NONLINEAR OBLIQUE BOUNDARY VALUE PROBLEMS FOR TWO DIMENSIONAL HESSIAN AND CURVATURE EQUATIONS

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Our aim here is to describe some recent results concerning nonlinear oblique boundary value problems for two closely related classes of nonlinear elliptic equations—Hessian and curvature equations.

Hessian equations are equations of the form

$$F(D^2u) = g(x, u, Du) \tag{1}$$

where the function is of a special type. It is given by

$$F(D^2u) = f(\lambda(D^2u)) \tag{2}$$

where $\lambda(D^2u)$ denotes the vector of eigenvalues of the Hessian D^2u of a C^2 real valued function u and f is a smooth real valued symmetric function defined on a region $\Sigma \subset \mathbb{R}^n$. Equations of this type arise in many situations. In differential geometry Hessian equations arise in connection with the Minkowski, Christoffel and similar problems (see [16]). Typical examples are Poisson's equation

$$\Delta u = g\,, \tag{3}$$

and the Monge–Ampère equation

$$\det D^2u = g\,. \tag{4}$$

These correspond to the choices

$$f(\lambda) = \sum_{i=1}^{n} \lambda_i \tag{5}$$

and

$$f(\lambda) = \prod_{i=1}^{n} \lambda_i \tag{6}$$

respectively. Many other symmetric functions f give rise to interesting partial differential equations, for example,

$$f(\lambda) = \sigma_m(\lambda) = \mathop{}_{1\le i_1<\cdots<i_m\le n} \lambda_{i_1}\cdots\lambda_{i_m} \tag{7}$$

for m an integer between 1 and n. The cases (5) and (6) of course correspond to the choices $m = 1$ and $m = n$ respectively. Suitable powers and ratios of such functions are also of interest.

Curvature equations are obtained in a similar way to Hessian equations—in equation (2) we simply replace the eigenvalues of D^2u by the principal curvatures of the graph of u. Thus a typical curvature equation can be written as

$$F[u] = g(x, u, Du) \tag{8}$$

where F is given by

$$F[u] = f(\kappa)\,, \tag{9}$$

f is a symmetric function as above and $\kappa = \kappa(u)$ is the vector of principal curvatures of the graph of u. These are the eigenvalues of the second fundamental form

$$h_{ij} = \frac{D_{ij}u}{\sqrt{1+|Du|^2}} \tag{10}$$

relative to the metric

$$g_{ij} = \delta_{ij} + D_iuD_ju\,. \tag{11}$$

Equivalently, they are the eigenvalues of the matrix

$$a_{ij} = h_{ik}\,g^{kj} \tag{12}$$

where g^{ij} denotes the inverse of g_{ij}, and so is given by

$$g^{ij} = \delta_{ij} - \frac{D_iuD_ju}{1+|Du|^2}\,. \tag{13}$$

Notice that a_{ij} is generally not symmetric, but if desired, it can be replaced by a symmetric matrix having the same eigenvalues. Corresponding to the functions given by (5) and (6) we get the mean curvature equation

$$\sum_{i=1}^{n} D_i \left(\frac{D_iu}{\sqrt{1+|Du|^2}}\right) = g\,, \tag{14}$$

and the Gauss curvature equation

$$\frac{\det D^2u}{(1+|Du|^2)^{\frac{n+2}{2}}} = g\,. \tag{15}$$

The intermediate functions given by (7) with $1 < m < n$ also give rise to interesting partial differential equations, of which the scalar curvature equation ($m = 2$) is the most important.

To obtain existence results for equations of the above types it is reasonable to impose restrictions which force (1) and (8) to be elliptic. In the present setting this means that the matrix $F_{ij} = \frac{\partial F}{\partial r_{ij}}$ is positive definite. Here r_{ij} denotes a replacement variable for the second derivatives of u. It is not difficult to check that the positivity of $[F_{ij}]$ is equivalent to a condition on f, namely

$$(16) \qquad f_i = \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{on} \quad \Sigma \quad \text{for each} \quad i\,.$$

Thus we should seek solutions u which are *admissible* in the sense that $\lambda(D^2u)$, respectively $\kappa(u)$, belongs to a region Σ on which (16) holds. Suitable regions Σ are usually symmetric cones containing the positive cone $\Gamma_+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \ \forall\ i\}$. For f given by (7) we may take $\Sigma = \Gamma_m$ where for each integer m between 1 and n Γ_m is the component of the set $\{\lambda \in \mathbb{R}^n : \sigma_m(\lambda) > 0\}$ containing Γ_+. We then find that each Γ_m is an open symmetric convex cone with vertex at the origin, and

$$\Gamma_+ = \Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1 = \{\lambda \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i > 0\}\,.$$

If we seek solutions of this type, we must clearly take g positive.

