Shadowing and approximation in dynamical systems*

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1 Introduction

Smoothness and hyperbolicity of a mapping $f : \Omega \to \Omega \subseteq \Re^d$ implies that a C^r dynamical system, generated by f, preserves many of its structural properties under small smooth perturbations. For instance, structural stability is present for large classes of smooth hyperbolic mappings, and the Shadowing Lemma will hold.

However, complicated behaviour of the orbits of f is often investigated computationally. Then f is replaced by a *computer realization* $\tilde{f}$. This realization involves some or all of the effects of

- finite machine arithmetic;
- a computational method;
- approximate evaluation of f.

In such a situation, it is important that exact orbits can be closely modelled during computation. That is, that the orbits of f and $\tilde{f}$ are close in

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some sense. *Shadowing* means that each trajectory of $\tilde{f}$ is close to some orbit of f. An inverse of this would determine that a given orbit of f is close to at least one orbit of $\tilde{f}$.

Computer modelling of $f : \Re^d \to \Re^d$ involves a finite discretized space $\mathbf{L} \subset \Re^d$ and a discretized mapping $f_{\mathbf{L}} : \mathbf{L} \to \mathbf{L}$ which is close to the restriction of f to $\mathbf{L}$. Since C^r approximation is not involved, it is not possible to use standard forms of the shadowing lemma, nor structural stability theorems for the analysis of computer models. On the other hand, for orbits of what we call *semi–hyperbolic* mappings, the inverse shadowing property holds in the C^0 sense. It is also possible to prove shadowing theorems for this class of mappings. Moreover, the criterion of semi–hyperbolicity is computationally simpler to treat than hyperbolicity itself.

2 Shadowing, Computer Robustness

A sequence $\{y_k\}_{k=a}^{k=b}$ in $\Re^n$ is a δ*–pseudo–orbit* of $f : \Re^n \to \Re^n$ if

$$||y_{k+1} - f(y_k)|| \leq \delta \quad \text{for all} \quad a \leq k \leq b\,.$$

An orbit $\{f^k(x)\}$ ϵ*–shadows* the δ–pseudo–orbit $\{y_k\}$ if

$$||f^k(x) - y_k|| \leq \epsilon \quad \text{for all} \quad a \leq k \leq b\,.$$

Shadowing Lemma. *Let Λ be a closed hyperbolic set. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ–pseudo–orbit in Λ is ϵ–shadowed.*

That is, each trajectory of a realization $\tilde{f}$ is near to some orbit of f. Even though computation does not produce the true orbit at the initial point, it does approximate to some true orbit.

To closely model any given exact orbit γ, we require that if $\tilde{f}$ is sufficiently close to f in some sense, then $\tilde{f}$ has at least one orbit close to γ. Certainly, if f is structurally stable and $\tilde{f}$ is close in some C^r, $r > 1$ topology, then topological properties of trajectories of f are preserved by the realization $\tilde{f}$. Various classes of hyperbolic mappings have this property. However, it is very difficult to verify that a set is uniformly hyperbolic and it is convenient to introduce another concept.

A four–tuple of nonnegative real numbers

$$\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u), \qquad \lambda_s \leq \lambda_u,$$

is called a *split* if the eigenvalues δ_1 and δ_2 of

$$\Delta = \begin{pmatrix} \lambda_s & \mu_s \\ \mu_u & \lambda_u \end{pmatrix}$$

are real and satisfy $|\delta_1| < 1 < |\delta_2|$. Clearly, $(\lambda_s, \lambda_u, \mu_s, \mu_u)$ is a split if and only if

$$\lambda_s < 1 < \lambda_u \tag{1}$$

and

$$(1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u. \tag{2}$$

Given some split s and a positive real number h, the map f is called (s, h)*–hyperbolic on the set* Ω if for any $x \in \Omega$ there exists a decomposition

$$T_x \Re^d = E_x^s \oplus E_x^u$$

with corresponding projectors P_x^s and P_x^u which satisfy the following inequalities:

