TWO NOTES ON SUPERCRITICAL PROBLEMS ON LONG DOMAINS

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N.S.W. 2006, Australia

In this short paper, we discuss two results for the problem

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{on } D_n = \widetilde{\Omega} \times (-n, n) \\ u &= 0 \qquad\quad \text{on } \partial D_n \end{aligned} \tag{1}$$

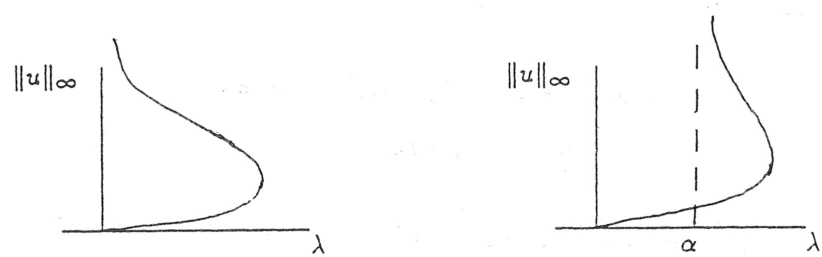
for large n. Here $\widetilde{\Omega}$ is a bounded domain in R^{m-1} with smooth boundary. In [2], we showed that the unstable solutions (unstable for the natural corresponding parabolic) are largely determined by a problem on the infinite string $\widetilde{\Omega} \times R$. Thus there seems no reason that they should behave like the solutions of the corresponding problem on $\widetilde{\Omega}$. (As seen in [2], this contrasts with the stable solutions.) Here we present an example for each m with $m \geq 3$ where the unstable solutions of (1) for all n are quite different to those of the lower dimensional problem and this holds uniformly in n. To do this, we consider f's which are asymptotically like y^q for large y where the nonlinearity is subcritical on $\widetilde{\Omega}$ but supercritical on D_n. Note that for many f's general results in nonlinear functional analysis ensure a general resemblance between the solution structure on $\widetilde{\Omega}$ and on D_n. Secondly, we show that on certain symmetric domains, we can bound the branch of symmetric solutions uniformly in n where the nonlinearity grows faster than the critical nonlinearity.

Here we choose q such that $(m+2)/(m-2) < q < (m+1)/(m-3)$ and consider f smooth convex such that $f(0) > 0$, $f(y) > 0$ for $y > 0$ and $f(y) \sim y^q$ as $y \to \infty$.

(We do not really need to assume that f is convex.) We also assume $\tilde{\Omega}$ is star shaped. We will prove that the problem

$$-\Delta' u = \lambda f(u) \text{ in } \tilde{\Omega}, \quad u = 0 \text{ on } \partial\tilde{\Omega} \tag{2}$$

(where Δ' denotes the Laplacian on R^{m-1}) has a unique small positive solution $u_1(\lambda)$ for all small positive λ. Any other positive solutions for small λ has large sup norm and at least one solution of the latter type for all small positive λ. On the other hand, we will prove that there exists $\alpha > 0$ independent of n such that for large n, (1) has a unique positive solution $u_1^n(\lambda)$ for $0 \le \lambda < \alpha$ and these solutions are uniformly bounded in λ and n in $C(D_n)$. Thus the two problems behave quite differently. See Fig. 1. Note that we really need to prove the result on D_n uniformly in n to ensure that the two solutions branches are not asymptotically alike.