The most basic boundary value problem for elliptic equations of the above types is the Dirichlet problem, where we seek a solution which takes prescribed values on the boundary $\partial\Omega$ of a region $\Omega \subset \mathbb{R}^n$. This problem has been resolved in recent years in varying degrees of generality for both Hessian and curvature equations. The Dirichlet problem for a very general class of Hessian equations was solved by Caffarelli, Nirenberg and Spruck [3]. For the special cases (7) some results were previously obtained by Ivochkina [7]. In addition, recent work of Krylov [10, 11, 12] embraces Hessian equations as special cases of a much larger family of equations. The theory for curvature equations is less developed. At the present time the Dirichlet problem for curvature equations corresponding to the function f given by (7) has been solved by Ivochkina [8, 9] and Trudinger [18, 19] under geometrically natural conditions on the data. Also, Caffarelli, Nirenberg and Spruck [4] have treated the Dirichlet problem for more general curvature equations, but only for zero boundary data on uniformly convex domains. In addition, the special cases (14) and (15) have been studied in their own right, using techniques which are specific to each of these cases; the contributions are too numerous to list here (see [6] for references).

Next we want to describe some recent work [20, 21] on other boundary value problems for Hessian and curvature equations. This is less developed than the corresponding theory for the Dirichlet problem, in that so far we have been able to treat only two dimensional problems. However, we hope to deal with higher dimensional problems in the future. So far these have been successfully handled only in certain special cases, for example [14] for the Neumann problem for Monge–Ampère equations, and [17] for the Neumann problem for general Hessian equations, but only on balls.

We now state our results in detail. From now on we will be interested only in the two dimensional case, so below Γ_+ denotes the positive cone in $\mathbb{R}^2$. First we consider boundary value problems for Hessian equations, of the form

$$F(D^2u) = g(x,u) \quad \text{in} \quad \Omega\,, \tag{17}$$

$$b(x,u,Du) = 0 \quad \text{on} \quad \partial\Omega\,. \tag{18}$$

We assume that $f \in C^\infty(\Gamma_+) \cap C^0(\overline{\Gamma}_+)$ is a positive function such that

$$f_i = \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{on} \quad \Gamma_+ \quad \text{for} \quad i = 1,2\,, \tag{19}$$

$$f \ \text{ is concave on } \ \Gamma_+\,, \tag{20}$$

$$f \equiv 0 \quad \text{on} \quad \partial\Gamma_+\,, \tag{21}$$

$$\sum f_i(\lambda)\,\lambda_i \geq 0 \quad \text{on} \quad \Gamma_+\,, \tag{22}$$

and

$$\mathcal{T} = \sum f_i(\lambda) \geq \sigma_0 \quad \text{on} \quad \{\lambda \in \Gamma_+ : f(\lambda) \leq \mu\} \tag{23}$$

for any $\mu > 0$ and some positive constant $\sigma_0 = \sigma_0(f,\mu)$.

These hypotheses (with Γ_+ replaced by a larger convex symmetric cone Γ with vertex at the origin) turn out to be natural and essentially optimal for Hessian and curvature equations, in all dimensions. Occasionally we need to add some extra conditions (such as condition (39) below). It can be shown that these conditions are satisfied by $f = {\sigma_m}^{1/m}$ on Γ_m, where σ_m is given by (7). In the two dimensional case $f(\lambda) = \sqrt{\lambda_1\lambda_2}$ and $f(\lambda) = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$ are examples on Γ_+.

The assumption that f is defined on Γ_+ rather than on a larger cone may seem restrictive at first sight, but it is appropriate in two dimensions. To see this we observe that the matrix $[F_{ij}(D^2u)]$ is diagonal if D^2u is diagonal, and its eigenvalues are f_1 and f_2. Evidently (f_1, f_2) is normal to the level lines of f, which are asymptotic to $\partial\Gamma_+$. If Γ_+ is replaced by a larger cone Γ, the level lines of f will be asymptotic to $\partial\Gamma$, so f_1/f_2 will be bounded between two positive constants depending on the aperture of the cone. This implies that the equation is uniformly elliptic. Nonlinear oblique boundary value problems for uniformly elliptic equations have been studied by Lieberman and Trudinger [13] in all dimensions, so we do not consider this case. In our situation, however, the equation is necessarily quite strongly nonuniformly elliptic, and results for this case turn out to be qualitatively different from the uniformly elliptic case, as well as requiring different techniques.

