϕ by a sequence of smooth functions $\{\phi_j\}$, solving the Dirichlet problem on B with boundary data ϕ_j, and passing to a limit with the help of the interior version of the estimate (4.13) and a suitable barrier argument. We can then adapt the Perron method to obtain a Perron solution of (4.12) on any bounded domain Ω, and, using (4.13), this solution can be shown to belong to $C^{2,\alpha}(\bar{\Omega})$ if $\partial\Omega \in C^{2,\alpha}$ and $\phi \in C^{2,\alpha}(\bar{\Omega})$.

We now turn to the estimate (4.13). As mentioned above, the estimate (4.13) is a consequence of interior and partially interior estimates similar to those derived for solutions of Poisson's equation in the previous section. In fact, they are derived from these by a perturbation argument, which we now describe.

Let x_0 be a point of Ω. By making a linear transformation of coordinates we may assume that $a^{ij}(x_0) = \delta^{ij}$. We now write equation (4.1) in the form

$$\Delta u = (a^{ij}(x_0) - a^{ij}(x))D_{ij}u - b^i(x)D_iu - c(x)u + f(x) = \tilde{f}(x) \tag{4.21}$$

and apply Theorem 3.1 to get

$$\begin{aligned} R^{2+\alpha}[D^2u]_{\alpha;B_1} &\le C(n,\alpha)(|u|_{0;B_2} + R^2|\tilde{f}|'_{0,\alpha;B_2}) \\ &\le \tilde{C}(n,\alpha,\Lambda)\Big\{|u|_{0;B_2} + R^2(R^\alpha|D^2u|_{0;B_2} + |Du|_{0;B_2} + |u|_{0;B_2}) \\ &\quad + R^{2+\alpha}(R^\alpha[D^2u]_{\alpha;B_2} + |D^2u|_{0;B_2} + [Du]_{\alpha;B_2} \\ &\quad + |Du|_{0;B_2} + [u]_{\alpha;B_2} + |u|_{0;B_2}) + |f|'_{0,\alpha;B_2}\Big\} \end{aligned} \tag{4.22}$$

for any two concentric balls $B_1 = B_R(x_0)$ and $B_2 = B_{2R}(x_0) \subset\subset \Omega$; to keep the dependence on R a little simpler we have assumed that $R \le 1$. Here we have also used the easily verified inequalities $[f+g]_\alpha \le [f]_\alpha + [g]_\alpha$ and $[fg]_\alpha \le |f|_0[g]_\alpha + [f]_\alpha|g|_0$. We now use the following interpolation inequality: for each $\epsilon > 0$ and any integers $k \ge l \ge 1$ there is a constant $C = C(\epsilon, \alpha, n, k)$ such that for any $u \in C^{k,\alpha}(B_R)$ we have

$$R^l|D^lu|_{0;B_R} \le \epsilon R^{k+\alpha}[D^ku]_{\alpha;B_R} + C|u|_{0;B_R}\,. \tag{4.23}$$

Using this and the inequality

$$[D^ku]_{\alpha;B_R} \le R^{1-\alpha}|D^{k+1}u|_{0;B_R} \tag{4.24}$$

in (4.22) we obtain

$$[D^2u]_{\alpha;B_1} \le C\left(R^\alpha[D^2u]_{\alpha;B_2} + R^{-(2+\alpha)}(|u|_{0;B_2} + R^2|f|'_{0,\alpha;B_2})\right) \tag{4.25}$$

where C depends on n, α and Λ. The important point here is that the coefficient of $[D^2u]_{\alpha;B_2}$ can be made small by making R small. This alone is not sufficient to absorb the term $CR^\alpha[D^2u]_{\alpha;B_2}$ into the left hand side of the inequality, since the seminorm on the left is taken over a smaller ball. However, this can be achieved with the help of a somewhat technical covering argument (see [S], Lecture 6, Lemma 2). Alternatively, it is possible to work with weighted Hölder spaces and use an interpolation inequality similar to (4.23) in these spaces. In any case, we finally conclude that

$$|u|'_{2,\alpha;B_1} \le C(|u|_{0;B_2} + R^2|f|'_{0,\alpha;B_2})\,, \tag{4.26}$$

where C depends on n, α and Λ.

A similar argument can be used to get the boundary estimate

$$|u|_{2,\alpha;B_1\cap\Omega} \le C(|u|_{0;B_2\cap\Omega} + |\phi|_{2,\alpha;B_2\cap\Omega} + R^2|f|'_{0,\alpha;B_2\cap\Omega}) \tag{4.27}$$

for any two concentric balls $B_1 = B_R(x_0)$ and $B_2 = B_{2R}(x_0)$ of sufficiently small radius centred at a point $x_0 \in \partial\Omega$. The constant C depends on n, α, Λ, R and $\partial\Omega$. The details are a little more complicated, since we first need to flatten $\partial\Omega$ in a neighbourhood of x_0.

