Measurable conformal mappings in space

(The regularity theory of quasiregular mappings)

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Abstract

We survey some recent results in the theory of quasiregular mappings in higher dimensions, particularly those related to the regularity theory of the nonlinear Cauchy–Riemann and Beltrami systems.

1 Introduction

Geometric Function Theory in higher dimensions is largely concerned with generalisations to $\mathbf{R}^n$, $n \geq 3$ of aspects of the planar theory of holomorphic or conformal mappings (particularly the geometric and function theoretic aspects). In this sense it has been quite a successful theory with diverse applications. The category of maps that one usually considers in the theory are the so–called quasiregular mappings, or quasiconformal mappings if one wants injections. Both classes of mappings have the characteristic property of "bounded distortion". A useful idea when studying these mappings and their properties is to view quasiregular mappings as conformal maps between measurable conformal structures (ellipse fields) on subdomains of $\mathbf{R}^n$. This is the point of view which we shall adopt here and hence my title.

Here is a sample of successful and diverse applications of the theory (in no particular order of importance).

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- Gehring's versions of the Liouville Theorem and improved regularity theorems.
- Resetnyak's stability and rigidity phenomena.
- Mostow rigidity – uniqueness up to isometry of hyperbolic structures on closed manifolds ($n \geq 3$)
- Sullivan's Uniformisation Theorem – the existence of quasiconformal structures on topological n–manifolds ($n \neq 5$).
- Rickman's versions of the Picard Theorem and Nevanlinna Theory.
- Nonlinear potential theory, $\mathcal{A}$–harmonic function theory.
- Tukia–Väisälä's solution to the lifting problem and "quasiconformal geometric topology"
- Quasiconformal group actions as generalisations of conformal group actions.
- Donaldson and Sullivan's "Quasiconformal Yang–Mills Theory".
- Removable singularity theorems.

There are of course many more beautiful results and ideas in the theory. We direct the reader to the references [1], [14], [20] and [21] for the basic theory and an explanation and proof of some of the results cited above. However here I want to concentrate on the close relationships with nonlinear PDE theory. This connection is through the defining equations. However before getting to those we will need to make a few definitions. We define the Sobolev space $W^1_{p,loc}(\Omega, \mathbf{R}^n)$, $1 \leq p < \infty$, to be the space of functions defined in a subdomain Ω of euclidean n–space and valued in $\mathbf{R}^n$ and for which first derivatives are locally L^p–integrable. We now define a quasiregular mapping as follows.

Let Ω be a subdomain of $\mathbf{R}^n$ and $f : \Omega \to \mathbf{R}^n$. Then we say that f is *weakly K-quasiregular*, $1 \leq K < \infty$, if

1. $f \in W^1_{p,loc}(\Omega, \mathbf{R}^n)$,
2. $J_f(x) \geq 0$ a.e. Ω,

3. $\max_{|h|=1} |Df(x)h| \leq K \min_{|h|=1} |Df(x)h|$ a.e. Ω.

Here $Df(x)$ is the Jacobian derivative of f and $J_f(x)$ is the Jacobian determinant. The number K is referred to as the *dilatation* of f. We say that f is *K–quasiregular* if the Sobolev index $p = n$. Notice that $p = n$ is the natural exponent for *any* geometric theory since it implies integrability of the Jacobian. The defining assumptions are necessary for a good theory, they imply that change of variables holds, that sets of measure zero are preserved, $|E| = 0$ if and only if $|f(E)| = 0$ and that our mappings preserve orientation. We also say that f is *K-quasiconformal* if f is an injective K-quasiregular mapping. The basic connection between quasiregular mappings, nonlinear PDE's and nonlinear potential theory is via the *Beltrami system.* Let $S(n)$ denote the space of symmetric positive definite $n \times n$ matrices of determinant equal to 1. Given Ω a subdomain of $\mathbf{R}^n$ and $G : \Omega \to S(n)$ a bounded measurable mapping we define the Beltrami equation as

$$Df^t(x)Df(x) = J_f(x)^{2/n} G(x), \quad a.e.\ \Omega \tag{1}$$

In the special case that $G(x) \equiv Id$, the identity matrix, (or equivalently if $K = 1$), we have the *Cauchy–Riemann* system

$$Df^t(x)Df(x) = J_f(x)^{2/n} Id, \quad a.e.\ \Omega \tag{2}$$

That weakly quasiregular mappings satisfy (for suitable choice of G) the Beltrami equation is of course a tautology. However we want to examine properties and obtain geometric information about quasiregular mappings as viewed as solutions to this PDE. Notice that the distortion condition (3) above guarantees

$$K^{-2}|\zeta|^2 \leq \langle G(x)\zeta, \zeta\rangle \leq K^2|\zeta|^2 \tag{3}$$

so that G is uniformly elliptic and has bounded image in $S(n)$. However the Beltrami equation is fully nonlinear in dimension $n \geq 3$. When $n = 2$ we can use complex variables and rewrite the equation as

$$\frac{\partial}{\partial \overline{z}} f = \mu(z) \frac{\partial}{\partial z} f \tag{4}$$

where μ is a element of the unit ball of L^∞. The above (linear!) equation is known as the Beltrami equation. When $\mu = 0$ we obtain the Cauchy–Riemann equation. The existence of solutions to this equation and their regularity properties is fairly well–known, see [1]. Virtually nothing is known about the existence of solutions to the Beltrami system in dimension $n \geq 3$ except that it is not always possible to solve. To find good conditions on $G(x)$ which guarantee the existence of a solution is perhaps the most important outstanding problem in the area (and probably the hardest!). There are many important applications in geometry for a "good" solution to this problem.