on $\tilde{\Omega}$ on D_n

Fig. 1

We first prove the results for $\tilde{\Omega}$. By standard arguments, there exist $\tilde{\alpha}_1, \varepsilon > 0$ such that (2) has a unique positive solution $u_1(\lambda)$ with $\|u_1(\lambda)\|_\infty \le \varepsilon$ for each λ in $[0, \tilde{\alpha}_1]$. Moreover, if λ is small any other positive solution has large sup norm. Let us consider in a little more detail the construction of $u_1(\lambda)$. It is a fixed point of the mapping $A_\lambda : C_0(\tilde{\Omega}) \to C_0(\tilde{\Omega})$ defined by $A_\lambda(u) = \lambda(-\Delta')^{-1} f(u)$. Here $C_0(\tilde{\Omega})$ denotes the set of continuous functions on the closure of $\tilde{\Omega}$ vanishing on $\partial\tilde{\Omega}$. Let K denote the set of non-negative functions in $C_0(\tilde{\Omega})$. Since $u_1(\lambda) \to 0$ as $\lambda \to 0$, it is easy to prove that $i_K(A_\lambda, u_1(\lambda)) = 1$, for small λ where i_K denotes the fixed point index on K. (For example, one could apply Theorem 1 in [4]).

Next we prove that for each $\lambda > 0$, the sum of the indices of the positive solutions is zero (counted relative to the cone K). By the homotopy invariance of the degree, it suffices to prove that (2) has no positive solution for λ large and to prove that on any interval $[\mu, \nu]$ (where $\mu > 0$) there is an apriori bound in $C_0(\widetilde{\Omega})$ for positive solutions of (2) which holds uniformly in λ for $\lambda \in [\mu_1, \nu]$. To prove the first of these claims, we note that our assumptions on f ensure that there exists $\beta > 0$ such that $f(y) \geq \beta y$ on $[0, \infty)$. Hence, if λ is large ($\lambda > \gamma$), $\lambda f(y) > \widetilde{\lambda}_1 y$ on $(0, \infty)$ where $\widetilde{\lambda}_1$ is the first eigenvalue of $-\Delta'$ on $\widetilde{\Omega}$ for Dirichlet boundary conditions. It follows easily by scalar multiplying (2) by $\widetilde{\phi}$ (where $\widetilde{\phi}$ is the eigenfunction corresponding to $\widetilde{\lambda}_1$) that (2) has no positive solution for $\lambda > \gamma$. This proves the first claim of this paragraph. To prove the other claim, that is the apriori bound, we use a simple blowing up argument very similar to the proof of Lemma 1 in [3]. As in Remark 1 after Lemma 1 in [3], it is easy to prove the uniformity in λ on $[\mu, \nu]$. This completes the proof that the sum of the indices of the positive solutions is zero.

Since the sum of the indices of the positive solutions is zero for each $\lambda > 0$ and since $u_1(\lambda)$ has index 1 for small positive λ, it follows that there must be at least one other positive solution. By our earlier remarks, this other solution must have large sup norm (tending to infinity as $\lambda \to 0$). This proves our claim for $\widetilde{\Omega}$.

Remarks.

1. A slightly more careful blowing up argument shows that any large positive solution for small λ is of the form $\lambda^{-1/(q-1)}(v + o(1))$ where v is a non–trivial positive solution of

$$\begin{aligned} -\Delta' w &= a w^q \quad \text{in } \widetilde{\Omega} \\ w &= 0 \qquad \text{on } \partial\widetilde{\Omega}\,. \end{aligned} \tag{3}$$

In particular, if all the positive solutions of (3) are non–degenerate, and if $y^{1-q} f'(y)$ has a limit as $y \to \infty$, then it is not difficult to prove that the number of large positive solutions of (2) for small λ is equal to the number of non–trivial positive solutions of (3). In particular if $\widetilde{\Omega}$ is a disc (or is C^2 close to a disc), it follows from [5], [6] and [7] that (2) has a unique large solution for small λ. On the other hand by using the results in [6], one can construct star–shaped $\tilde{\Omega}$'s where (3) has many positive solutions for all small positive λ.

2. We used that $q < (m+3)/(m-1)$ in the blowing up argument. For this part of our argument we do not need that $\tilde{\Omega}$ is star shaped.