$$\|P_{f(x)}^s Df_x u\| \leq \lambda_s \|u\|, \quad u \in E_x^s\ ; \tag{3}$$

$$\|P_{f(x)}^s Df_x v\| \leq \mu_s \|v\|, \quad v \in E_x^u\ ; \tag{4}$$

$$\|P_{f(x)}^u Df_x v\| \geq \lambda_u \|v\|, \quad v \in E_x^u\ ; \tag{5}$$

$$\|P_{f(x)}^u Df_x u\| \leq \mu_u \|u\|, \quad u \in E_x^s\ ; \tag{6}$$

$$\|P_x^s\|, \|P_x^u\| \leq h\ . \tag{7}$$

If a map is (s, h)–hyperbolic for at least one split it will be called *semi–hyperbolic.* This allows "leakage" rather than full invariance, and is on an open set containing the attractor.

Let $f : \Re^n \to \Re^n$ have a trajectory

$$\mathbf{x} = x_0, x_1, \ldots, x_N,\ x_k = f(x_{k-1}),\ k = 1, \ldots, N. \tag{8}$$

Let $||f - \phi||_\infty = \sup_{t \in \Re^n} ||f(t) - \phi(t)||$. Let α be a positive real number. The trajectory $\mathbf{x}$ will be called α*–robust* if there exists $\varepsilon_0 > 0$ such that any continuous mapping φ satisfying

$$||f - \varphi||_\infty \leq \varepsilon_0 \tag{9}$$

has at least one trajectory $y_0, y_1, \ldots, y_N$ such that

$$||y_n - x_n|| \leq \alpha ||f - \varphi||_\infty, \quad n = 0, 1, \ldots N. \tag{10}$$

This is a form of *inverse shadowing.*

Any trajectory $\mathbf{x}$ is $(1+L+\ldots+L^N)$–robust if the mapping f is Lipschitz with constant L in a neighbourhood of $\mathbf{x}$. Semi–hyperbolicity allows the robustness constant α to be independent of N uniformly throughout the domain of semi–hyperbolicity Ω.

Theorem 1. *Let $f : \Omega \to \Omega$ be semi-hyperbolic on an open set $\Omega \subseteq \Re^n$. Then there exists $\alpha > 0$ such that every finite trajectory $\mathbf{x} \subset \Omega$ is α–robust.*

Theorem 2. *Let f, Ω be as in Theorem 1 and suppose that Df is continuous on $\overline{\Omega}$. Then for every sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that every δ–pseudo–orbit $\{y_n\}$ of f is ε–shadowed by a true orbit $\{x_n\}$. In fact, each $(\mathbf{s}, h)$–hyperbolic trajectory of f is α–robust for every*

$$\alpha > \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h .$$

3 Proof of Theorem 1

Theorem 1 will follow from the following result:

Theorem 1a. *Each $(\mathbf{s}, h)$–hyperbolic trajectory of a smooth mapping f is α–robust for every*

$$\alpha > \alpha_*(\mathbf{s}, h) = \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h . \tag{11}$$

PROOF. Let $\mathbf{x} = x_0, x_1, \ldots, x_N$, be the given trajectory of the mapping f,

$$x_{n+1} = f(x_n), \qquad n = 0, 1, \ldots, N.$$

Denote by $\mathcal{B}$ the space of N–sequences

$$\mathbf{z} = z_0, z_1, \ldots, z_N, \quad z_n \in \Re^d, \tag{12}$$

satisfying

$$P^s_{x_0} z_0 = P^u_{x_N} z_N = 0. \tag{13}$$

The set $\mathcal{B}$ can be treated as a subspace of the Nd–dimensional vector space $\Re^d \times \ldots \times \Re^d$ (N times), with the norm

$$\|\mathbf{z}\| = \max_{0 \le n \le N} \|z_n\| \ .$$

Let $\varphi : \Re^d \to \Re^d$ be a given mapping. Define an operator $W_\varphi : \mathcal{B} \to \mathcal{B}$, which transforms every sequence (12) into a sequence $\mathbf{w} = w_0, w_1, \ldots, w_N$ defined by the initial conditions (13) and the relations