The global Schauder estimate (4.13) follows by combining (4.26) and (4.27) and using a covering argument. The higher order estimates follow by successively applying (4.13) to the elliptic equation satisfied by each derivative of u. Some care is needed here since u is not *a priori* sufficiently smooth to differentiate equation (4.1) the required number of times, but this difficulty can be overcome by approximating derivatives of u by difference quotients.

5 SOBOLEV THEORY

In this section our aim is to develop the theory of linear elliptic equations in a class of spaces known as Sobolev spaces. The Hölder spaces are suited to the theory of classical solutions, i.e., solutions which are (at least) twice continuously differentiable. However, these spaces are neither reflexive nor separable, which precludes the application of certain techniques of functional analysis, especially Hilbert space methods, to the proof of existence of solutions. In the Sobolev space theory we weaken the notion of solution so that we can solve the Dirichlet problem in a separable Hilbert space. This requires much less work than the procedure described in the previous three sections. The price we pay for this is that we need to do further work to show that the solution we obtain is in fact a classical solution.

We begin with the definitions of Sobolev spaces. For $1 \leq p < \infty$ the space of measurable functions whose p-th power is integrable on Ω is denoted by $L^p(\Omega)$ (with the usual convention that functions which agree almost everywhere are regarded as the same). The norm on $L^p(\Omega)$ is given by

$$\|u\|_{L^p(\Omega)} = \|u\|_{p;\Omega} = \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}} . \tag{5.1}$$

For $p = \infty$ $L^\infty(\Omega)$ denotes the space of bounded measurable functions with the norm

$$\|u\|_{L^\infty(\Omega)} = \|u\|_{\infty;\Omega} = \sup_\Omega |u| \tag{5.2}$$

with the supremum understood to be the essential supremum.

The L^p spaces are Banach spaces. They are separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. Furthermore, $L^2(\Omega)$ is a Hilbert space under the inner product

$$(u, v) = \int_\Omega uv \, dx . \tag{5.3}$$

The space of functions which are locally p-integrable on Ω is denoted by $L^p_{loc}(\Omega)$.

Let u be locally integrable on Ω and let α be any multi-index. Then a locally integrable function v is said to be the α-th *weak derivative* of u if it satisfies

$$\int_\Omega \phi v \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi \, dx \quad \text{for all } \phi \in C_0^{|\alpha|}(\Omega). \tag{5.4}$$

It is clear from the definition that the α-th weak derivative is unique if it exists. Furthermore, if $u \in C^{|\alpha|}(\Omega)$, then the weak α-th derivative of u coincides with the classical derivative $D^\alpha u$, by integration by parts. Thus weak differentiability is an extension of the classical concept. We denote the α-th weak derivative of u by $D^\alpha u$.

Many properties of classical derivatives remain true for weak derivatives, for example, the usual Leibniz rule for differentiating products is valid, as is a form of the chain rule (see [GT], Chapter 7). We will not explicitly use these here, although these properties are used in proving some of the results we will state.

We can now define the *Sobolev spaces*. For $p \geq 1$ and any nonnegative integer k we define

$$W^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}. \tag{5.5}$$

$W^{k,p}(\Omega)$ is a Banach space under the norm

$$\|u\|_{k,p;\Omega} = \|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha|\leq k} |D^\alpha u|^p \, dx \right)^{\frac{1}{p}} \tag{5.6}$$

if $1 \leq p < \infty$. $W^{k,\infty}(\Omega)$ is also a Banach space if the norm is defined in the obvious way. $W^{k,p}(\Omega)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. Furthermore, $W^{k,2}(\Omega)$ is a Hilbert space under the inner product

$$(u, v) = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha u D^\alpha v \, dx . \tag{5.7}$$

We denote by $W^{k,p}_{loc}(\Omega)$ the space of functions belonging to $W^{k,p}(\Omega')$ for any $\Omega' \subset\subset \Omega$.

An important result is the following.

Theorem 5.1 ([GT], Theorem 7.9) *The subspace $W^{k,p}(\Omega) \cap C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

The closure of $C^\infty_0(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $W^{k,p}_0(\Omega)$. It is also a Banach space. An equivalent norm on $W^{k,p}_0(\Omega)$ is given by

$$\|u\|_{W^{k,p}_0(\Omega)} = \left(\int_\Omega \sum_{|\alpha|=k} |D^\alpha u|^p \, dx \right)^{\frac{1}{p}} . \tag{5.8}$$

This is a consequence of the following *Sobolev embedding theorem*. It is a fundamental tool in the theory of partial differential equations.