We now proceed to the remaining hypotheses. We assume that Ω is a smooth, uniformly convex domain in $\mathbb{R}^2$ and $g \in C^\infty(\bar{\Omega} \times \mathbb{R})$ is a positive function satisfying

$$g_z \geq 0 \quad \text{on} \quad \Omega \times \mathbb{R}. \tag{24}$$

For the boundary condition (18) obliqueness means that

$$b_p(x,z,p) \cdot \gamma(x) > 0 \tag{25}$$

for all $(x,z,p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2$, where γ denotes the inner unit normal vector field to $\partial\Omega$. It follows that any smooth oblique boundary condition (18) can be written in the form

$$D_\gamma u + \phi(x,u,\delta u) = 0 \quad \text{on} \quad \partial\Omega, \tag{26}$$

where ϕ is a smooth function defined on $\partial\Omega \times \mathbb{R} \times \mathbb{R}^2$ and δu denotes the tangential gradient of u relative to $\partial\Omega$ given by

$$\delta_i u = (\delta_{ij} - \gamma_i\gamma_j) D_j u .$$

It turns out to be convenient to consider several types of boundary conditions separately. For the semilinear boundary condition

$$D_\beta u + \phi(x,u) = 0 \quad \text{on} \quad \partial\Omega, \tag{27}$$

we assume that β is a smooth unit vector field on $\partial\Omega$ satisfying the strict obliqueness condition

$$\beta \cdot \gamma > 0 \quad \text{on} \quad \partial\Omega, \tag{28}$$

together with the structure condition

$$\left[-2\left(1+\left(\frac{\beta\cdot\tau}{\beta\cdot\gamma}\right)^2\right)\delta_i\beta_j(x) - \phi_z(x,z)\,\delta_{ij}\right]\tau_i\tau_j > 0 \tag{29}$$

for all $(x,z) \in \partial\Omega \times \mathbb{R}$, where τ is a unit tangent vector to $\partial\Omega$ at x.

We also assume that $\phi \in C^\infty(\partial\Omega \times \mathbb{R})$ satisfies

$$\phi_z < 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}, \tag{30}$$

$$\phi(x,z) < 0 \quad \text{for all} \quad x \in \partial\Omega \quad \text{and all} \quad z \geq N \tag{31}$$

for some constant N, and

$$\phi(x,z) \to \infty \quad \text{as} \quad z \to -\infty \tag{32}$$

uniformly for $x \in \partial\Omega$. Notice that (29) is automatically satisfied if $\beta \equiv \gamma$, or more generally if β is a vector field with constant normal and tangential components; this follows easily from (30) and the uniform convexity of Ω.

We then have the following result.

Theorem 1 *Under the above hypotheses the boundary value problem* (17), (27) *has a unique convex solution* $u \in C^\infty(\bar{\Omega})$.

For the fully nonlinear boundary condition (26) we assume that $\phi \in C^\infty(\partial\Omega \times \mathbb{R} \times \mathbb{R}^2)$ satisfies the conditions

$$\phi_z < 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R} \times \mathbb{R}^2, \tag{33}$$

$$\phi(x, z, 0) < 0 \quad \text{for all} \quad x \in \partial\Omega \quad \text{and all} \quad z \geq N \tag{34}$$

for some constant N, and

$$\phi(x, z, p^T) \to \infty \quad \text{as} \quad z \to -\infty \tag{35}$$

uniformly for (x, p) lying in any compact subset of $\partial\Omega \times \mathbb{R}^2$, where $p^T = p - (p \cdot \gamma(x))\,\gamma(x)$. We also assume that ϕ satisfies the concavity condition

$$\phi_{p_i p_j}(x, z, p^T)\,\tau_i \tau_j < 0 \tag{36}$$

for all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2$ where τ is a unit tangent vector to $\partial\Omega$ at x.

Theorem 2 *Under the above hypotheses the boundary value problem* (17), (26) *has a unique convex solution* $u \in C^\infty(\bar{\Omega})$.