$$\begin{aligned} P^s_{x_n} w_n &= P^s_{x_n}(\varphi(x_{n-1} + z_{n-1}) - x_n) \ , \\ P^u_{x_{n-1}} w_{n-1} &= (U_n)^{-1}(P^u_{x_n} z_n - P^u_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1} + \\ & \quad P^u_{x_n}(-\varphi(x_{n-1} + z_{n-1}) + x_n + Df_{x_{n-1}} z_{n-1})) \ , \end{aligned}$$

where $U_n : E^u_{x_{n-1}} \to E^u_{x_n}$, defined by $U_n v = P^u_{x_n} Df_{x_{n-1}} v$, is surjective. Note that $(U_n)^{-1}$ is well–defined by virtue of the inequality (5). The following lemma is immediate.

Lemma 1 *W_φ is continuous. For any fixed point $\mathbf{z}^* = z_0^*, z_1^*, \ldots, z_N^*$ of W_φ, the sequence*

$$\mathbf{y}^* = x_0 + z_0^*, \ x_1 + z_1^*, \ \ldots, \ x_N + z_N^*$$

is a trajectory of the mapping φ.

We require a few more notations and definitions. For any $\beta > 0$, denote by $\delta_\beta(\varepsilon)$ the largest positive value δ such that, for any $\|z\| \le \delta$, the following inequality is valid:

$$\|x_n + Df_{x_{n-1}} z - f(x_{n-1} + z)\| \ \le \ \beta \, \varepsilon.$$

For each $\mathbf{z} \in \mathcal{B}$ define the pair of real numbers

$$V^s(\mathbf{z}) = \max_{0 \le n \le N} \|P^s_{x_n} z_n\| \ , \tag{14}$$

$$V^u(\mathbf{z}) = \max_{0 \le n \le N} \|P^u_{x_n} z_n\| \ , \tag{15}$$

and denote by $\mathbf{V}(\mathbf{z})$ the two-dimensional column vector with coordinates $V^s(\mathbf{z})$, $V^u(\mathbf{z})$. Define the matrix

$$M = \begin{pmatrix} \lambda_s & \mu_s \\ \mu_u/\lambda_u & 1/\lambda_u \end{pmatrix}, \tag{16}$$

and the column vector

$$\mathbf{h} = (h, h/\lambda_u)^T .$$

Lemma 2 *Let $\beta > 0$. Then for each continuous mapping φ and each $\mathbf{z}$ from the set $\mathcal{W}_{\varphi,\beta} = \{\mathbf{z} \in \mathcal{B} : \|\mathbf{z}\| \leq \delta_\beta(\|f - \varphi\|_\infty)\}$,*

$$\mathbf{V}(W_\varphi(\mathbf{z})) \leq M\,\mathbf{V}(\mathbf{z}) + (1+\beta)\|f-\varphi\|_\infty \mathbf{h} . \tag{17}$$

PROOF. First, estimate the value of $V^s(W_\varphi(\mathbf{z}))$. By definition

$$V^s(W_\varphi(\mathbf{z})) = \max_{0\leq n\leq N} \|v_n^s\| , \tag{18}$$

where

$$v_n^s = P_{x_n}^s(\varphi(x_{n-1} + z_{n-1}) - x_n) . \tag{19}$$

Rewrite (19) as

$$v_n^s = I_1 + I_2 + I_3 + I_4 , \tag{20}$$

where

$$\begin{aligned} I_1 &= P_{x_n}^s Df_{x_{n-1}} P_{x_{n-1}}^s z_{n-1} , \\ I_2 &= P_{x_n}^s Df_{x_{n-1}} P_{x_{n-1}}^u z_{n-1} , \\ I_3 &= P_{x_n}^s(\varphi(x_{n-1}+z_{n-1}) - f(x_{n-1}+z_{n-1})) , \\ I_4 &= P_{x_n}^s(f(x_{n-1}+z_{n-1}) - (f(x_{n-1}) + Df_{x_{n-1}} z_{n-1})) . \end{aligned}$$