Notice that a nonconstant solution to the Cauchy–Riemann system has the property that a.e. Ω the derivative of f is a scalar multiple of an orthogonal matrix. Thus infinitesimally the mapping f maps round objects to round objects. That is f is *conformal.* In the more general case of the Beltrami system we view $G(x)$ as an ellipse field on Ω. We declare that the ellipses defined by G are "round" and then $f : (\Omega, G) \to (\mathbf{R}^n, Id)$ maps round objects to round objects. We say that f is G–conformal, or when G is understood we say f is a measurable conformal mapping. Notice that we could also prescribe the measurable conformal structure in the target space, that is replace $(\mathbf{R}^n, Id)$ by $(\mathbf{R}^n, H)$ for some uniformly elliptic bounded measurable map $H : \mathbf{R}^n \to S(n)$. However this addition complication adds little to the theory for we essentially obtain the same class of mappings, the only thing changing is the exact value of the dilatation when measured against the background conformal structure $(\mathbf{R}^n, Id)$.

Before getting on to the more interesting and important aspects of the regularity theory of solutions to the Beltrami system, which is what this paper is mainly about, let us recall a few basic facts. If $f = (f^1, f^2, \ldots, f^n)$ is a solution to the Beltrami system, then $u = f^i$ satisfies the following equation of degenerate elliptic type:

$$div(\langle G^{-1}(x)\nabla u, \nabla u\rangle^{(n-2)/2} G^{-1}(x)\nabla u) = 0. \tag{5}$$

Thus u is a prototypical $\mathcal{A}$–harmonic function. Notice that in this divergence form it is not really necessary that $f \in W^1_{n,loc}(\Omega)$. Actually the above equations arise from consideration of the variational equation for the conformally invariant integral

$$I(f) = \int_\Omega \langle G^{-1}(x) Df^t(x), Df^t(x)\rangle^{n/2} dx \tag{6}$$

When $K = 1$ the equation (5) above reduces to the well–known n–harmonic equation,

$$div(|\nabla u|^{n-2}\nabla u) = 0, \tag{7}$$

which when $n = 2$ reduces to the usual Laplace equation. Actually, if f is a solution of the Beltrami system, then $u = \log|f|$ also satisfies an $\mathcal{A}$–harmonic type equation. In any case the standard theory of solutions to the $\mathcal{A}$–harmonic equation implies that quasiregular mappings are discrete (the inverse image of a point in the range is discrete in the domain Ω) and open. Roughly, openness follows from a weak maximum principle and discreteness follow from the fact that the polar sets $(\{x : \log|f(x)| = -\infty\})$ have conformal capacity zero, therefore Hausdorff dimension zero and therefore are totally disconnected. A topological degree argument gives discreteness from total disconnectedness of the pre–image. There are some other properties of quasiregular mapping which are worth recording here. First the property of local to global distortion control (Mori's Theorem). Suppose $f : \mathbf{R}^n \to \mathbf{R}^n$ is K-quasiregular, then there is $H = H(n, K)$ such that

$$\frac{\max_{|h|=r} |f(x+h) - f(x)|}{\min_{|h|=r} |f(x+h) - f(x)|} \leq H \tag{8}$$

for all x and all r. Notice that the definition of a quasiregular mapping only gives a bound (independent of x) on the limit supremum as $r \to 0$ of the left hand side. There is a local version of this result as well. Next we would like to mention a purely higher dimension phenomenon discovered by Zŏrich. Namely a locally injective globally defined function is globally injective. That is if $f : \mathbf{R}^n \to \mathbf{R}^n$, $n \geq 3$, is locally injective, then f is injective. A counterexample when $n = 2$ is given by the 1-quasiregular mapping e^z. Finally we would like to mention a stability result of Resetnyak. We recall that a Möbius transformation of $\overline{\mathbf{R}^n}$ is an orientation preserving mapping which is the finite composition of reflections in spheres and hyperplanes. The collection of all Möbius transformations is the index two orientation preserving subgroup of the group of transformations of $\overline{\mathbf{R}^n}$ generated by the similarity group and the inversion $x \to x/|x|^2$ in the unit sphere. It is clear every such mapping satisfies the Cauchy–Riemann equations. We shall see later that these mapping essentially exhaust the class of conformal mappings of space. Roughly speaking Resetnyak's results says that as $K \to 1$, K-quasiregular mappings are uniformly well approximated by conformal mappings. That is small dilatation (close to 1) implies close to Möbius.

2 The regularity theory of solutions to the Beltrami system

The major new idea that is utilised here in studying the solutions to the Beltrami system is to lift the equation to the exterior bundle and study the induced equations at that level. The systematic study of nonlinear equations for differential forms seems to have begun with work of L.M. Sibner and R.B. Sibner [17] and K. Uhlenbeck [19], see too [4]. In particular we mention Uhlenbeck' proof of the C^α–regularity theory for $\mathcal{A}$–harmonic tensors, proved using the Nash–Di Giorgi–Moser technique. Our initial investigations were motivated by the paper of S. Donaldson and D. Sullivan [3]. They used the standard Calderón–Zygmund theory of singular integrals to study the "quasiconformal Yang–Mills" theory. I think a lot of new ideas still remain to be extracted from that paper and used in a more general setting.

When lifted to the level of exterior algebra simplification can often occur. For instance, as we shall see, in certain circumstances equations may linearise. The results presented here were first given by T. Iwaniec and myself [10], [11], [12] in the case of even dimensions. In this case it is more or less possible to give the exact regularity theory and the best possible results. Iwaniec extended these results in a qualitative form to odd dimensions as well [8]. A principal feature of that work is the development of the Calderón–Zygmund theory for certain nonlinear p-harmonic operators. Many of the ideas used come from nonlinear Hodge theory. It is also clear that these techniques apply to a much wider class of mappings. Recently applications of these ideas have been found in nonlinear elasticity and nonlinear potential theory, see [9], [13], [16].

We define $\Lambda = \oplus \Lambda^k$ to be the graded Grassman algebra of exterior forms. Thus if $\alpha \in \Lambda$ we have $\alpha = \sum_{|I|=k} \alpha_I e^I$ where the sum is over multi–indices of length k, $I = (i_1, i_2, \ldots, i_k)$ and $e^I = e_{i_1} \wedge e_{i_2} \wedge \ldots e_{i_k}$. In this case α is a k-form. If $\beta = \sum_{|I|=k} \beta_I dx^I$ is another k–form, the inner product $\langle \alpha, \beta \rangle$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{|I|=k} \alpha_I \beta_I$$

The Hodge star operator $* : \Lambda^k \to \Lambda^{n-k}$ is defined by $*1 = e_1 \wedge e_2 \ldots \wedge e_n$ and $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle (*1)$. Recall that $** = (-1)^{n(n-k)} : \Lambda^k \to \Lambda^k$. The exterior algebra goes over pointwise to the agebra of differential forms.