Theorem 5.2 ([GT], Theorems 7.10, 7.17) *Let Ω be a bounded domain in $\mathbb{R}^n$.*
(i) If $p < n$, the space $W^{1,p}_0(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, $p^* =$

$np/(n-p)$, and compactly embedded in $L^q(\Omega)$ for any $q < p^*$. Furthermore, we have

$$\|u\|_{np/(n-p)} \le C(n,p)\|Du\|_p \,. \tag{5.9}$$

(ii) If $p > n$, the space $W_0^{1,p}(\Omega)$ is continuously embedded in $C^{0,\alpha}(\bar\Omega)$ for $\alpha = 1 - n/p$, and compactly embedded in $C^{0,\beta}(\bar\Omega)$ for any $\beta < \alpha$. Furthermore, we have

$$|u|_{0,\alpha;\Omega} \le C(n,p)[1 + (\operatorname{diam}\Omega)^\alpha]\|Du\|_p \,. \tag{5.10}$$

To see why Sobolev spaces are useful let's return to Poisson's equation

$$\Delta u = f \,. \tag{5.11}$$

Suppose $u \in C^2(\Omega)$ is a solution of (5.11). Then, by integration by parts we have

$$-\int_\Omega Du \cdot Dv \, dx = \int_\Omega fv \, dx \quad \text{for any } v \in C_0^1(\Omega) \,. \tag{5.12}$$

But (5.12) makes sense even if u belongs only to $W^{1,2}(\Omega)$ rather than to $C^2(\Omega)$. Thus we define a function $u \in W^{1,2}(\Omega)$ to be a *weak* or *generalized solution* of (5.11) if it satisfies (5.12) for all $v \in C_0^1(\Omega)$.

We now turn our attention to the Dirichlet problem

$$\Delta u = f \text{ in } \Omega \,, \quad u = \phi \text{ on } \partial\Omega \,. \tag{5.13}$$

A generalized solution of (5.11) is not necessarily continuous, so it is not clear whether the boundary condition in (5.13) has any meaning for such solutions. However, we can give a weak notion of this as well: if $u, \phi \in W^{1,2}(\Omega)$, we say $u = \phi$ on $\partial\Omega$ if $u - \phi \in W_0^{1,2}(\Omega)$. A weak definition of inequality on $\partial\Omega$ can also be given. In particular, $u \le 0$ on $\partial\Omega$ if $u^+ = \max\{u, 0\} \in W_0^{1,2}(\Omega)$.

Once we have these notions we can prove a *weak maximum principle* for $W^{1,2}$ solutions.

Theorem 5.3 ([GT], Theorem 8.1) *Let $u \in W^{1,2}(\Omega)$ satisfy $\Delta u \ge 0 (\le 0)$ in a bounded domain Ω. Then*

$$\sup_\Omega u \le \sup_{\partial\Omega} u^+ \qquad \left(\inf_\Omega u \ge \inf_{\partial\Omega} u^-\right) \,. \tag{5.14}$$

Proof By approximation and Theorem 5.1, (5.12) is valid for any $v \in W_0^{1,2}(\Omega)$. Therefore we may choose $v = \max\{u - l, 0\}$ where $l = \sup_{\partial\Omega} u^+$, in the weak form of $\Delta u \ge 0$, namely

$$\int_\Omega Du \cdot Dv \, dx \le 0 \quad \text{for all nonnegative } v \in W_0^{1,2}(\Omega) \,,$$

to obtain

$$\int_{\{u>l\}} |Du|^2\,dx \le 0\,.$$

This implies (5.14).

As a corollary of this we see that generalized solutions of the Dirichlet problem (5.12) are unique.

We now turn to the existence question. Here the fact that $W_0^{1,2}(\Omega)$ is a Hilbert space makes this relatively easy. As in the classical case we can reduce to the case of zero boundary values by replacing u by $u-\phi$. We must then replace f by $\tilde{f} = f - \Delta\phi$. Notice that this is not a function in general, since ϕ belongs only to $W^{1,2}(\Omega)$. Set $f^i = -D_i\phi$. Then the equation

$$\Delta u = f + D_i f^i \tag{5.15}$$

can be interpreted in weak form, namely

$$-\int_\Omega Du\cdot Dv\,dx = \int_\Omega (fv - f^i D_i v)\,dx \quad \text{for any } v \in W_0^{1,2}(\Omega). \tag{5.16}$$

In view of this it is reasonable to assume that $f, f_i \in L^2(\Omega)$.