From (3),

$$\|I_1\| \leq \lambda_s \|P_{x_{n-1}}^s z_{n-1}\| , \tag{21}$$

and from (4),

$$\|I_2\| \leq \mu_s \|P_{x_{n-1}}^u z_{n-1}\| . \tag{22}$$

The relations (7) imply that

$$\|I_3\| \leq h\,\|f-\varphi\|_\infty . \tag{23}$$

Lastly, the relations (7) and the definition of $\delta_\beta(||f-\varphi||_\infty)$ imply that

$$||J_4|| \;\leq\; h\beta\,||f-\varphi||_\infty\,. \tag{24}$$

From (20) and (21)–(24) it follows that

$$||v_n^s|| \;\leq\; \lambda_s\,||P^s_{x_{n-1}}z_{n-1}|| + \mu_s\,||P^u_{x_{n-1}}z_{n-1}|| + (1+\beta)\,||f-\varphi||_\infty h\,. \tag{25}$$

By (18) we can rewrite (25) as

$$V^s(W_\varphi(\mathbf{z})) \;\leq\; \lambda_s\,(V^s(\mathbf{z}) + \mu_s\,V^s(\mathbf{z}) + (1+\beta)\,||f-\varphi||_\infty\,h)\,. \tag{26}$$

Now estimate the value of $V^u(W_\varphi(\mathbf{z}))$. By definition,

$$V^u(W_\varphi(\mathbf{z})) = \max_{0\leq n\leq N}||v_n^u||\,, \tag{27}$$

where

$$\begin{aligned} v^u_{n-1} \;&=\; (U_n)^{-1}(P^u_{x_n}z_n - P^u_{x_n}Df_{x_{n-1}}P^s_{x_{n-1}}z_{n-1} + \\ &\quad P^u_{x_n}(-\varphi(x_{n-1}+z_{n-1}) + f(x_{n-1}) + Df_{x_{n-1}}z_{n-1}))\,. \end{aligned}$$

Rewrite this last equation as

$$v^u_{n-1} = (U_n)^{-1}J_1 + (U_n)^{-1}J_1 + (U_n)^{-1}J_2 + (U_n)^{-1}J_3 + (U_n)^{-1}J_4\,, \tag{28}$$

with

$$J_1 \;=\; P^u_{x_n}z_n\,, \tag{29}$$

$$J_2 \;=\; -P^u_{x_n}Df_{x_{n-1}}P^s_{x_{n-1}}z_{n-1}\,, \tag{30}$$

$$J_3 \;=\; P^u_{x_n}(-\varphi(x_{n-1}+z_{n-1}) + f(x_{n-1}+z_{n-1})\,, \tag{31}$$

$$J_4 \;=\; -f(x_{n-1}+z_{n-1}) + (f(x_{n-1}) + Df_{x_{n-1}}z_{n-1})\,. \tag{32}$$

The relations (5) and (29) imply that

$$||(U_n)^{-1}J_1|| \;\leq\; \lambda_u^{-1}\,||P^u_{x_n}z_n||\,, \tag{33}$$

while the relations (5), (6) and (30) imply that

$$||(U_n)^{-1}J_2|| \;\leq\; \lambda_u^{-1}\mu_u\,||P^s_{x_{n-1}}z_{n-1}||\,. \tag{34}$$

The relations (5), (7), (31) give

$$\|(U_n)^{-1}J_3\| \;\le\; \lambda_u^{-1}h\,\|f-\varphi\|_\infty\,. \tag{35}$$

Finally, the relations (5), (7), (32) and the definition of $\delta_\beta(\|f-\varphi\|_\infty)$ imply

$$\|(U_n)^{-1}J_4\| \;\le\; \lambda_u^{-1}h\beta\,\|f-\varphi\|_\infty\,. \tag{36}$$

From (28) and (33)–(36) it follows that

$$\|v^u_{n-1}\| \;\le\; \lambda_u^{-1}\left(\|P^u_{x_n}z_n\| + \mu_u\,\|P^s_{x_{n-1}}z_{n-1}\| + (1+\beta)\,\|f-\varphi\|_\infty\,h\right). \tag{37}$$

By (27) we can rewrite (37) as

$$V^u(W_\varphi(\mathrm{z})) \;\le\; \lambda_u^{-1}\left(V^u(\mathrm{z}) + \mu_u\,V^s(\mathrm{z}) + (1+\beta)\,\|f-\varphi\|_\infty\,h\right). \tag{38}$$