Given a subdomain Ω of $\mathbf{R}^n$ we define $\Lambda(\Omega)$ to be the graded exterior algebra of differential forms whose coefficients are regular distributions. That is

$$\alpha = \sum_{|I|=k} \alpha_I(x) dx^I \tag{9}$$

and all the coefficients α_I are in $L^p_m(\Omega, \mathbf{R})$. Here we are really identifying the differential k-form α with an $L^p_m(\Omega, \Lambda)$ function defined on Ω and valued in the Grassman algebra Λ. The inner product between differential forms (of the same degree) $\alpha \in L^p(\Omega, \Lambda)$ and $\beta \in L^q(\Omega, \Lambda)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$, is defined as

$$(\alpha, \beta) = \int_\Omega \langle \alpha(x), \beta(x) \rangle dx$$

We recall the usual exterior derivative

$$d : \Lambda^{k-1} \to \Lambda^k$$

and the Hodge operator

$$d^* : \Lambda^k \to \Lambda^{k-1}$$

defined by $d^* = (-1)^{n(n-k)} * d *$. Then we define the Sobolev space of differential forms

$$W^1_p(\Omega, \Lambda) = L^p(\Omega, \Lambda) \cap L^p_1(\Omega, \Lambda) \tag{10}$$

In this setting we wish to point out that

$$\Delta = dd^* + d^*d : W^2_p(\Omega, \Lambda) \to L^p(\Omega, \Lambda) \tag{11}$$

is the usual Laplacian acting on coefficients

$$\Delta\alpha = \sum_I \Delta\alpha_I dx^I \tag{12}$$

Also $d^2 = (d^*)^2 = 0$. Here are a couple of basic facts.

Theorem 2.1 *Let* $\alpha \in L^p_1(\mathbf{R}^n, \Lambda^k)$ *and* $\beta \in L^q_1(\mathbf{R}^n, \Lambda^k)$, $1 < p, q < \infty$, $1/p + 1/q = 1$. *Then*

$$\int_{\mathbf{R}^n} (d\alpha, d^*\beta) dx = 0 \tag{13}$$

Another key result is the following Hodge decomposition theorem. We shall see below its application in the linear theory, however it is a stability property of the decomposition that actually enables the nonlinear theory to go through (though the results are somewhat less explicit).

Theorem 2.2 *For each* $\omega \in L^p(\mathbf{R}^n, \Lambda)$, $1 < p < \infty$, *there exist differential forms*

$$\alpha \in \ker d^* \cap L_1^p(\mathbf{R}^n, \Lambda)$$

and

$$\beta \in \ker d \cap L_1^p(\mathbf{R}^n, \Lambda)$$

such that

$$\omega = d\alpha + d^*\beta.$$

The forms $d\alpha$ *and* $d^*\beta$ *are unique and satisfy the uniform estimate*

$$\|\alpha\|_{L_1^p(\mathbf{R}^n)} + \|\beta\|_{L_1^p(\mathbf{R}^n)} \leq C_p(n)\|\omega\|_{L^p(\mathbf{R}^n)}$$

for some constant $C_p(n)$ *independent of* ω.

If f is a function in the Sobolev space $W^1_{pk,loc}(\Omega, \mathbf{R}^n)$ we can define the pull back via f of a differential (k-1)–form with smooth coefficients,

$$f^* : C^\infty(\Omega, \Lambda^{k-1}) \to L^p_{loc}(\Omega, \Lambda^{k-1})$$

by the rule

$$f^*\alpha(x) = \sum_I \alpha^I(f(x)) df^{i_1} \wedge df^{i_2} \wedge \ldots \wedge df^{i_{k-1}} \tag{14}$$

We then have the following theorem.

Theorem 2.3 *Let* $\alpha \in \Lambda^{k-1}(\mathbf{R}^n)$ *be a differential (k-1)–form with linear coefficients and* $f \in W^1_{pk,loc}(\Omega, \mathbf{R}^n)$, $p \geq 1$. *Then*

$$d(f^*(\alpha)) = f^*(d\alpha) \tag{15}$$

where the left hand side is understood in the sense of distributions.

A fundamental point to note here is that we may choose p and k so that $pk < n$. Thus when we lift a differential equation, such as the Beltrami system, up to the level of differential forms we may not need as much regularity to make sense of the induced equations. We have set up a bit of machinery and glossed over some of the more difficult technical details but in order to show the utility of these ideas let's work through an important example. Namely the Cauchy–Riemann system. Here we have $f \in W^1_{s,loc}(\Omega, \mathbf{R}^n)$, $f = (f^1, f^2, \ldots, f^n)$ and choose

$$s = \max\{k, n-k\} \tag{16}$$

Define the two functions u and v as follows

$$\begin{aligned} u &= f^k df^1 \wedge df^2 \wedge \ldots \wedge df^{k-1} = f^*(x^k dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{k-1}) \\ v &= f^{k+1} df^{k+2} \wedge df^{k+3} \wedge \ldots \wedge df^n \end{aligned}$$

Then from the Cauchy–Riemann system (via purely algebraic manipulations, see [10]) we find that u and v satisfy the following two equations

$$\begin{aligned} |du|^{p-2} du &= d^*v, \quad p = n/k \\ |d^*v|^{q-2} d^*v &= du, \quad q = n/(n-k) \end{aligned}$$