Now consider the Hilbert space $\mathcal{H} = W_0^{1,2}(\Omega)$ equipped with the inner product

$$(u,v)_{\mathcal{H}} = \int_\Omega Du\cdot Dv\,dx\,. \tag{5.17}$$

We denote the corresponding norm by $\|\cdot\|_{\mathcal{H}}$. As noted above, it is equivalent to the usual $W^{1,2}$ norm, so that $\mathcal{H}$ is indeed a Hilbert space. If we now define

$$F(v) = -\int_\Omega (fv - f^i D_i v)\,dx\,, \tag{5.18}$$

for $v \in \mathcal{H}$, then by Hölder's inequality and (5.9) we have

$$\begin{aligned} |F(v)| &\le \left(\|f\|_{L^2(\Omega)} + \sum_{i=1}^n \|f^i\|_{L^2(\Omega)}\right)\|v\|_{W^{1,2}(\Omega)} \\ &\le C\left(\|f\|_{L^2(\Omega)} + \sum_{i=1}^n \|f^i\|_{L^2(\Omega)}\right)\|v\|_{\mathcal{H}} \end{aligned} \tag{5.19}$$

so that F is a bounded linear functional on $\mathcal{H}$. It follows from the Riesz representation theorem for Hilbert spaces that there is a unique element $u \in \mathcal{H}$ such that

$$F(v) = (u,v)_{\mathcal{H}} \quad \text{for all } v \in \mathcal{H}. \tag{5.20}$$

This says exactly that the Dirichlet problem

$$\Delta u = f + D_i f^i \text{ in } \Omega\,, \quad u = 0 \text{ on } \partial\Omega\,, \tag{5.21}$$

is uniquely solvable in $W_0^{1,2}(\Omega)$ for any $f, f^i \in L^2(\Omega)$. Furthermore, for the particular choice $v = u$ in (5.19) and (5.20) we obtain the estimate

$$\|u\|_{W^{1,2}(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \sum_{i=1}^{n} \|f^i\|_{L^2(\Omega)}\right). \tag{5.22}$$

Returning to the original boundary condition in the usual way we see that we have proved the following.

Theorem 5.4 *Let Ω be a bounded domain in $\mathbb{R}^n$. Then the Dirichlet problem*

$$\Delta u = f + D_i f^i \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega, \tag{5.23}$$

has a unique generalized solution in $W^{1,2}(\Omega)$ for any $f, f^i \in L^2(\Omega)$ and $\phi \in W^{1,2}(\Omega)$. Furthermore, we have

$$\|u\|_{W^{1,2}(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \sum_{i=1}^{n} \|f^i\|_{L^2(\Omega)} + \|\phi\|_{W^{1,2}(\Omega)}\right). \tag{5.24}$$

We can extend all of these ideas to more general second order elliptic equations. The notion of generalized solution is based on integration by parts, so it is natural to require that the equation be written in a form suitable for this procedure. Thus we assume now that L has the form

$$Lu = D_i(a^{ij}(x)D_j u + b^i(x)u) + c^i(x)D_i u + d(x)u, \tag{5.25}$$

where the coefficients are measurable functions on Ω. We also need to assume the ellipticity condition

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{5.26}$$

where λ is a positive constant. We also assume that the coefficients are bounded:

$$|a^{ij}(x)|, |b^i(x)|, |c^i(x)|, |d(x)| \leq \Lambda \quad \text{for all } x \in \Omega \tag{5.27}$$

for another positive constant Λ. Finally, corresponding to the nonpositivity of the coefficient of u for an equation of the form (4.1), we assume

$$\int_\Omega (dv - b^i D_i v)\, dx \leq 0 \quad \text{for all nonnegative } v \in W_0^{1,2}(\Omega). \tag{5.28}$$

Proceeding essentially as before, we can prove the following.

Theorem 5.5 ([GT], Theorem 8.3) *Let Ω be a bounded domain in $\mathbb{R}^n$, and suppose the operator L given by* (5.25) *satisfies conditions* (5.26), (5.27) *and* (5.28). *Then for any $f, f^i \in L^2(\Omega)$ and $\phi \in W^{1,2}(\Omega)$ the Dirichlet problem*

$Lu = f + D_i f^i$ in Ω, $u = \phi$ on $\partial\Omega$ has a unique generalized solution in $W^{1,2}(\Omega)$, and we have

$$(5.29) \qquad \|u\|_{W^{1,2}(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \sum_{i=1}^{n} \|f^i\|_{L^2(\Omega)} + \|\phi\|_{W^{1,2}(\Omega)}\right),$$

where C depends on n, λ, Λ and Ω.

The final question we will consider in this section is the regularity of the solutions obtained in Theorems 5.4 and 5.5. The following theorem answers this.