Inequalities (26) and (38) are equivalent to the assertion of the lemma. ∎

Let us return to and complete the proof of Theorem 1a. The spectral radius $\sigma(M)$ of the matrix

$$M = \begin{pmatrix} \lambda_s & \mu_s \\ \dfrac{\mu_u}{\lambda_u} & \dfrac{1}{\lambda_u} \end{pmatrix}$$

is just

$$\sigma(M) = \frac{1}{2}\left(\left(\frac{1}{\lambda_u} + \lambda_s\right) + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s\right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}}\right).$$

The entries of the matrix M are positive. Therefore by the Perron-Frobenius theorem the spectral radius $\sigma(M)$ is the maximal eigenvalue and the corresponding eigenvector has positive coordinates. Without loss of generality, assume that this eigenvector takes the form $(1,\gamma)^t$, where

$$\gamma = \frac{1}{2\mu_s}\left(\left(\frac{1}{\lambda_u} - \lambda_s\right) + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s\right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}}\right).$$

It follows that

$$\begin{pmatrix} \lambda_s & \mu_s \\ \dfrac{\mu_u}{\lambda_u} & \dfrac{1}{\lambda_u} \end{pmatrix}\begin{pmatrix} 1 \\ \gamma \end{pmatrix} = \sigma(M)\begin{pmatrix} 1 \\ \gamma \end{pmatrix}.$$

In $\Re^2$ introduce the auxiliary norm $\|\cdot\|_*$ by $\|(y_1, y_2)^t\|_* = \max\{\gamma|y_1|, |y_2|\}$. Clearly, the corresponding norm $\|M\|_*$ of the linear operator with the matrix (16) coincides with the spectral radius of M. Therefore, $\|My\|_* \leq \sigma(M)\|y\|_*$ for all $y \in \Re^2$. Hence, by Lemma 2, for any positive β we have

$$\|\mathbf{V}(W_\varphi(\mathbf{z}))\|_* \;\leq\; \sigma(M)\,\|\mathbf{V}(\mathbf{z})\|_* + (1+\beta)\,\|f-\varphi\|_\infty\,\|\mathbf{h}\|_*\,, \quad \mathbf{z} \in \mathcal{W}_{\varphi,\beta}\,. \tag{39}$$

Choose a fixed real number $\alpha > \alpha_*(\mathbf{s}, h)$, where $\alpha_*(\mathbf{s}, h)$ is defined by (11), and write $\beta = \alpha/\alpha_*(\mathbf{s}, h) - 1$. Note that by (1) and (2)

$$\sigma(M) \;<\; 1\,. \tag{40}$$

Clearly there exists $\varepsilon_0 > 0$ such that for

$$\|f-\varphi\|_\infty \;\leq\; \varepsilon_0 \tag{41}$$

we have the inclusion

$$\left\{\mathbf{z}:\; \|\mathbf{V}(\mathbf{z})\|_* \;\leq\; \frac{1+\beta}{1-\sigma(M)}\,\|\mathbf{h}\|_*\,(\|f-\varphi\|_\infty)\right\} \;\subseteq\; \mathcal{W}_{\varphi,\beta}\,.$$

By (39) and (40), for any f satisfying (41), the set

$$\mathcal{V}_{f,\beta} = \left\{\mathbf{z}:\; \|\mathbf{V}(\mathbf{z})\|_* \;\leq\; \frac{1+\beta}{1-\sigma(M)}\,\|\mathbf{h}\|_*\,(\|f-\varphi\|_\infty)\right\}$$

is invariant for the operator W_φ. Then, because of the continuity of W_φ (see Lemma 1), there exists a point $\mathbf{z}^*$ satisfying $W_\varphi \mathbf{z}^* = \mathbf{z}^*$, such that

$$\mathbf{z}^* \in \mathcal{W}_{\varphi,\beta}\,. \tag{42}$$

From (42) and (17) it follows that

$$\mathbf{V}(W_\varphi(\mathbf{z}^*)) \;\leq\; M\,\mathbf{V}(\mathbf{z}^*) + (1+\beta)\,\|f-\varphi\|_\infty\,\mathbf{h}\,,$$

and moreover that

$$\mathbf{V}(\mathbf{z}^*) \;\leq\; \frac{1+\beta}{1-M}\,\|f-\varphi\|_\infty\,\mathbf{h}\,.$$