Now du and d^*v are forms whose coefficients are regular distributions. Applying d to the first equation, and d^* to the second equation we get the two equations

$$d^*(|du|^{p-2} du) = 0 \tag{17}$$

$$d(|d^*v|^{q-2} d^*v) = 0 \tag{18}$$

(see also (26)) Such functions u (and v) are called p–harmonic tensors. In both cases u and v are stationary points of the functional

$$\int_\Omega (|du|^2 + |d^*v|^2)^{p/2} \tag{19}$$

subject to closed forms v and coclosed forms u. Now notice that if n is even, $n = 2k$, then $p = q = 2$ and the two equations above are linear. We see that in fact u and v are weakly harmonic k-forms, the assumed degree of integrability of f is $s = n/2$. Since u and v are then harmonic by Weyl's lemma we see that the Jacobian determinant of f is the product of harmonic forms and is therefore real analytic. In fact notice that if we were to permute the component functions of f, we would see that the determinant of every $k \times k$, $k = n/2$, minor of the Jacobian matrix of f is harmonic. Expanding the determinant and using the Cauchy–Riemann equations again, we can write the Jacobian as

$$\binom{n}{k} J_f(x) = \sum_{J\,K} \left| \frac{\partial f^K}{\partial x^J} \right|^2 \tag{20}$$

so that $\sqrt{J_f(x)}$ is locally a Lipschitz function. This is enough now for the usual proofs of the Liouville Theorem to work, see [5], [2]. This will show that our 1-weakly quasiregular mapping is also quasiregular and actually the restriction (to the domain of definition) of a Möbius transformation. It is not too difficult to construct examples to show that the Sobolev exponent $n/2$ is best possible (in fact these examples can be constructed in all dimensions). We have therefore the following version of the Liouville Theorem [10].

Theorem 2.4 *Let $n = 2k > 2$. Then every weakly 1-quasiregular mapping f of Sobolev class $W^1_{k,loc}(\Omega, \mathbf{R}^n)$, is either constant or the restriction to Ω of a Möbius transformation of $\mathbf{R}^n$. The Sobolev exponent k is the lowest possible for the result to hold.*

One can see how this result and the ideas used in its proof are part of a more broad spectrum of results for general quasiregular mappings. Notice too that the regularity theory for the system is better than the regularity theory for the related n–harmonic equation.

Let $H : \Omega \to \mathcal{L}(\Lambda)$ be a bounded measurable function defined on Ω and valued in the symmetric linear transformations of the Grassman algebra $\Lambda = \Lambda(\mathbf{R}^n)$, and assume that H satisfies the ellipticity condition

$$\lambda^{-1}|\zeta|^2 \leq \langle H(x)\zeta, \zeta\rangle \leq \lambda|\zeta|^2 \tag{21}$$

for $(x,\zeta) \in \Omega \times \Lambda$ and where $\lambda \geq 1$ is a positive constant. In analogy with the classical case of conformally invariant integrals [7] we consider the functional

$$I[u] = \frac{1}{p}\int_\Omega \langle H(x)du, du\rangle^{p/2}, \quad u \in L^p_1(\Omega, \Lambda). \tag{22}$$

The derivative with respect to ζ of the integrand is the nonlinear mapping $\mathcal{A} : \Omega \times \lambda \to \Lambda$ defined by

$$\mathcal{A}(x,\zeta) = \langle H(x)\zeta, \zeta\rangle^{(p-2)/2} H(x)\zeta \tag{23}$$

Then $u \in L^p_1(\Omega, \Lambda)$ is a stationary point of I[u] if and only if it satisfies the $\mathcal{A}$–harmonic equation

$$d^*\mathcal{A}(x, du) = 0. \tag{24}$$

Putting $u = *v$ in the above leads to the dual $\mathcal{A}$–harmonic equation.

$$d\mathcal{A}^*(x, d^*v) = 0, \tag{25}$$

the subspace $D_s(L^2(R^n)) \cap H^m(R^n)$. By Theorem 1, $\int_{R^n} \frac{|\hat{f}(x)|^2}{|x|^{2s}} dx < \infty$. Let $k = \max\{2s+1, n\}$. Then adapting the proof of Theorem 3 in [4], it can be shown that there are vectors $a_1, a_2, \ldots a_k$ in R^n such that $\int_{R^n} \frac{|\hat{f}(x)|^2 (1+|x|^2)^m}{\sum_{j=1}^{k} \sin^{2s} \frac{<a_j, x>}{2}} dx < \infty$. For each $j \le k$, let g_j be the function defined by $g_j(x) = \frac{\hat{f}(x)(|1-e^{-i<a_j,x>}|^s)}{(1-e^{-i<a_j,x>})^s \sum_{j=1}^{k} |1-e^{-i<a_j,x>}|^s}$ a.e., then g_j is a function in $L^2(R^n)$. By the Plancherel Theorem, for each $j \le k$, there is f_j in $L^2(R^n)$ such that $\hat{f}_j = g_j$. Apply an argument similar to the proof of Theorem 6, it can be shown that for each $j \le k$, f_j is in $H^m(R^n)$. By the Fourier Inversion Theorem, $f = \sum_{j=1}^{k} (\delta_0 - \delta_{a_j})^s * f_j$ a.e. Thus, f is in $D_s(H^m(R^n))$. So, $D_s(L^2(R^n)) \cap H^m(R^n)$ is contained in $D_s(H^m(R^n))$. Hence, the result follows.

By the properties of Fourier Transform and Theorem 8, we can have the followings.

Theorem 9

Let $s > 0$ and $m \ge 0$ be given integers. Then $D_s(H^m(R^n))$ is a Hilbert space in the norm $\|\cdot\|_{m,s}$ given by $\|f\|_{m,s} = (\int_{R^n} |\hat{f}(x)|^2 (1+|x|^2)^m (1+|x|^{-2})^s dx)^{1/2}$.

Theorem 10

Let m be a given positive integer. Suppose f is a function in the Sobolev space $H^m(R)$ and f' is the distributional derivative of f. Then there are constants a_1, a_2, a_3 in R and functions f_1, f_2, f_3 in $H^{m-1}(R)$ such that $f' = \sum_{j=1}^{3} (\delta_0 - \delta_{a_j}) * f_j$ a.e.