We now prove our claims for the positive solutions of (1) on D_n (and uniformly in n). The existence and local uniqueness of $u_1^n(\lambda)$ for $0 \le \lambda \le \delta$ where δ is independent of n) follows by a simple contraction mapping argument in $C_0(D_n)$ if we prove that $(-\Delta)^{-1}$ is uniformly bounded in n as a linear map of $C(D_n)$ into itself. It suffices to prove a uniform bound for the solution of

$$-\Delta u = 1 \quad \text{in } D_n$$
$$u = 0 \quad \text{on } \partial D_n$$

(by the positivity and since 1 is an order unit for $C(D_n)$.) This is obvious since $0 \le u(x', t) \le u_0(x')$ where u_0 is the solution of $-\Delta' u_0 = 1$ in $\tilde{\Omega}$, $u_0 = 0$ on $\partial\tilde{\Omega}$. Here $x' \in \tilde{\Omega}$, $t \in [-n, n]$. (To see the second inequality, one can use that u_0 is a supersolution for the equation for u.) This completes the proof of the uniform boundedness of $(-\Delta)^{-1}$.

This result has two other useful consequences. Firstly simple sup estimates applied to our equation imply that any positive solution other than $u_1^n(\lambda)$ must have large sup norm for λ small and this holds uniformly in n. Secondly, it follows easily from our estimate for $(-\Delta)^{-1}$ and a contraction mapping argument that $u_1^n(\lambda)$ are uniformly bounded for all large n and for $0 \le \lambda \le \delta$. By now applying standard $W^{2,p}$ estimates on sets $C_\beta = \tilde{\Omega} \times (\beta, \beta+1)$, we deduce that $u_1^n(\lambda)$ are uniformly bounded (in n, β and δ) in $W^{2,p}(C_\beta)$. Thus $u_1^n(\lambda)$ are uniformly bounded in the C^1 norm (by the Sobolev embedding theorem). We need this below.

It remains to prove that there is $\tilde{\delta} > 0$ independent of n such that (1) has no large positive solution for $0 \le \lambda \le \tilde{\delta}$ and large n. For fixed n, this is a nice result of Schaaf [12]. See also McGough [11]. We need to check that it can be done uniformly in n by examining the proof in [12]. We need to examine carefully the proof of Theorem 1 in [12]. It is necessary to have a copy of [12] to read this part. The proof proceeds by using a change of variable $u = u_1^n(\lambda) + v$. Thus v is a solution of

$$-\Delta v = \lambda f(u_1^n(\lambda)(x) + v) - \lambda f(u_1^n(\lambda)(x)) \equiv \lambda g_n(x, v, \lambda) \quad \text{in } D_n$$

$$v = 0 \quad \text{on } \partial D_n .$$

Note that v is non-negative because $u_1^n(\lambda)$ is the minimal positive solution. By our assumptions on f and by the uniform boundedness of the u_1^n's for $\lambda \leq \delta$, it is easy to check that the assumptions of (17) in [12] hold uniformly in n. Moreover, since we can choose $h_\delta(x) = \frac{1}{m}(x - x_0)$ (where $x_0 \in \widetilde{\Omega}$) by our star shapedness assumption and since $u_1^n(\lambda)$ are uniformly bounded in C^1, it is easy to check that (26) in [12] holds uniformly in n and that the $R(\alpha)$ there can be defined uniformly in n. It is also easy to check that (21) in [12] holds uniformly in n and that the α_2 defined after (27) in [12] can be chosen independent of n. (This uses a number of our remarks above on when inequalities hold uniformly in n.) It is also easy to check that the $r(\alpha)$ defined after (27) in [12] is bounded uniformly in n. Since $\lambda_1(\widetilde{\Omega} \times [-n, n]) = \lambda_1(\widetilde{\Omega})$ for large n (by separating variables), we eventually obtain by repeating the derivation of (28) in [12] that

$$\lambda r(\alpha_2) \int_{0 \leq v \leq \alpha} v^2 \geq \lambda_1(\widetilde{\Omega}) \left(\frac{1}{2} - \frac{1}{m} - R(\alpha_2) - 2\delta\right) \int_{D_n} v^2 .$$