In particular, $V^s(\mathbf{z}^*) + V^u(\mathbf{z}^*) \leq \alpha\|f-\varphi\|_\infty$. Further,

$$\max_{0\leq n\leq N} \|\mathbf{z}_n^*\| \;\leq\; \alpha\,\|f-\varphi\|\,. \tag{43}$$

By (43) and Lemma 1, for any continuous mapping φ satisfying (9) the sequence $x_0 + z_0^*$, $x_1 + z_0^*$, $\ldots$, $x_N + z_N^*$ is a trajectory of φ and satisfies (10). That is, the trajectory (8) is α–robust and the theorem is proved. ∎

4 Proof of Theorem 2

Let (s, h), $\mathrm{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ denote a split and associated projection norm bound. Denote by $\vartheta(\mathrm{s}, h)$ the largest constant ϑ satisfying

$$\begin{aligned}\vartheta &\leq h^{-2}\min\{1-\lambda_s, \lambda_u - 1\}\ ,\\ \vartheta &\leq \kappa = h^{-2}\frac{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u}{\lambda_s + \lambda_u + \mu_s + \mu_u}\ .\end{aligned}$$

Denote by $\ell^\infty(\Re^d)$ the space of all bounded sequences of vectors $x_n \in \Re^d$ with norm $||\{x_n\}||_{\ell^\infty} = \sup_n ||x_n||$.

We need two results to prove the theorem.

Inversion Lemma. *Let A_n be a sequence of $d \times d$, (s, h)-hyperbolic matrices. Let $\rho = \max\{\lambda_s + \mu_s\mu_u/(\lambda_u - \lambda_s), [\lambda_u - \mu_s\mu_u/(\lambda_u - \lambda_s)]^{-1}\}$. Then the bounded linear operator $\mathcal{L} : \ell^\infty(\Re^d) \to \ell^\infty(\Re^d)$ defined by*

$$(\mathcal{L}x)_n = x_{n+1} - A_n x_n$$

has a bounded right inverse $\mathcal{L}^{-1}$ satisfying

$$\left\|\mathcal{L}^{-1}\right\|_{\ell^\infty} \leq \frac{1+\tau}{1-\rho}\ ,$$

where $\tau = \max\{\mu_u/(\lambda_u - \lambda_s), \mu_s/(\lambda_u - \lambda_s)\}$.

Fixed Point Lemma. *Let E be a Banach space and $\mathcal{F} : E \to E$ be a C^1 map. let $y \in E$ be a point such that $D\mathcal{F}(y)^{-1}$ is a bounded linear right inverse of $D\mathcal{F}(y)$ and let $\varepsilon_0 > 0$ be chosen so that*

$$||D\mathcal{F}(x) - D\mathcal{F}(y)||_{\ell^\infty} \leq \frac{1}{2||D\mathcal{F}(y)^{-1}||_{\ell^\infty}}$$

for $||x - y|| \leq \varepsilon_0$. If $0 < \varepsilon \leq \varepsilon_0$ and

$$||F(y)||_{\ell^\infty} \leq \frac{\varepsilon}{2||D\mathcal{F}(y)^{-1}||_{\ell^\infty}}$$

then the equation $F(x) = 0$ has a unique solution x such that $||x - y|| \leq \varepsilon$.

The proof of the Fixed Point result can be found in [4]. It remains to prove the Inversion Lemma, before proceeding to the proof of Theorem 2.