Theorem 11

Let $m \geq 2$ and $n \leq 5$ be given positive integers. Let Δ be the Laplace operator on $H^m(R^n)$. Suppose f is a function in the Sobolev space $H^m(R^n)$. Then there are vectors a_1, a_2,...,a_5 in R^n and functions f_1, f_2,..., f_5 in $H^{m-2}(R^n)$ such that

$$\Delta f = \sum_{j=1}^{5} f_j - \frac{(\delta_{-a_j} + \delta_{a_j})}{2} * f_j \quad \text{a.e.}$$

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where $\mathcal{A}^*(x,\zeta) = *\mathcal{A}(x,*\zeta)$. In practise the mapping H will be the k^{th} exterior power of $G^{-1}(x)$ where $G(x)$ is defined by the Beltrami equation, $G(x) = J_f(x)^{-2/n}Df^t(x)Df(x)$, for a weakly quasiregular mapping f. (The exterior power of an $n \times n$ matrix B is found as follows: view the exterior basis $\{e_1, e_2, \ldots, e_n\}$ as n–vectors, then for $I = (i_1, 1_2, \ldots, i_k)$ we set

$$B_{\#}e^I = Be_{i_1} \wedge Be_{i_2} \wedge \ldots \wedge Be_{i_k}.$$

Now extend linearly to all of Λ). Given a weakly quasiregular mapping f we define (k-1)-forms u and v (in $L^{n/(n-1)}_{loc}(\Omega, \Lambda)$) as before from the components f^i of f. In [10] we derived the following relations from the Beltrami equations:

$$H(x)du^I = J_f(x)^{(2k-n)/n}d^*v^J, \tag{26}$$

which together with the identity $J_f(x) = \langle du^I, d^*v^J\rangle$ yields

$$J_f(x) = |du^I|^p_H = |d^*v^J|^q_{H^{-1}} \tag{27}$$

where $|\zeta|_H = \langle H(x)\zeta, \zeta\rangle^{1/2}$. This enables us to eliminate the determinant from equation (26) above from which we deduce the following first order differential equation.

$$\mathcal{A}(x, du^I) = d^*v^J \quad p = n/k \tag{28}$$

and the dual equation takes the form

$$\mathcal{A}^{-1}(x, d^*v^J) = du^I \quad q = n/(n-k) \tag{29}$$

Again applying the operators d and d^* as before we wind up with the p,q-harmonic equations

$$\begin{aligned} d^*\mathcal{A}(x, du^I) &= 0 \\ d\mathcal{A}^{-1}(x, d^*v^J) &= 0 \end{aligned}$$

Notice too that in the case $n = 2k$ both of these equations are linear. In this case one can apply the usual Calderón–Zygmund theory of singular integral operators to study the solutions of this problem. We shall just sketch what the results are, how they are obtained and point out a few of the ideas that are important for the nonlinear setting.

Associated with the Hodge decomposition is a singular integral operator

$$S : L^p(\mathbf{R}^n, \Lambda) \to L^p(\mathbf{R}^n, \Lambda)$$

defined by

$$S\omega = d\alpha - d^*\beta$$

where $\omega = d\alpha + d^*\beta$. Formally $S = (dd^* - d^*d) \circ \Delta^{-1}$ and so is represented by a $2^n \times 2^n$ matrix of products Riesz transforms. It is therefore of weak type (1,1). The theory of this operator, which we call the Beurling–Ahlfors transform in analogy with the classical two dimensional case, is of independent interest. Here is an explicit convolution formula for the operator S, see [10].

$$(S\omega)(x) = \left(1 - \frac{2k}{n}\right)\omega(x) - \frac{\Gamma(1 + n/2)}{\pi^{n/2}} \int_{\mathbf{R}^n} \frac{\Omega(x-y)\omega(y)}{|x-y|^n} dy$$

where

$$\Omega(\zeta) = M_{\#}(\zeta) + \left(\frac{2k}{n} - 1\right) Id : \Lambda^k \to \Lambda^k$$

and $M_{\#}$ is the k^{th} exterior power of the $n \times n$ matrix whose entries are homogeneous polynomials of degree 0 which we can write in tensor notation as

$$M(\zeta) = Id - 2|\zeta|^{-2}(\zeta \otimes \zeta)$$

There are a couple of interesting things to note. Firstly the obvious simplification which occurs when $n = 2k$. Secondly $M(\zeta) = |\zeta|^2 D\Phi(\zeta)$, where $D\Phi$ is the Jacobian matrix of the (anti–conformal) inversion in the unit sphere,

$$\Phi(x) = \frac{x}{|x|^2}.$$

It is the p-norms of the Beurling–Ahlfors transform S which control the regularity theory of solutions to the Beltrami equation. It is not difficult to see from the Hodge decomposition and duality that

$$\|S\|_2 = 1, \quad \|S\|_p = \|S\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{30}$$

We conjecture that $\|S\|_p = p - 1$, for $p > 2$. This would probably make the regularity results stated below (in the even dimensional case) best possible.

This is another significant unsolved problem in the theory (even in the planar case these norms are unknown, though there are estimates). We were able to show in [12] that $\|S\|_p \leq (n+1)\|A\|_p$ where A is the two dimensional Beurling–Ahlfors (or complex Hilbert) transform

$$A\omega(z) = -\frac{1}{2\pi \imath}\int\int_{\mathbf{C}} \frac{\omega(\zeta)d\zeta \wedge d\overline{\zeta}}{(z-\zeta)^2} \tag{31}$$

Here is the result which describes the regularity theory for solutions to the Beltrami system in higher even dimensions.