It is known that for uniformly elliptic equations (29) is not necessary for the existence of smooth solutions—all we require is that β be a sufficiently smooth vector field satisfying (28). Likewise, condition (36) does not appear in the uniformly elliptic theory. Notice that (36) excludes the well known capillary boundary condition

$$D_\gamma u + \theta(x, u)\sqrt{1 + |Du|^2} = 0 \quad \text{on} \quad \partial\Omega, \tag{37}$$

unless θ is negative, a condition which is impossible to satisfy in our setting since we are seeking convex solutions (recall that γ is the inner normal). The capillary boundary condition is a typical nonlinear boundary condition studied in connection with uniformly elliptic equations, as well as with some nonuniformly elliptic ones, such as the mean curvature equation (14). In view of these remarks, it may appear that conditions (29) and (36) are an artifice of our method of proof, and are not really necessary. This is not so, for it is possible to construct examples for which the second derivatives of the solution become unbounded precisely where the conditions (29) and (36) fail (see [20], Section 6). We do not have such examples for the boundary condition (37); the classical solvability of (17) subject to (37) under the natural (in our context) restriction $\theta > 0$ remains an open problem.

Next we mention some analogous results for curvature equations of the form

(38) $$F[u] = g(x, u).$$

In addition to the above hypotheses we now also need an extra hypothesis on f, namely

(39) $$\mathcal{T} \to \infty \quad \text{as} \quad \lambda_2 \to \infty \quad \text{on} \quad \{\lambda \in \Gamma_+ : f(\lambda) \leq \mu\}$$

for any $\mu > 0$. This condition is undesirable in view of the fact that it is not satisfied by one of the more interesting examples, the harmonic mean curvature, which corresponds to the choice $f(\lambda) = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$. The reason we need this condition is that the left hand side of (38) has a gradient dependence. In fact, this condition (and sometimes an even stronger condition) is also necessary in the Hessian case if g depends on Du. Corresponding to Theorems 1 and 2 for Hessian equations we have the following analogous results for curvature equations.

Theorem 3 *If in addition to the above hypotheses there is a convex subsolution $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of equation* (38), *then each of the boundary value problems* (38), (26) *and* (38), (27) *has a unique convex solution $u \in C^\infty(\bar{\Omega})$.*

The hypothesis concerning the existence of a convex subsolution is required to ensure that the domain does in fact support a convex solution of the equation. For example, the only convex solution of the prescribed Gauss curvature equation (15) with $g \equiv 1$ on the unit ball B is $u(x) = -\sqrt{1 - |x|^2}$, up to an additive constant; furthermore, there are no convex solutions with $g \geq 1$ on any domain which properly contains B. The existence of a subsolution does not need to be assumed in Theorems 1 and 2 because it is trivial to construct convex subsolutions of (17).

It turns out that the proof of Theorem 3 for the semilinear boundary condition can be modified slightly to obtain some existence results without assuming the structure condition (29). Instead of (29) we need to impose an additional condition on f.

Theorem 4 *Suppose that the hypotheses of Theorem* 3 *are satisfied, with the exception of* (29), *and in addition we have*

(40) $$\lambda_1 \lambda_2 \to \infty \quad \text{as} \quad \lambda_2 \to \infty \quad \text{on} \quad \{\lambda \in \Gamma_+ : \mu_1 \leq f(\lambda) \leq \mu_2\}$$

for any positive constants $\mu_1 \leq \mu_2$. Then the boundary value problem (38), (27) *has a unique convex solution $u \in C^\infty(\bar{\Omega})$.*

There is also an analogue of Theorem 4 for Hessian equations. Its proof, however, depends on Theorem 4—we do not have a direct proof.

We now mention another type of boundary condition which is natural for convex solutions. For smooth uniformly convex functions the gradient map $Du : \Omega \to \mathbb{R}^2$ is one to one. Thus a natural problem is to prescribe the gradient image of u, i.e.,

$$Du(\Omega) = \Omega^* \tag{41}$$

for a given uniformly convex domain $\Omega^* \subset \mathbb{R}^2$. This can be reformulated in a more conventional way as

$$h(Du) = 0 \quad \text{on} \quad \partial\Omega \tag{42}$$

where h is a uniformly concave defining function for Ω^*, i.e., $\Omega^* = \{p \in \mathbb{R}^2 : h(p) > 0\}$ and $Dh \neq 0$ on $\partial\Omega^*$. It is not difficult to verify that (42) is a degenerate oblique boundary condition on convex functions, i.e.,

$$h_p(Du) \cdot \gamma \geq 0 \quad \text{on} \quad \partial\Omega\,. \tag{43}$$

However, we can show that for solutions of (17) and (38), (42) is a strictly oblique boundary condition in two dimensions, in the sense that there exists a positive constant c_0, depending only on $\Omega, \Omega^*, f, g, h$ and $\|u\|_{C^1(\bar\Omega)}$, such that

$$h_p(Du) \cdot \gamma \geq c_0 \quad \text{on} \quad \partial\Omega\,. \tag{44}$$

Once we know this the boundary condition (42) is very similar to the condition (26) with the concavity assumption (36).