PROOF OF INVERSION LEMMA. We may suppose that s is *minimal*, that is, λ_s, μ_s, μ_u are the least possible and λ_u the greatest possible for the split inequalities to work. Let the splitting associated with A_n be $\Re^d = T\Re^d = E_n^s \oplus E_n^u$, with projection $Q_n : \Re^d \to E_n^s$. The inequalities (3)–(7) become

$$\begin{aligned} ||Q_{n+1}A_nQ_nx|| &\leq \lambda_s||Q_nx|| \,, \\ ||Q_{n+1}A_n(I-Q_n)x|| &\leq \mu_s||(I-Q_n)x|| \,, \\ ||(I-Q_{n+1})A_nQ_nx|| &\leq \mu_u||Q_nx|| \,, \\ ||(I-Q_{n+1})A_n(I-Q_n)x|| &\geq \lambda_u||(I-Q_n)x|| \,, \\ ||Q_n|| \leq h \,,\ ||I-Q_n|| &\leq h \,, \end{aligned} \tag{44}$$

and hold for all $n = 0, 1, \dots$. We show that $\mathcal{L}$ is surjective. Let $z = \{z_n\} \in \ell^\infty(\Re^d)$ and define $\{x_n\}$ by

$$x_{n+1} = A_nx_n + z_n \,.$$

It is sufficient to show that $x \in \ell^\infty(\Re^d)$. Put $\xi_n = Q_nx_n$ and $\eta_n = (I-Q_n)x_n$. Then

$$\begin{aligned} \xi_{n+1} &= Q_{n+1}A_n\xi_n + Q_{n+1}A_n\eta_n + Q_{n+1}z_n \,, \\ \eta_{n+1} &= (I-Q_{n+1})A_n\xi_n + (I-Q_{n+1})A_n\eta_n + (I-Q_{n+1})z_n \,. \end{aligned}$$

Writing $u_n = ||\xi_n||$, $v_n = ||\eta_n||$ and using (44), obtain

$$u_{n+1} \leq \lambda_su_n + \mu_sv_n + \alpha_n \,,\ \alpha_n = ||Q_{n+1}z_n|| \,, \tag{45}$$

$$v_{n+1} \geq -\mu_uu_n + \lambda_uv_n - \beta_n \,,\ \beta_n = ||(I-Q_{n+1})z_n|| \,. \tag{46}$$

Clearly, $x \in \ell^\infty(\Re^d)$ if and only if $\{(u_n, v_n)^T\} \in \ell^\infty(\Re^{+2})$ (here ^{T} denotes the transpose). First, suppose that $\mu_s\mu_u = 0$. Then, if $\mu_s = 0$, (3) becomes

$$u_{n+1} \leq \lambda_su_n + \alpha_n \,,$$

which gives

$$u_{n+1} \leq \lambda_s^{n+1}u_0 + \sum_{k=0}^{n}\lambda_s^k\alpha_{n-k} = \gamma_n \leq \sum_{k=0}^{\infty}\lambda_s^k\alpha_{n-k} < \infty \,.$$

Hence, (4) may be rewritten as

$$v_n \leq \frac{1}{\lambda_u}(v_{n+1} + \mu_u\gamma_n + \beta_n) \,,$$

giving $v_n \leq \sum_{k=0}^{\infty} \lambda_u^{-k}(\mu_u\gamma_{n+k} + \beta_{n+k}) < \infty$. So $x \in \ell^\infty(\Re^d)$. If $\mu_u = 0$, the argument is very much the same. So, suppose that $\mu_s\mu_u \neq 0$. Let $\widehat{\lambda}_s$, $\widehat{\lambda}_u$ be the eigenvalues of the 2×2 matrix

$$\Delta_1 = \begin{pmatrix} \lambda_s & \mu_s \\ -\mu_u & \lambda_u \end{pmatrix} ,$$

and note that

$$\begin{aligned} \widehat{\lambda}_s &< \lambda_s + \frac{\mu_s\mu_u}{\lambda_u - \lambda_s} < \lambda_s + \frac{(1-\lambda_s)(\lambda_u - 1)}{\lambda_u - \lambda_s} \\ &= 1 - \frac{(1-\lambda_s)^2}{\lambda_u - \lambda_s} < 1 . \end{aligned}$$