Theorem 5.6 ([GT], Theorems 8.8, 8.10, 8.12, 8.13) *(i) Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $Lu = f$ in Ω where L is strictly elliptic in Ω, the coefficients $a^{ij}, b^i \in C^{k,1}(\bar{\Omega})$, the coefficients $c^i, d \in C^{k-1,1}(\bar{\Omega})$ ($L^\infty(\Omega)$ if $k = 0$), and $f \in W^{k,2}(\Omega)$, where k is a nonnegative integer. Then for any subdomain $\Omega' \subset\subset \Omega$ we have $u \in W^{k+2,2}(\Omega')$ and*

$$(5.30) \qquad \|u\|_{W^{k+2,2}(\Omega')} \le C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{W^{k,2}(\Omega)})$$

where C depends only on $n, \lambda, k, \operatorname{dist}(\Omega', \partial\Omega)$ and K, where

$$K = \max\{\|a^{ij}, b^i\|_{C^{k,1}(\bar{\Omega})}, \|c^i, d\|_{C^{k-1,1}(\bar{\Omega})}\}.$$

In particular, if $a^{ij}, b^i, c^i, d, f \in C^\infty(\Omega)$, then u belongs to $C^\infty(\Omega)$.
(ii) Suppose in addition that $\partial\Omega \in C^{k+2}$ and that there exists a function $\phi \in W^{k+2,2}$ such that $u - \phi \in W_0^{1,2}(\Omega)$. Then $u \in W^{k+2,2}(\Omega)$ and

$$(5.31) \qquad \|u\|_{W^{k+2,2}(\Omega)} \le C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{W^{k,2}(\Omega)} + \|\phi\|_{W^{k+2,2}(\Omega)})$$

where C depends only on n, λ, k, K and $\partial\Omega$. In particular, if $a^{ij}, b^i, c^i, d, f, \phi \in C^\infty(\bar{\Omega})$ and $\partial\Omega \in C^\infty$, then u belongs to $C^\infty(\bar{\Omega})$.

This is proved by applying the estimate (5.29) to the derivatives of u, or more precisely, to suitable difference quotients of u, since we do not know *a priori* that u is sufficiently smooth to differentiate directly.

6 PARABOLIC EQUATIONS

In this section we will describe some results for parabolic equations. A *parabolic* equation is one of the form

$$u_t = \frac{\partial u}{\partial t} = Lu \quad \text{on} \quad \Omega \times (0,T)\,, \tag{6.1}$$

where Ω is a domain in $\mathbb{R}^n$ and L is an elliptic operator for each time $t \in (0,T)$. Thus the general linear second order parabolic equation has the form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a^{ij}(x,t)D_{ij}u + \sum_{i=1}^{n} b^i(x,t)D_iu + c(x,t)u + f(x,t) \tag{6.2}$$

where

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j > 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n\,,\ \xi \neq 0 \tag{6.3}$$

for each $(x,t) \in (0,T)$.

The simplest example of a parabolic equation is the *heat equation*

$$u_t = \Delta u\,. \tag{6.4}$$

We will restrict our attention to this model equation in this lecture. However, the theory of much more general parabolic equations can be developed very much along the lines of elliptic theory, with the heat operator playing the central role in place of the Laplacian. Roughly speaking, there is a parabolic version of most results of the elliptic theory. Thus there are mean value equalities for solutions of the heat equation, parabolic weak and strong maximum principles, and parabolic versions of the Schauder and Sobolev theories. Of course, there are also aspects which have no elliptic analogues, such as the asymptotic behaviour of solutions as $t \to \infty$, but even these may have close connections to the elliptic theory.

The basic boundary value problem for the heat equation, and also for general second order parabolic equations, is to prescribe the value of the solution on the *parabolic boundary* of $Q = \Omega \times (0,T)$ (we always take $T > 0$), which is given by $\partial' Q = (\{0\} \times \Omega) \cup (\partial\Omega \times [0,T])$. This is known as the *first initial-boundary value problem.* It is called the *Cauchy problem* in the special case $\Omega = \mathbb{R}^n$.

We begin with the *weak maximum principle* for the heat equation.

Theorem 6.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $Q = \Omega \times (0,T)$. Suppose that $u \in C^2(Q) \cap C^0(\bar{Q})$ satisfies $u_t - \Delta u \leq 0$ in Q. Then*

$$\sup_Q u = \sup_{\partial' Q} u\,. \tag{6.5}$$

Proof Let $\epsilon > 0$ and set $v = u - \epsilon t$, so that

$$v_t - \Delta v < 0 \quad \text{in } Q. \tag{6.6}$$

Let $T' \in (0, T)$ and set $Q' = \Omega \times (0, T')$. If v attains its maximum over Q' at an interior point, then $v_t = 0$ and $D^2 v \leq 0$ at that point, which contradicts (6.6). If v has its maximum over Q' at a point of $\Omega \times \{T'\}$, then $v_t \geq 0$ and $D^2 v \leq 0$ at that point, which also contradicts (6.6). It follows that v attains its maximum on $\partial' Q'$, and hence, letting $T' \to T$ and $\epsilon \to 0$, that u attains its maximum on $\partial' Q$.