Note that the derivation in [12] ensures that $\frac{1}{2} - \frac{1}{m} - R(\alpha_2) - 2\delta$ is positive and bounded below (uniformly in λ and n). If $r(\alpha_2) \leq 0$, it follows that $v \equiv 0$, while if $r(\alpha_2) > 0$, it follows that $v \equiv 0$ provided

$$\lambda < r(\alpha_2)^{-1} \lambda_1(\widetilde{\Omega}) \left(\frac{1}{2} - \frac{1}{m} - R(\alpha_2) - 2\delta\right) .$$

Thus in all cases we have a $\widetilde{\delta} > 0$ such that the original problem has a unique positive solution for $\lambda \leq \widetilde{\delta}$, as required.

Remarks.

1. With a little care, one could replace the star shapedness assumption by the condition that $M(\widetilde{\Omega}) < \frac{1}{2}$ for suitable q (with the notation of [12]). This follows since one can easily establish that $M(D_n) \leq M(\widetilde{\Omega})$ for all n.

2. If $m = 2$, our methods also cover the case of the Gelfand equation where $f(y) = \exp y$.
3. We could more easily obtain results when $f(0) = 0$ (where the minimal solutions are trivial) by using some of the ideas of Brezis and Nirenberg [1] (or as in §4 of Schaaf [12]). This covers some cases where the growth is critical on D_n rather than supercritical on D_n.
4. Our methods could also be used to obtain analogous results for even in t and decreasing in t (for $t \geq 0$) solutions u of $-\Delta u = \lambda f(u)$ in the infinite strip $\widehat{D} = \widetilde{\Omega} \times (-\infty, \infty)$ such that $u = 0$ on $\partial \widehat{D}$. For small λ, these solutions decay exponentially to the minimal solution which ensures that various integrals on the whole strip converge. This sort of result is to be expected because of the strong connection between the unstable solutions of (1) for large n and the problem on the infinite strip.

Secondly, we want to give a simple case where part of the branches continues all the way back to $\lambda = 0$ (with estimates independent of n) where we are supercritical. We take $\widetilde{\Omega}$ to be a true annulus in R^{m-1} with centre at the origin ($\widetilde{\Omega} = \{x' \in R^{m-1} : \mu < \|x'\| < 1\}$) and choose any $q > 1$. On D_n, we prove that there is a uniform bound (in n) for the positive solutions u of (1) of the form $u = u(r,t)$ where $r = \|x'\|$ for $\lambda \geq \delta$. Standard continuation arguments (applied in the subspace of functions which are functions of r and t only) then ensures that the branch continues back to $\lambda = 0$. I stress that I do not claim anything about the behaviour of the solutions which are not radial in x' (and which we expect to exist). We prove the required bound by blowing up arguments. However, we first need to consider where a solution u on D_n achieves its maximum. By standard Gidas–Ni–Nirenberg results, the maximum of a solution u_n on D_n must occur on $t = 0$. (This will eventually mean that the length of the domain in the t direction is unimportant.) Suppose by way of contradiction that u_n are solutions of (1) on D_n for $\lambda = \lambda_n$ such that $\lambda_n \geq a > 0$ for all n and $\{\|u_n\|_{\infty,D_n}\}$ is not bounded (with the obvious notation). By choosing a subsequence, we may assume that $\|u_n\|_{\infty,D_n} \to \infty$ as $n \to \infty$. Let $v_n(x) = (\|u_n\|_{\infty,D_n})^{-1} u_n((\|u_n\|_{\infty,D_n})^{2-q} x)$. By a simple and standard calculation v_n satisfies $-\Delta v_n = v_n^q + o(1)$ on a tube domain