Similarly, $\widehat{\lambda}_u > \lambda_u - \mu_s\mu_u/(\lambda_u - \lambda_s) > 1$. Consequently, there exists an invertible matrix C diagonalising Δ_1 and

$$C = \begin{pmatrix} 1 & c_{12} \\ c_{21} & 1 \end{pmatrix} ,$$

with

$$c_{12} > \mu_s/(\lambda_u - \lambda_s) \quad \text{and} \quad 0 < c_{21} < \mu_u/(\lambda_u - \lambda_s) . \tag{47}$$

This last may be seen by computing the eigenvectors of Δ_1 and using the inequalities for $\widehat{\lambda}_s$, $\widehat{\lambda}_u$. Writing $(u_n, v_n)^T = C(\widehat{u}_n, \widehat{v}_n)^T$, $(\alpha_n, \beta_n)^T = C(\widehat{\alpha}_n\widehat{\beta}_n)^T$, from (3) and (4) we obtain

$$\widehat{u}_{n+1} \leq \widehat{\lambda}_s\widehat{u}_n + \widehat{\alpha}_n \leq \widehat{\lambda}_s^{n+1} + \sum_{k=0}^{n} \widehat{\lambda}_s^k \widehat{\alpha}_{n-k} , \tag{48}$$

$$\widehat{v}_n \leq \widehat{\lambda}_u^{-1}\widehat{v}_{n+1} + \widehat{\lambda}_u^{-1}\widehat{\beta}_n \leq \sum_{k=n}^{\infty} \widehat{\lambda}^{n-k-1}\widehat{\beta}_k . \tag{49}$$

Here we have taken cognisance of the inequalities (47), which preserve the appropriate direction of the inequalities (3) and (4). Consequently, $\{(\widehat{u}_n, \widehat{v}_n)^T\} \in \ell^\infty(\Re^{2+})$ and hence so also is $\{(u_n, v_n)^T\}$. That is, $\{x_n\} \in \ell^\infty(\Re^d)$. The inequality for the norm of $\mathcal{L}^{-1}$ now follows immediately from the relations (6) and (7). ∎

PROOF OF SHADOWING THEOREM. The proof adapts and slightly simplifies those of [4], [5]. Define the nonlinear mapping $\mathcal{F} : \ell^\infty(\Re^d) \to \ell^\infty(\Re^d)$ by

$$(\mathcal{F}(x))_n = x_{n+1} - f(x_n) .$$

Then $\mathcal{F}$ is C^1 with derivative

$$(D\mathcal{F}(x)u)_n = u_{n+1} - Df_{x_n}u_n \,.$$

For $t \geq 0$, define

$$\omega(t) = \sup\left\{||Df(y) - Df(x)|| \;:\; x, y \in \overline{\Omega},\; ||y - x|| \leq t\right\} .$$

Now $\omega(t) \to 0+$ as $t \to 0+$. Define ε to be the largest positive number satisfying

$$\omega(\varepsilon) \leq \frac{1}{2L} , \text{ where } L = \frac{1+\tau}{1-\rho} ,$$

and ρ, τ are the constants appearing in the Inversion Lemma. From that lemma, $D\mathcal{F}(y)$ thus has a right inverse $D\mathcal{F}(y)^{-1}$ and

$$\left\|D\mathcal{F}(y)^{-1}\right\|_{\ell^\infty} \leq \frac{1+\tau}{1-\rho} = L \,.$$

Choose $\delta = \varepsilon/(2L)$. Now, if y is the pseudo-orbit sequence and x is any $\ell^\infty(\Re^d)$ sequence satisfying $||x - y||_{\ell^\infty} \leq \varepsilon$, then

$$\begin{aligned} ||D\mathcal{F}(x) - D\mathcal{F}(y)||_{\ell^\infty} &\leq \sup_n ||Df_{x_n} - Df_{y_n}|| \\ &\leq \omega(\varepsilon) \leq \frac{1}{2L} \leq \frac{1}{2\,||D\mathcal{F}(y)^{-1}||_{\ell^\infty}} \,. \end{aligned}$$

Hence, by the Fixed Point Lemma, $\mathcal{F}(x) = 0$ has a unique solution satisfying $||x - y|| \leq \varepsilon$. That is, the pseudo-orbit $\{y_n\}$ is ε−shadowed by the actual trajectory $\{x_n\}$. ■

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