Remark It is clear that this argument generalizes to more general parabolic equations. However, there is also a mean value equality for solutions of the heat equation from which we may obtain the maximum principle, as we did in Section 2 for harmonic functions. There is also a *strong maximum principle* for solutions of (6.4) (which can also be deduced from the mean value equality) which states that if u attains its maximum at a point of $\bar{Q} - \partial' Q$, then u is constant in Q.

From Theorem 6.1 we immediately obtain the following uniqueness result for the first initial-boundary problem.

Theorem 6.2 *Let Ω be a bounded domain in $\mathbb{R}^n$. Then there is at most one solution $u \in C^2(Q) \cap C^0(\bar{Q})$ of the problem*

$$u_t - \Delta u = f \text{ in } Q, \quad u = \phi \text{ on } \partial' Q. \tag{6.7}$$

Next we find an analogue of the fundamental solution and a related representation formula for solutions of the Cauchy problem

$$u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty), \quad u = \phi \text{ on } \mathbb{R}^n. \tag{6.8}$$

The quickest way to do this is to take the Fourier transform of the heat equation. But the following method (taken from [E]) is more elementary. First, if u solves the heat equation (6.4) on $\mathbb{R}^n \times (0, \infty)$, then so does the function w given by

$$u_\lambda(x, t) = u(\lambda x, \lambda^2 t)$$

for any $\lambda \in \mathbb{R}$. This scaling property suggests that we look for a solution of the form

$$u(x, t) = v\left(\frac{r^2}{t}\right) \quad r = |x|,\ t > 0\,,$$

where v is to be determined. This eventually leads to what we want, but it turns out to be simpler to look for solutions of the form

$$u(x, t) = w(t) v\left(\frac{r^2}{t}\right), \tag{6.9}$$

where both v and w are to be determined. After a little computation we find that

$$\begin{aligned} u_t - \Delta u &= w'(t)v\left(\frac{r^2}{t}\right) - w(t)v'\left(\frac{r^2}{t}\right)\frac{r^2}{t^2} \\ &\quad -w(t)v''\left(\frac{r^2}{t}\right)\frac{4r^2}{t^2} - w(t)v'\left(\frac{r^2}{t}\right)\frac{2n}{t}. \end{aligned}$$

Consequently, if u is to solve (6.4), we require

$$w'(t)v\left(\frac{r^2}{t}\right) - \frac{w(t)}{t}\left[v''\left(\frac{r^2}{t}\right)\frac{4r^2}{t} + v'\left(\frac{r^2}{t}\right)\frac{r^2}{t} + v'\left(\frac{r^2}{t}\right)2n\right] = 0. \tag{6.10}$$

If we choose

$$v(z) = e^{-z/4}, \tag{6.11}$$

then $4v''(z) + v'(z) = 0$, so the first two terms inside the brackets cancel. Consequently, (6.10) reduces to

$$w'(t) + \frac{nw(t)}{2t} = 0$$

which has the solution

$$w(t) = t^{-n/2}. \tag{6.12}$$

Combining (6.9), (6.11) and (6.12) we see that

$$u(x,t) = \frac{1}{t^{n/2}}e^{-|x|^2/4t}, \tag{6.13}$$

up to additive and multiplicative constants. The function

$$K(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/4t} \tag{6.14}$$

is called the *fundamental solution of the heat equation* or the *heat kernel.*
Remark The normalization factor $(4\pi)^{-n/2}$ is introduced so that

$$\int_{\mathbb{R}^n} K(x,t)\,dx = 1 \quad \text{for each } t > 0. \tag{6.15}$$

We can now use the heat kernel to obtain a representation formula for solutions of the Cauchy problem (6.8). To do this we observe that the heat equation is translation invariant, so $K(x-y,t)$ is a solution for each fixed $y \in \mathbb{R}^n$. In addition, the sum of finitely many solutions is also a solution. This suggests that

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t)\phi(y)\,dy \tag{6.16}$$

should also be a solution.

Theorem 6.3 *Suppose* $\phi \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*, and let* u *be defined by* (6.16)*. Then*
(i) u *belongs to* $C^\infty(\mathbb{R}^n \times (0,\infty))$*;*
(ii) u *satisfies the heat equation in* $\mathbb{R}^n \times (0,\infty)$*;*
(iii) $\lim_{(x,t)\to(x_0,0)} u(x,t) = \phi(x_0)$ *for each* $x_0 \in \mathbb{R}^n$*.*

Proof (i) and (ii) follow from the fact that $t^{-n/2}e^{-|x|^2/4t}$ is C^∞ with derivatives of all orders uniformly bounded on $\mathbb{R}^n \times [\delta,\infty)$ for each $\delta > 0$; this allows us to differentiate under the integral. To prove (iii) let $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$. By continuity of ϕ there is a $\delta > 0$ such that

$$(6.17) \qquad |\phi(y) - \phi(x_0)| < \epsilon \quad \text{if } |y - x_0| < \delta, \quad y \in \mathbb{R}^n.$$