Theorem 2.5 *Let $n = 2k$, $\Omega \subset \mathbf{R}^n$ and $G : \Omega \to S(n)$ bounded and measurable. Let*

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

denote the ordered eigenvalues $\lambda_i = \lambda_i(x)$ of $G(x)$ and let

$$|\mu| = \sup_{x\in\Omega} \frac{\lambda_n\lambda_{n-1}\ldots\lambda_{k+1} - \lambda_k\lambda_{k-1}\ldots\lambda_1}{\lambda_n\lambda_{n-1}\ldots\lambda_{k+1} + \lambda_k\lambda_{k-1}\ldots\lambda_1} < 1$$

Suppose $f \in W^1_{pk}(\Omega, \mathbf{R}^n)$, $1 \leq p \leq 2$, is a solution of the Beltrami system

$$Df^t(x)Df(x) = J_f(x)^{2/n}G(x) \quad a.e.\ \Omega$$

and that

$$|\mu|\|S\|_p < 1$$

Then $f \in W^1_{qk}(\Omega, \mathbf{R}^n)$, where $q = p/(p-1) \geq 2$, is a quasiregular mapping.

A few things to note are that the assumption $|\mu| < 1$ together with f being a solution to the Beltrami system is equivalent to the assumption that f is weakly quasiregular. Actually $|\mu|$ is the norm of an operator μ which can be thought of as the Beltrami coefficient for f, again in complete analogy with the planar setting. Also, since $\|S\|_2 = 1$ and $|\mu| < 1$ the result always has nontrivial content. We start off with an assumption of integrability *below* the ambient dimension and deuce higher integrability *above* the ambient dimension. In particular this result includes (in even dimensions) Gehring's theorem on the higher integrability of the partial derivatives of quasiconformal mapping [6]. Further note that as $G \to Id$, $|\mu| \to 0$ and we see "in the limit" the regularity result stated for the Cauchy–Riemann system. From the regularity theory above (more particularly the method of proof) one can obtain Caccioppoli-type estimates *below the ambient dimension.* Such estimates have important geometric consequences, such as removability theorems. However these applications are the same in the linear setting (even dimensions) and the nonlinear setting (odd dimensions). So we shall discuss how one obtains qualitative versions of the above theorem in the nonlinear case.

In order to deal with the nonlinear case it is necessary to consider the nonhomogeneous $\mathcal{A}$-harmonic equation. This is arrived at in much the same way as the homogeneous case by looking at the variation of the integral

$$I(g,h)[u] = \int \langle H(g+du), g+du\rangle^{p/2} - p\langle du, h\rangle \tag{32}$$

for $g \in L^p(\mathbf{R}^n, \Lambda)$ and $h \in L^q(\mathbf{R}^n, \Lambda)$ with $1 < p, q < \infty$ conjugate indices. Existence and uniqueness of du follow from standard variational principles. The solution $u \in L_1^p(\mathbf{R}^n, \Lambda)$ is found from the Euler–Lagrange equation

$$d^*\mathcal{A}(x, g+du) = d^*h \tag{33}$$

which is what we call the nonhomogeneous $\mathcal{A}$-harmonic equation (here $\mathcal{A}$ is defined as before). The definition of the dual equation (the $\mathcal{A}^*$-harmonic equation) is similar. Fairly standard arguments lead to the usual estimates of the form

$$\|du\|_p^p \le C(n,p,\lambda)(\|g\|_p^p + \|h\|_p^p)$$

(where we recall that λ is the ellipticity constant associated to H). What we really want here is to obtain estimates like the above, but for exponents different from p. The following theorem is the crucial ingredient, [8].

Theorem 2.6 *For each $\mathcal{A}$-harmonic equation*

$$d^*\mathcal{A}(x, g+du) = d^*h$$

there is a constant $\nu = \nu(n,p,\lambda) > 0$ such that every weak solution u, with $du \in L^s(\mathbf{R}^n, \Lambda)$ and $p-\nu < s < p+\nu$, satisfies

$$\int_{\mathbf{R}^n} |du|^s \le C(p,\lambda) \int_{\mathbf{R}^n} \left(|g|^s + |h|^{s(p-1)}\right) \tag{34}$$

Of course similar estimates are valid for the $\mathcal{A}^*$–harmonic equation. Here the exponent p is regarded as the natural exponent for such estimates, the key feature of this estimate again is the fact that we are able to find uniform estimates above and below this natural exponent. Notice that the proof of such a result is not going to follow by using test function type arguments. What we mean by a weak solution u is that $du \in L^s(\Omega, \Lambda)$ and for each test form $\alpha \in L_1^{s/(s-p+1)}(\mathbf{R}^n, \Lambda)$ we have

$$\int \langle \mathcal{A}(x, g + du), d\alpha \rangle = \int \langle h, d\alpha \rangle.$$

From this we see that a desirable test for would be ηu for some C^∞ smooth cutoff function η. But this form fails to be in the space $L_1^{s/(s-p+1)}(\mathbf{R}^n, \Lambda)$. The idea of the proof is to consider what happens to nonlinear perturbations of exact forms and their Hodge decompositions. In particular if we are given $du \in L^s(\Omega, \Lambda)$ and we find the Hodge decomposition of $\omega = |du|^\epsilon du$ as $\omega = d\alpha + d^*\beta$, we would like to get good estimates relating the quantities ω, $d\alpha$ and $d^*\beta$ for small ϵ. That one can achieve these estimates is what I referred to earlier as the stability property of the Hodge decomposition. The result is proved using properties of harmonic functions and (to replace the Calderón–Zygmund theory) the Hardy–Littlewood maximal function. Roughly, one obtains under certain restrictions (on r and ϵ) that if $\eta \in L^{r(1+\epsilon)}(\mathbf{R}^n, \Lambda)$ and if $|\eta|^\epsilon \eta = d\alpha + d^*\beta$ with $\alpha, \beta \in L_1^r(\mathbf{R}^n, \Lambda)$, then we get the estimates

$$\begin{aligned} \|d^*\beta\|_r &\leq C(n) r |\epsilon| \|\eta\|_{r(1+\epsilon)}^{1+\epsilon} \quad \text{if } d\eta = 0 \\ \|d\alpha\|_r &\leq C(n) r |\epsilon| \|\eta\|_{r(1+\epsilon)}^{1+\epsilon} \quad \text{if } d^*\eta = 0 \end{aligned}$$