To obtain existence results for (42) we need slightly different hypotheses on f and g. In addition to the hypotheses required above we also need to assume

$$\lambda_1\lambda_2 \leq G(f(\lambda)) \quad \text{for} \quad \lambda \in \Gamma_+ \tag{45}$$

for some real valued, continuous, increasing function G on $[0,\infty)$ with $G(0) = 0$. We also assume that g is a smooth positive function on $\bar\Omega \times \mathbb{R}$ satisfying

$$g(x,z) \to \infty \quad \text{as} \quad z \to \infty \tag{46}$$

and

$$g(x,z) \to 0 \quad \text{as} \quad z \to -\infty\,, \tag{47}$$

uniformly for all $(x,z) \in \bar\Omega \times \mathbb{R}$. We then have the following results.

Theorem 5 *Under the above hypotheses each of the boundary value problems* (17), (42) *and* (38), (42) *has a convex solution u belonging to $C^\infty(\bar\Omega)$. If in addition*

$$g_z > 0 \quad on \quad \Omega \times \mathbb{R}\,, \tag{48}$$

the solution is unique.

The boundary condition (41) for Monge–Ampère equations has received considerable attention. Pogorelov [15] proved the existence of generalized (i.e., not necessarily C^2) solutions. In two dimensions the interior regularity of the solution is then a consequence of well known results for two dimensional Monge–Ampère equations. The global regularity of solutions in two dimensions was established by Delanoë [5]. For higher dimensional Monge–Ampère equations subject to (41), the interior and global $C^{1,\alpha}$ regularity was recently proved by Caffarelli [1, 2]. Higher global regularity in dimensions greater than two remains an open problem.

We conclude this survey with a brief sketch of the ideas involved in the proof of Theorems 1 to 5. For simplicity we consider only the Hessian equation (17) with the semilinear boundary condition (27).

It is well known (see [6], Chapter 17) that to prove Theorem 1 it suffices to establish an *a priori* estimate

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \tag{49}$$

for some $\alpha > 0$ for any convex solution u of (17), (27). Of the various steps involved in proving (49) the only one which is neither easy nor a consequence of the uniformly elliptic theory is the second derivative bound, so we discuss only this step. For this we of course assume that u and Du have already been bounded.

First we need to obtain some second derivative bounds on $\partial\Omega$. If we differentiate the boundary condition (27) once in a tangential direction τ at any $x_0 \in \partial\Omega$, we get

$$|Du_{\tau\beta}u(x_0)| \leq C\,. \tag{50}$$

Next we estimate $D_{\beta\beta}u$ on $\partial\Omega$. To do this we need to derive a suitable differential inequality for $B = D_\beta u + \phi(x,u)$. After some computation we find that

$$|F_{ij}D_{ij}B| \leq C\mathcal{T} \quad \text{in} \quad \Omega\,, \tag{51}$$

and clearly $B = 0$ on $\partial\Omega$. If ψ is a uniformly convex defining function for Ω, then for A sufficiently large

$$F_{ij}D_{ij}(A\psi \pm B) \geq 0 \quad \text{in} \quad \Omega\,, \tag{52}$$

and consequently, by the maximum principle

$$D_\gamma(A\psi \pm B) \leq 0 \quad \text{on} \quad \partial\Omega\,. \tag{53}$$

This leads to

$$0 \leq D_{\beta\beta}u \leq C \quad \text{on} \quad \partial\Omega\,; \tag{54}$$

the lower bound is implied by the convexity of u.

