EVOLVING CONVEX HYPERSURFACES

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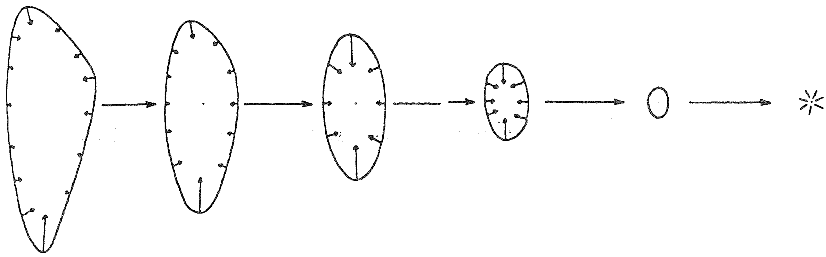
ABSTRACT. This talk summarises a variety of results concerning evolution equations for convex hypersurfaces: Simple proofs are sketched for some of the main results; a few open problems are described, along with some of the clues which suggest what the answers might be; and several applications to geometric problems are given.

1. INTRODUCTION.

The mean curvature flow is a well-known example of a parabolic evolution equation for hypersurfaces. The considerable attention which this equation has received in recent years is due in large measure to the elegant results proved for the case of convex hypersurfaces in the papers of Huisken [Hu1], Gage [Ga1–2] and Gage & Hamilton [GH]. In these results, one considers an initial convex hypersurface M_0, given by an embedding $\varphi_0 : M^n \to \mathbf{R}^{n+1}$, where M^n is some smooth compact manifold. The idea is to make this map evolve, producing a family of embeddings $\varphi : M^n \times [0,T) \to \mathbf{R}^{n+1}$, describing a family of hypersurfaces $\{M_t = \varphi_t(M)\}$. The evolution equation for the mean curvature flow is the following:

(1)
$$\frac{\partial}{\partial t}\varphi(z,t) = -H(x,t)\nu(x,t)$$
$$\varphi(z,0) = \varphi_0(z)$$

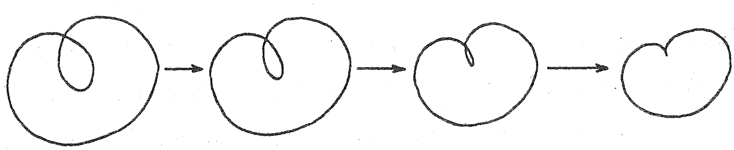
where H is the mean curvature, and ν is the outward unit normal.



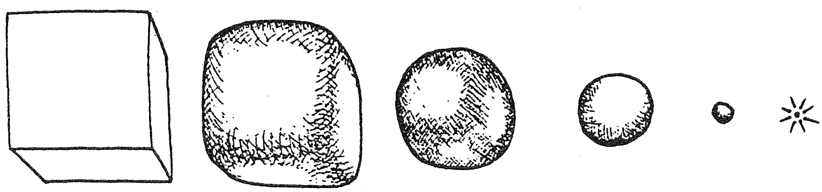
The main result describing the solutions of this equation (compiled from [Hu1] for the case $n \geq 2$ and [GH] for $n = 1$) is the following:

Theorem 1. *Let φ_0 be a smooth, strictly convex immersion. If $n = 1$, assume further that φ_0 is an embedding. Then there exists a unique, smooth solution φ to equation (1) on a finite maximal time interval $[0, T)$. φ converges uniformly to a single point p in $\mathbf{R}^{n+1}$ as t approaches T, and the rescaled solutions given by $\tilde{\varphi}_t = V(M_t)^{-\frac{1}{n+1}}(\varphi_t - p)$ converge in C^∞ to an diffeomorphism $\varphi_\infty : M^n \to S^n \subset \mathbf{R}^{n+1}$ as t approaches T. Here $V(M_t)$ is the enclosed volume of the hypersurface $M_t = \varphi_t(M)$.*

Remarks. The requirement of embeddedness for $n = 1$ cannot be avoided, as there are well-known examples where cusp-like singularities develop for immersed convex curves.



In higher dimensions, there are no non-embedded convex immersions, so we can combine the conditions in a natural way: Let D be an arbitrary bounded, convex region in $\mathbf{R}^{n+1}$, and let M_0 be the boundary of D. Then there exists a smooth solution φ : $M^n \times (0, T)$, unique up to smooth diffeomorphisms of M^n, such that $\varphi_t(M^n)$ converges to M_0 in the Minkowski distance as t approaches zero, and φ_t converges to a point, becoming round as in the theorem, as t approaches T. The existence, uniqueness, and regularity of a solution with such boundary data follows from interior estimates proved by Ecker and Huisken [EH]. These, together with the strong parabolic maximum principle, show that the solution immediately becomes strictly convex and smooth, so that we can apply the theorem as stated above.



This result serves as a model of perfection for many of the later results in the field. Results of similar generality have been obtained some other situations — most notably the result of Grayson [Gr], which removes the assumption of convexity in the case $n = 1$ of the above theorem, and the results of Urbas [U2] and Gerhardt [Ge] concerning "expansion flows" of star-shaped hypersuraces. However, in general circumstances no such nice result holds.