Because Theorem 2.6 is so central, let us sketch the proof for the case $p \geq 2$ (consideration of the $\mathcal{A}^*$–harmonic equation gives the case $1 \leq p \leq 2$, but it is not immediate). We first observe that for all $\zeta, \zeta' \in \Lambda(\mathbf{R}^n)$ and $s \geq p - 1 \geq 1$ we have (from the ellipticity of the matrix H) that

$$|\zeta|^s \leq (2\lambda)^p \langle \mathcal{A}(x, \zeta + \zeta'), |\zeta|^{s-p}\zeta \rangle + 4^s \lambda^{4s} |\zeta'|^s \tag{35}$$

(I don't claim that this is obvious, see [8] Lemma 9.1.) This means we can write

$$\int |du|^s \leq (2\lambda)^p \int \langle \mathcal{A}(x, g + du), |du|^{s-p} du \rangle + 4^s \lambda^{4s} \int |g|^s$$

Now look at the Hodge decomposition of $|du|^{s-p}du$. Since du is closed we have the advertised estimate on $\|d^*\beta\|_r$ in terms of $\|du\|_s^{s+1-p}$ which in turn gives a similar estimate on $\|d\alpha\|_r$. Substituting in the Hodge decomposition and using the definition of a weak solution we find that

$$\int |du|^s \leq (2\lambda)^p \left(\int \langle \mathcal{A}(x, g+du), d^*\beta\rangle \ + \ \langle h, d\alpha\rangle\right) + 4^s\lambda^{4s}\int |g|^s$$

Then applying both the Cauchy–Schwarz and Hölder inequalities, together with the estimates on $\|d^*\beta\|_r$ and $\|d^\alpha\|_r$ mentioned above, we obtain

$$\|du\|_s^s \leq C(n,p,\lambda)\left(\nu r\|g+du\|_s^{p-1}\|du\|_s^{s+1-p} + \|h\|_{s/(p-1)}\|du\|_s^{s+1-p} + \|g\|_s^s\right)$$

We now need to separate out the du term. This is done by means of Young's inequality. We first observe

$$\|g+du\|_s^{p-1} \leq 2^{p-2}(\|g\|_s^{p-1} + \|du\|_s^{p-1})$$

so that

$$\|g+du\|_s^{p-1}\|du\|_s^{s+1-p} \leq 2^{p-1}(\|g\|_s^s + \|du\|_s^s)$$

Also by Young's inequality

$$\|h\|_{s/(p-1)}\|du\|_s^{s+p-1} \leq 2^{4s-p-1}\lambda^{2s-p}\|h\|_{s/(p-1)}^{s/(p-1)} + 2^{p-3}\lambda^{-p}\|du\|_s^s$$

Putting this all together (and being a little more careful with the constants than I have been) gives

$$\|du\|_s^s \leq C(n,p,\lambda,s)\left(\|g\|_s^s + \|h\|_{s/(p-1)}^{s/(p-1)}\right) + \frac{1}{4}\|du\|_s^s + C(n,\lambda,p)r\nu\|du\|_s^s$$

which yields the desired result as soon as ν is chosen small enough.

Now we show how to apply the result to the theory of quasiregular mappings. As before we have $f = (f^1, f^2, \ldots, f^n) \in W_r^1(\Omega, \mathbf{R}^n)$. Choose an integer k and set

$$\begin{aligned} u_0 &= f^k df^1 \wedge df^2 \wedge \ldots \wedge df^{k-1} \\ v_0 &= *f^{k+1}df^{k+2} \wedge df^{k+3} \wedge \ldots \wedge df^n \end{aligned}$$

So $du_0 \in L_{loc}^{r/k}(\Omega, \Lambda^k)$ and $d^*v_0 \in L_{loc}^{r/(n-k)}(\Omega, \Lambda^k)$. Let ϕ be a smooth test function compactly supported in Ω. We recall the equation $\mathcal{A}(x, du_0) = d^*v_0$ and multiply both sides by ϕ^{n-k} to obtain

$$\mathcal{A}(x, \phi^k du_0) = \phi^{n-k}d^*v_0 \tag{36}$$

We now set $u = \phi^k u_0$ and $v = \phi^{n-k} v_0$. Then $\phi^k du_0 = g + du$ and $\phi^{n-k} d^* v_0 = h + d^* v$ with $g \in L^{r/k}(\mathbf{R}^n, \Lambda)$ and $h \in L^{r/(n-k)}(\mathbf{R}^n, \Lambda)$. Putting this in the above equation is how we arrive at the nonhomogeneous $\mathcal{A}$–harmonic equation,

$$d^* \mathcal{A}(x, g + du) = d^* h \tag{37}$$

Recall that the operator $\mathcal{A}$ is only defined on Ω, but we extend it to all of $\mathbf{R}^n$ by extending H via the identity. Now applying the estimate on weak solutions to the $\mathcal{A}$–harmonic equation we have

$$\int_{\mathbf{R}^n} |du|^s \le C(n, K) \left(\int_{\mathbf{R}^n} |g|^s + |h|^{s/(p-1)} \right) \tag{38}$$

if $|p - s| < \nu$ (recall K is the dilatation of f and so controls the ellipticity constants of G and H defining $\mathcal{A}$). We can replace du by $g + du$ in the above, from which we obtain

$$\int |\phi^k du_0|^{r/k} \le C(n, K) \int \left(|\phi^{k-1} u_0| |\nabla \phi|^{r/k} + |\phi^{n-k-1} v_0| |\nabla \phi|^{r/(n-k)} \right)$$

Since f is K-quasiregular we have pointwise estimates of the form

$$\begin{aligned} |u_0| &\le |f||Df|^{k-1} \\ |v_0| &\le |f||Df|^{n-k-1} \\ |du_0| &\le |Df|^k \le K^k |du_0| \\ |d^* v_0| &\le |Df|^{n-k} \le K^{n-k} |d^* v_0| \end{aligned}$$

This, together with the above, gives

$$\int |\phi^k Df|^r \le C(n, K) \int \left(|\phi Df|^{k-1} |f| \nabla \phi|^{r/k} + |\phi Df|^{n-k-1} |f| |\nabla \phi|^{r/(n-k)} \right)$$

To which we apply Hölder's inequality and a little computation to see

$$\|\phi Df\|_{r,\Omega} \le \| f |\nabla \phi| \|_{r,\Omega} \tag{39}$$

Thus we are able to obtain Caccioppoli estimates above and below the ambient dimension for quasiregular mappings, see (39). Using the estimates of equation (39) and a refinement of Gehring's lemma (reverse Hölder inequality) one obtains the following result which holds in *all* dimensions, see [8] Theorem 3.