Next we consider the function

$$w = w(x, \xi) = D_{\eta\eta} u + A|x|^2 , \tag{55}$$

where η ranges over all directions in $\mathbb{R}^2$. After some computation we find that for A fixed large enough

$$F_{ij} D_{ij} w \geq 0 \quad \text{in} \quad \Omega \tag{56}$$

for any η, so w has its maximum on $\partial\Omega$, say at $x_0 \in \partial\Omega$ and a direction ξ. By replacing ξ by its negative if necessary we may suppose that ξ points into Ω at x_0. Then (56) implies, by the maximum principle,

$$D_\beta w(x_0) \leq 0 , \quad D_\xi w(x_0) \leq 0 , \tag{57}$$

which can be written as

$$D_{\xi\xi\beta} u(x_0) \leq C , \quad D_{\xi\xi\xi} u(x_0) \leq C . \tag{58}$$

To proceed further we need to relate these inequalities to the boundary condition. If we tangentially differentiate (27) twice we find, after using the estimates (50) and (54), that

$$c_0 D_{\tau\tau} u(x_0) \leq D_{\tau\tau\beta} u(x_0) + C , \tag{59}$$

where τ is tangential to $\partial\Omega$ at x_0 and c_0 is a positive constant. In fact, c_0 is precisely the left hand side of (29) evaluated at $x = x_0 , z = u(x_0)$. Clearly, an upper bound for $D_{\tau\tau} u(x_0)$ follows if we can show

$$D_{\tau\tau\beta} u(x_0) \leq C . \tag{60}$$

This would follow from the first inequality of (58) if we knew that $\xi = \tau$. Unfortunately, we cannot assert this, nor have we been successful in modifying the choice of w to ensure that at a boundary maximum point ξ is necessarily tangential (although this can be done in the special case that Ω is a ball, see[17]). Our approach, therefore, is to show that the two inequalities (58) imply the bound (60), provided $D_{\xi\xi} u(x_0)$ is large enough. Essentially, the idea is to express $D_{\tau\tau\beta} u$ as a linear combination of $D_{\xi\xi\beta} u$ and $D_{\xi\xi\xi} u$ with positive bounded coefficients, modulo some well behaved correction terms. To do this we need to use the obliqueness of β and the once differentiated equation

$$F_{ij} D_{ijk} u = g_{x_k} + g_z D_k u , \quad k = 1 , 2 , \tag{61}$$

which gives us some crucial relations between certain third derivatives. Once we have bounded $D_{\tau\tau} u(x_0)$, all the second derivatives are bounded at x_0, by the obliqueness of β, and hence everywhere in $\bar{\Omega}$.

This approach does not appear to work in higher dimensions as there are too many third derivatives.

In the case of either of the fully nonlinear boundary conditions (26) and (42) the strategy is similar. The estimation of $D_{\tau\beta}u$ on $\partial\Omega$ is the same as before, (where now β denotes the vector field $b_p/|b_p|$ with b given by either (26) or (42)), while the estimation of $D_{\beta\beta}u$ on $\partial\Omega$ is a little more difficult. In place of (59) we obtain, by virtue of the concavity condition (36) or the uniform concavity of h, an even stronger estimate, namely

$$c_0\,(D_{\tau\tau}u(x_0))^2 \leq D_{\tau\tau\beta}u(x_0) + C\,, \tag{62}$$

where c_0 is a positive constant.

For curvature equations the ideas are similar in principle, but technically more difficult. One of the major differences arises in the reduction to a boundary estimate. It no longer suffices, as above, to consider a simple function such as (55). Instead we consider

$$w(x,\eta) = e^{\alpha(v-u)}\kappa(x,\eta) \tag{63}$$

where α is a positive constant to be determined, $\kappa(x,\eta)$ denotes the normal curvature of the graph of u at $(x,u(x))$ in a tangential (to the graph of u) direction η, and v is the unique convex solution of

$$\begin{aligned} F(D^2v) &= \delta_0 \quad \text{in} \quad \Omega\,, \\ D_\beta(v+\rho\psi) + \phi(x, v+\rho\psi) &= 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{64}$$

where δ_0 is a small positive constant and ψ is a uniformly convex defining function for Ω. It is in solving (64) that condition (29) is used—it is not needed anywhere else in the proof.

To conclude we mention that once we have the basic existence theorems stated above, we may obtain existence results under a variety of other hypotheses, but we still need to retain the essential structure conditions (29) and (36) on the boundary condition (except in Theorem 4 and its analogue for Hessian equations). We can obtain some results without assuming the monotonicity conditions (24), (30) or (36), and we can obtain $C^{1,1}$ solutions in certain degenerate situations, for example if g is nonnegative rather than positive, or if we do not have strict ineqality in (19).

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