For $|x - x_0| < \delta/2$ we have

$$\begin{aligned} |u(x,t) - \phi(x_0)| &= \left| \int_{\mathbb{R}^n} K(x-y,t)(\phi(y) - \phi(x_0))\,dy \right| \\ &\le \int_{B_\delta(x_0)} K(x-y,t)|\phi(y) - \phi(x_0)|\,dy \\ &\quad + \int_{\mathbb{R}^n - B_\delta(x_0)} K(x-y,t)|\phi(y) - \phi(x_0)|\,dy\,. \end{aligned}$$

The first integral is less than ϵ, by (6.15), and the second can easily be shown to go to zero as $t \to 0$, since ϕ is bounded. Thus we obtain $|u(x,t) - \phi(x_0)| < 2\epsilon$ if $|x - x_0| < \delta/2$ and $t > 0$ is small enough.

Remarks (i) Since we are not on a bounded domain, the weak maximum principle cannot be used as before to deduce the uniqueness of solutions of the Cauchy problem. In fact, uniqueness does not hold in general, but it can be shown that for any given positive constants A and α there is only one solution which satisfies the bound $|u(x,t)| \le Ae^{\alpha|x|^2}$.

(ii) Theorem 6.3 can be generalized to yield a representation formula for solutions of the Cauchy problem for the inhomogeneous heat equation $u_t - \Delta u = f$.

The next result we will derive is a representation formula for solutions of the inhomogeneous heat equation

$$(6.18) \qquad u_t - \Delta u = f \quad \text{in } Q = \Omega \times (0,\infty)\,,$$

where Ω is bounded with $\partial\Omega \in C^1$, which is analogous to the representation formula (2.18). Suppose that $u, v \in C^2(\bar{Q})$ and u solves (6.19). By integration by parts we find that

$$\int_Q vf\,dx\,dt = \int_Q v(u_t - \Delta u)\,dx\,dt$$

$$(6.19)\qquad = -\int_Q u(v_t+\Delta v)\,dx\,dt + \int_{\Omega\times\{T\}} uv\,dx - \int_{\Omega\times\{0\}} uv\,dx - \int_0^T\int_{\partial\Omega}\left(v\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu}\right) ds\,dt\,.$$

We now fix $y \in \Omega$, let $\epsilon > 0$ and choose $v(x,t) = K(x-y, T+\epsilon-t)$; then v belongs to $C^\infty(\mathbb{R}^n \times [0,T])$ and solves the backward heat equation $v_t + \Delta v = 0$. By the proof of Theorem 6.3 (iii) we have

$$(6.20)\qquad \int_{\Omega\times\{T\}} uv\,dx = \int_\Omega K(x-y,\epsilon)u(x,T)\,dx \to u(y,T) \quad \text{as } \epsilon\to 0.$$

Since $K(x-y, T+\epsilon-t)$ is uniformly continuous with respect to ϵ, x, t for $\epsilon \geq 0$, $x \in \partial\Omega$, $t\in[0,T]$ and for $x\in\Omega$, $t=0$, we see from (6.19) that

$$(6.21)\qquad u(y,T) = \int_Q K(x-y,T-t)f(x,t)\,dx\,dt + \int_\Omega K(x-y,T)u(x,0)\,dx + \int_0^T\int_{\partial\Omega}\left(K(x-y,T-t)\frac{\partial u}{\partial\nu}(x,t) - u(x,t)\frac{\partial K}{\partial\nu}(x-y,T-t)\right) ds\,dt\,.$$

It follows from this that if u solves the heat equation on Q, then $u \in C^\infty(Q)$.

The integral

$$(6.22)\qquad \int_Q K(x-y,T-t)f(x,t)\,dx\,dt$$

is called the *heat potential* of the function f. It plays a similar role in the theory of parabolic equations as the Newtonian potential plays in the elliptic theory. In particular, the representation formula (6.21) can be used in a similar way to (2.18) to obtain estimates for solutions of (6.18), which can then be extended to more general parabolic equations by a perturbation argument similar to the one used in Section 4.

Finally, we give a brief description of one method of proving the existence of a solution of the first initial-boundary problem

$$(6.23)\qquad \begin{aligned} u_t &= \Delta u \text{ in } \Omega\times(0,\infty)\,,\\ u &= 0 \text{ on } \partial\Omega\times(0,\infty)\,,\\ u &= \phi \text{ on } \Omega\times\{0\}\,. \end{aligned}$$

For details see [F], Chapter 4, [Fr], [LSU], Section III.17 and [S], Lecture 12. Other methods of proving the existence of solutions are also discussed in [LSU]. For simplicity we consider only the homogeneous heat equation with zero data on the lateral boundary, but much more general problems can be treated by similar techniques.