Theorem 2.7 *For each $n \geq 2$ and $K \geq 1$ there are constants $q_{n,K} < n < p_{n,K}$ such that every weakly K-quasiregular mapping of class $W^1_{q_{n,K}}(\Omega, \mathbf{R}^n)$ belongs to the class $W^1_{p_{n,K}}(\Omega, \mathbf{R}^n)$. In particular f is K-quasiregular.*

It is to be noted that this result has nontrivial content even when $K = 1$!

As we mentioned earlier, the Caccioppoli estimates below the ambient dimension have important geometric implications. Here we give one such. It is the generalisation of the classical result of Besicovitch and Painlevé concerning removability sets for analytic mappings. A set $E \subset \Omega$ is said to be removable under bounded K-quasiregular mappings if every bounded K-quasiregular mapping $f : \Omega \backslash E \to \mathbf{R}^n$ extends to a K-quasiregular mapping of Ω. We denote by $dim_H(E)$ the Hausdorff dimension of the set E.

Theorem 2.8 *For each dimension $n \geq 2$ and $K \geq 1$, there is an $\epsilon_{n,K} > 0$ such that every closed set E of Hausdorff dimension $dim_H(E) < \epsilon_{n,K}$ is removable under bounded K-quasiregular mappings.*

Note the particular consequence that sets of Hausdorff dimension 0 are always removable (that is independently of K). Even this result is substantially stronger than any previous result on removability. S. Rickman [15] has constructed examples to show that this result is qualitatively best possible by constructing nonconstant bounded quasiregular mappings defined in the complement of a Cantor subset of $\mathbf{R}^n$. Such sets cannot be removable for these mappings. Here is the proof of that result. We need to use an analytic characterisation of Hausdorff dimension. A compact set $E \subset \mathbf{R}^n$ is said to have zero q–capacity, $1 < q \leq n$, if there is a sequence $\{\eta_j\}$ of smooth compactly supported functions such that

- $0 \leq \eta_j \leq 1$
- $\eta_j | E \equiv 1$
- $\lim_{j \to \infty} \eta_j(x) = \chi_E(x)$ for every $x \in \mathbf{R}^n$
- $\lim_{j \to \infty} \|\nabla \eta_j\|_q = 0$

A closed set has zero q–capacity if every compact subset has zero q–capacity. Now recall that for $q = n - \epsilon$, $0 < \epsilon < n - 1$, a closed set of Hausdorff dimension $dim_H(E) < \epsilon$ has zero q–capacity. We define $\epsilon_{n,K} = n - q_{n,K} > 0$, where $q_{n,K}$ is defined above in Theorem 2.7. So suppose $f : \Omega \backslash E \to \mathbf{R}^n$ and $dim_H(E) < \epsilon_{n,K}$. Then E has zero $q = n - \epsilon_{n,K}$ capacity. Since E has measure zero we can consider f to be a bounded measurable function defined on Ω. We first show $f \in W^1_{q,loc}(\Omega, \mathbf{R}^n)$. Choose a smooth compactly supported test function ϕ and consider the sequence of functions

$$\phi_j = (1 - \eta_j)\phi$$

Then $|\nabla \phi_j| \leq |\nabla \phi| + |\phi||\nabla \eta_j|$, and

$$D(\phi_j f) = \phi_j Df + f \otimes \nabla \phi_j \in L^q(\mathbf{R}^n, GL(n))$$

Now applying the Caccioppoli estimate (39) we have

$$\begin{aligned} \|D(\phi_j f)\|_q &\leq \||\nabla \phi_j| f\|_q + \|\phi_j Df\|_q \\ &\leq (1 + C(n,K))\||\nabla \phi_j| f\|_q \\ &\leq (1 + C(n,K))\|f|\nabla \phi|\|_q + \|\phi f|\nabla \eta_j|\|_q \end{aligned}$$

Now let $j \to \infty$. Since f is assumed bounded and $\phi_j \to \phi$ we have $\phi_j f \to \phi f$ in $L^q(\mathbf{R}^n, \mathbf{R}^n)$. Also $\phi f |\nabla \eta_j| \to 0$ in $L^q(\mathbf{R}^n, \mathbf{R}^n)$. So the inequalities above imply that the norms $\|D(\phi_j f)\|_q$ remain bounded as $j \to \infty$, so $\phi_j f$ converge weakly to ϕf in $W^1_q(\mathbf{R}^n, \mathbf{R}^n)$. Thus $\phi f \in W^1_q(\mathbf{R}^n, \mathbf{R}^n)$ and in the limit we obtain

$$\|D(\phi f)\|_q \leq C'(n,K)\|f|\nabla \phi|\|_q$$

Since ϕ was an arbitrary test function we find $f \in W^1_{q,loc}(\Omega, \mathbf{R}^n)$. Thus f is weakly K-quasiregular. Then the regularity theorem asserts that f is actually K-quasiregular, proving the result.

Finally let us point out that Besicovitch and Painlevé prove the result for bounded analytic mappings with the restriction that the linear measure of the set E should be zero. In even dimensions we are explicitly able to find $\epsilon_{n,K}$. If the conjecture concerning the p–norms of the Beurling–Ahlfors transform is correct, then $\epsilon_{n,K} \to n/2$ as $K \to 1$ (Notice the result is completely trivial in the case $K = 1$ in view of the Liouville Theorem). Thus the results are in accord (except that when $n = 2$ we have $n - 1 = n/2$! It is easy to construct bounded quasiregular mappings in the complement of sets of codimension 1 which cannot be extended).

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