In this paper I will discuss a family of evolution equations with a form similar to the mean curvature flow, concentrating on the case of convex hypersurfaces. I will show that the good results for the mean curvature flow are very robust, and can be extended to a wide range of evolution equations satisfying a few structural conditions. Good results can be obtained not only for hypersurfaces in Euclidean space, but also in very general Riemannian spaces, if the evolution equation is chosen carefully. These results lead to some satisfying applications to geometric problems.

The overall picture of the behaviour of these geometric parabolic equations is still not clear, however — even for the convex case. In particular, little is known about flows where the speed of the hypersurfaces is given as a function of curvature which is not homogeneous of degree one. I will describe some partial results which give an indication of what to expect, and also some more complete results for special cases.

The general form of the equations I will consider is as follows: As before, we consider a convex embedding $\varphi_0 : M^n \to \mathbb{R}^{n+1}$, and try to evolve this under an evolution equation of the following form:

$$\frac{\partial}{\partial t}\varphi(z,t) = -F(z,t)\nu(z,t); \qquad (2)$$
$$\varphi(z,0) = \varphi_0(z).$$

This time F is to be given as a function $F(\mathcal{W})$ of the Weingarten map $\mathcal{W} : TM^n \to TM^n$. $\mathcal{W}$ describes the curvature of $\varphi(M)$, and may be identified with the derivative of the Gauss map $\nu : M^n \to S^n$. The main further conditions we require on F are just those which ensure that the equation (2) is parabolic: Assume that F is a smooth function defined on the positive cone $\mathcal{S}_+$ of positive definite symmetric maps, such that

$$\frac{\partial}{\partial s}F(A+sB,\nu)\Big|_{s=0} > 0 \qquad (3)$$

for every A in $\mathcal{S}_+$, whenever B is a symmetric, nonzero, non-negative map. It should be noted that the form of F means that it can be written as a symmetric function of the principal curvatures: $F(\mathcal{W}) = f(\lambda_1,\ldots,\lambda_n)$. Then the condition (3) is just equivalent to the requirement that f be monotonic increasing in all n variables.

This leaves a lot of room for different examples of flows. Some useful examples are the following: The elementary symmetric functions are defined by

$$E_k(\mathcal{W}) = \frac{1}{\binom{n}{k}}\mathrm{tr}\Lambda^{(k)}\mathcal{W}, \quad \text{for } k = 1,\ldots,n,$$

where $\Lambda^{(k)}\mathcal{W}$ is the kth exterior power of $\mathcal{W}$ (that is, the map induced on k-planes by $\mathcal{W}$). The functions E_k satisfy the condition above, as do the quotients

$$\frac{E_k}{E_\ell} \quad \text{for } k > \ell.$$

Another interesting family is the power means, given by

$$H_r(\mathcal{W}) = \left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i^r\right)^{\frac{1}{r}} \quad \text{for } r \neq 0,$$

where $\lambda_1,\ldots,\lambda_n$ are the principal curvatures (eigenvalues of $\mathcal{W}$).

Any of these examples can be used to produce further examples by taking the signed powers

$$F_{(\alpha)} = \mathrm{sgn}\alpha F^{\alpha} \quad \text{for } \alpha \neq 0.$$

Some special instances of these are the mean curvature $H = nE_1 = nH_1$, the scalar curvature $R = n(n-1)E_2$, the Gauss curvature $K = E_n = H_0^n$, and the harmonic mean curvature $E_n/E_{n-1} = H_{-1}$.

In most cases one is interested in flows where the speed is homogeneous of some degree, so that a solution remains a solution if it is scaled up or down, as long as the time is also appropriately rescaled. The flows then divide naturally into two classes: The expansion flows, where the speed is of negative degree in the curvature, and the contraction flows, where the speed is of positive degree.

The expansion flows have been considered by Urbas [U1–2], Huisken [Hu3], and Gerhardt [Ge]. For speeds which are homogeneous of degree -1, it is shown that the hypersurfaces expand to infinity, converging to a sphere after rescaling as the elapsed time approaches infinity. The behaviour is similar if the degree of homogeneity is between -1 and 0, but the "rapid expansion flows", which have degree less than -1, are more difficult. For these flows the hypersurfaces reach infinity in finite time; it is not clear that the rescaled hypersurfaces always converge, but some evidence suggests that they do. Some special cases are known [A1, section III].

2. Optimal behaviour: Speed homogeneous of degree one.

The optimal results which have been obtained for the mean curvature flow can be extended to a wide class of other flow equations where the speed shares certain crucial characteristics with the mean curvature. Most importantly, it is found that the degree of homogeneity of the speed F as a function of the Weingarten curvature $\mathcal{W}$ plays an important role: We find that flows where the speed is homogeneous of degree one in the curvature, as it is for the mean curvature flow, tend to behave well. If the speed is homogeneous of some other degree, then the entire method of proof seems doomed, and we are forced to use other methods.