First we look for solutions of the heat equation of the form

$$u(x,t)=\gamma(t)\psi(x)\,. \tag{6.24}$$

Clearly, for such solutions we must have

$$\frac{\gamma_t}{\gamma}=\frac{\Delta\psi}{\psi}=-\lambda \tag{6.25}$$

wherever $u\neq 0$ where λ is a constant. It follows that

$$\gamma(t)=\gamma(0)e^{-\lambda t}\,, \tag{6.26}$$

and that ψ must be a nontrivial solution of

$$\Delta\psi+\lambda\psi=0 \text{ in } \Omega\,,\quad \psi=0 \text{ on } \partial\Omega\,. \tag{6.27}$$

The boundary condition on ψ is imposed to make $u=0$ on $\partial\Omega\times(0,\infty)$.

From the maximum principle for elliptic equations we that (6.27) can have a nontrivial solution only if $\lambda>0$. The existence theory for (6.27) in the case $\lambda>0$ is not covered by the results in Sections 4 and 5, but we may proceed as follows (see [GT], Section 8.12, or [S], Lecture 9).

Let $\mathcal{H}=W_0^{1,2}(\Omega)$ and define

$$\lambda_1=\inf_{u\in\mathcal{H}-\{0\}}\frac{\int_\Omega|Du|^2}{\int_\Omega|u|^2}\,.$$

It can be shown that this infimum is achieved and that the minimizing function ϕ_1 is a generalized solution of (6.27) with $\lambda=\lambda_1$. We normalize ϕ_1 so that $\int_\Omega\phi_1^2=1$. By a similar argument it can be shown that there is a $\phi_2\in\mathcal{H}$ which is orthogonal to ϕ_1 in $L^2(\Omega)$ (and also in $\mathcal{H}$) such that $\int_\Omega\phi_2^2=1$,

$$\int_\Omega|D\phi_2|^2=\lambda_2=\inf_{\{u\in\mathcal{H}-\{0\}:u\perp\phi_1\}}\frac{\int_\Omega|Du|^2}{\int_\Omega|u|^2}\,,$$

and ϕ_2 is a generalized solution of (6.27) with $\lambda=\lambda_2$. Continuing inductively we obtain a sequence $\{\phi_j\}\subset\mathcal{H}$ with $\phi_j\perp\phi_k$ if $j\neq k$, such that $\|\phi_j\|_{L^2(\Omega)}=1$ for each j,

$$\int_\Omega|D\phi_j|^2=\lambda_j=\inf_{\{u\in\mathcal{H}-\{0\}:u\perp\phi_1,\cdots,\phi_{j-1}\}}\frac{\int_\Omega|Du|^2}{\int_\Omega|u|^2}\,,$$

and each ϕ_j is a generalized solution of (6.27) with $\lambda=\lambda_j$. In fact, by elliptic regularity theory (Theorem 5.6) each ϕ_j belongs to $C^\infty(\Omega)$, and to $C^\infty(\bar{\Omega})$ if $\partial\Omega\in C^\infty$.

The numbers $\{\lambda_j\}$ are called the *eigenvalues* of Δ and $\{\phi_j\}$ are the corresponding *eigenfunctions*. It can also be shown that each eigenvalue has finite multiplicity (i.e., is repeated at most finitely many times), the first eigenvalue λ_1 has multiplicity one, and $\lambda_j \to \infty$ as $j \to \infty$. Furthermore, $\{\phi_j\}$ is a complete orthonormal set in $L^2(\Omega)$.

Returning now to the initial-boundary problem (6.23), we see that if $\phi \in L^2(\Omega)$, then ϕ has an expansion

$$\phi(x) = \sum_{j=1}^{\infty} a_j \phi_j(x),$$

and at least formally the solution of (6.23) should be given by

$$u(x,t) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j t} \phi_j(x). \tag{6.28}$$

This turns out to be the case: the series converges in $L^2(\Omega \times (0,T))$ for any $T > 0$, and converges in $W^{1,2}(\Omega)$ for any fixed $t > 0$. Furthermore, this convergence is uniform for $t \geq \epsilon$ for any $\epsilon > 0$. The limit function u given by (6.28) is therefore a generalized solution of (6.23). Better regularity of u can then be deduced by applying a parabolic analogue of Theorem 5.6. In particular, $u \in C^\infty(\Omega \times (0,\infty))$, and if $\partial\Omega \in C^\infty$, then $u \in C^\infty(\bar{\Omega} \times (0,\infty))$. If also $\phi \in C^\infty(\Omega)$, then $u \in C^\infty((\bar{\Omega} \times (0,\infty)) \cup (\Omega \times \{0\}))$. However, it is not generally true that $u \in C^\infty(\bar{Q})$ if $\phi \in C^\infty(\bar{\Omega})$; further conditions on ϕ are needed to conclude this.

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