$\widetilde{D}_n = \widetilde{\Omega}_n \times [-t_n, t_n]$, $\|v_n\|_{\infty,\widetilde{D}_n} = 1$, $v_n = 0$ on $\partial \widetilde{D}_n$ and the $o(1)$ term tends to zero uniformly as $n \to \infty$ (since $\|u_n\|_{\infty,D_n} \to \infty$ as $n \to \infty$.) Moreover, by the construction, $t_n \to \infty$ as $n \to \infty$ and $\widetilde{\Omega}_n = \{x' : \mu_n \leq \|x'\| \leq \nu_n\}$ where $\mu_n \to \infty$ and $\nu_n - \mu_n \to \infty$ as $n \to \infty$. In addition v_n is a function of r and t only and v_n achieves its maximum on $\widetilde{D}_n$ at points $(r_n, 0)$. Since $0 \leq v_n \leq 1$ on $\widetilde{D}_n$ standard local $W^{2,p}$ estimates ensure that v_n is bounded in C^1 uniformly in n away from the corners. Since r is uniformly large on $\widetilde{D}_n$ if n is large, it follows that $\frac{1}{r}\frac{\partial v_n}{\partial r} \to 0$ uniformly on $\widetilde{D}_n$ as $n \to \infty$ (away from the corners). Hence we find that $-\frac{\partial^2 v_n}{\partial r^2} - \frac{\partial^2 v_n}{\partial z^2} - v_n^q \to 0$ uniformly on $\widetilde{D}_n$ as $n \to \infty$ (away from the corners). By standard limiting arguments (cp. [8]) and by shifting the origin to $(r_n, 0)$, we find that a subsequence of $\{v_n\}$ converges uniformly on compact sets to a bounded positive solution v of $-\frac{\partial^2 v}{\partial r^2} - \frac{\partial^2 v}{\partial z^2} = v^q$ on R^2 or $\{(r,z) : r \geq \gamma\}$ or $\{(r,z) : r \leq \gamma\}$ and in the last two cases $v = 0$ when $r = \gamma$. In the last two cases, we can shift the r again (and in the last case replace r by $-r$), and obtain a bounded positive solution of $-\frac{\partial^2 v}{\partial r^2} - \frac{\partial^2 v}{\partial z^2} = v^q$ on the half space $T = \{(r,z) : r \geq 0\}$ such that $v = 0$ on ∂T. In the first case, we obtain a contradiction by applying Theorem 5.1 in Gidas and Spruck [9] while in the last two cases, we obtain a contradiction by applying Theorem 1.3 in Gidas and Spruck [9]. Thus we have the required apriori bound and the proof is complete.

Remarks.

1. Clearly this idea can be used in some other symmetric situations. Note however that our argument uses essentially that r is bounded away from zero on the annulus. If that fails, it is unclear what can be said.
2. If $m = 2$, we can also obtain analogous results where $f(x) = \exp y$. We merely sketch the proof. It suffices to obtain the apriori bound. Firstly, the Gidas, Ni–Nirenberg theorem implies that the maximum of v_n can not occur for r close to 1. Moreover we use an inversion in the x' variables only (cp. the proof on p.223 of [7]) one can show that the maximum of v_n can not occur near $r = \mu$. (It is here that we use $m = 2$. We also need it later.) Thus the maximum of v_n occurs at a point $(r_n, 0)$ not close to the boundary of ∂D_n. We can easily modify the arguments in Spruck [13] and Kielhofer

[10], §2 to bound $\exp(\frac{1}{2}(1-\alpha)u)$ ii. .. *locally* near $t = 0$ for $0 < \alpha < 1$, uniformly in n. The extra term $\frac{1}{r}\frac{\partial u}{\partial r}$ in our equation compared with theirs is easily seen to be rather harmless. By the Sobolev embedding theorem, it follows that $\exp u$ is bounded in $L^p_{\text{loc}}(D_n)$ near $t = 0$ for all p (uniformly in n). The regularity theory for $-\Delta$ then implies that $u \in L^\infty_{\text{loc}}(D_n)$ near $t = 0$ uniformly in n. The various increasing properties of u (including our earlier remarks) now implies a uniform (in n) bound for u in $L^\infty(D_n)$ (for $\lambda \geq \delta > 0$). Here our claim follows.

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