In this section I will sketch the results for degree-one homogeneous speeds, and the method of proof. In later sections I will describe some of the other techniques which can be applied (usually for special flows) for other degrees of homogeneity.

The importance of the degree of homogeneity is evident from several results which have appeared since that of Huisken: Chow [Ch1] considered evolution equations where the speed is a power of the Gauss curvature ($F = K^\alpha$, $\alpha > 0$), following work by Tso [T] on the Gauss curvature flow ($\alpha = 1$). He showed that in all cases solutions converge to points, and in the case $\alpha = \frac{1}{n}$ that the rescaled solutions converge to a sphere. Chow also considered the flow by square root of the scalar curvature ($F = R^{\frac{1}{2}}$), and showed (under some further assumptions on the initial hypersurface) that solutions contract to points and become round [Ch2].

The results for more general flows of this kind may be described as follows (note that in the case $n = 1$ the flow is uniquely determined by the degree of homogeneity, so we need only consider $n \geq 2$):

Theorem 2 [A2]. *Let $n \geq 2$, and assume that $F = F(\lambda_1, \ldots, \lambda_n)$ is a smooth symmetric function defined on the positive cone $(0, \infty)^n$, which is homogeneous of degree one and satisfies the parabolicity condition (3). Suppose also that either F is concave and approaches zero on the boundary of the positive cone, or F is convex. Then for any smooth, strictly convex initial immersion $\varphi_0 : M^n \to \mathbf{R}^{n+1}$, there exists a unique, smooth solution $\varphi : M^n \times [0, T) \to \mathbf{R}^{n+1}$ to equation (2) which converges to a point, becoming spherical as the final time is approached.*

Remark. In the case that F is concave, but not necessarily zero on the boundary of the positive cone, then the same conclusion holds provided that $\frac{F(\mathcal{W})}{H}$ on the initial hypersurface is greater than the maximum of $\frac{f}{\sum\lambda}$ on the boundary of the positive cone (or, more generally, the limit superior of this quantity as the boundary of the positive cone is approached). Note that we mean "φ becomes spherical" in the sense of Theorem 1.

In order to sketch the proof of this result, we first need to know how various quantities evolve. In particular, we need to know how the induced metric $g(u,v) = \langle T\varphi(u), T\varphi(v)\rangle$ varies for any given tangent vectors u and v, where $T\varphi$ is the tangent mapping of φ. We also need to know the evolution of the unit normal ν, and of the second fundamental form $I\!I(u,v) = -\langle D_u D_v \varphi, \nu\rangle$ and the associated Weingarten map $\mathcal{W}(u) = D_u \nu$. This calculation can be carried out in a very general setting:

Lemma 3. *Suppose $\varphi : M^n \times [0,T) \to \mathbb{R}^{n+1}$ satisfies $\frac{\partial}{\partial t}\varphi(z,t) = -F(z,t)\nu(z,t)$, where $F : M^n \times [0,T) \to \mathbb{R}^{n+1}$ is smooth, and ν gives the unit normal to $\varphi(M)$ at each point. Then the following evolution equations hold:*

$$\begin{aligned}
&\frac{\partial}{\partial t} g(u,v) = -2F I\!I(u,v)\\
&\frac{\partial}{\partial t}\nu = T\varphi\,(\nabla F)\\
&\frac{\partial}{\partial t} I\!I(u,v) = \nabla_u\nabla_v F - F I\!I^2(u,v)\\
&\frac{\partial}{\partial t}\mathcal{W}(u) = \nabla_u \nabla F + F\mathcal{W}^2(u)
\end{aligned}$$

for any tangent vectors u and v to M. Here ∇ is the covariant derivative given by the Levi-Civita connection of the metric g.

Now if F is given as above by a function of the Weingarten curvature $\mathcal{W}$, we can immediately deduce a good-looking evolution equation for the speed itself:

$$\frac{\partial}{\partial t} = \dot{F}^i_j g^{jk}\nabla_i\nabla_k F + F\dot{F}^i_j \mathcal{W}^p_i \mathcal{W}^j_p. \tag{4}$$

Here $\dot{F}^i_j$ is the derivative of F with respect to the component $\mathcal{W}_i^j$ of the Weingarten curvature (working in a local coordinate system). The parabolicity assumption ensures that this is a positive definite map whenever $\mathcal{W}$ is positive definite. Hence the leading term on the right hand side is an elliptic operator acting on F.

Some further effort is required to obtain a good evolution equation for the Weingarten curvature $\mathcal{W}$. If we take the evolution equation above, and expand the first term on the right hand side, we obtain:

$$\nabla_i\nabla_k F = \dot{F}^\ell_m \nabla_i\nabla_k \mathcal{W}_\ell^m + \ddot{F}^{\ell m,pq}\nabla_i \mathcal{W}_{\ell m}\nabla_k \mathcal{W}_{pq}.$$

The second term here has a definite sign, on account of the concavity or convexity of F in the curvatures. The first term looks like it could be elliptic, if only we could exchange some indices around. In fact we can: We can apply the following identity, which is a generalisation of a result of Simons [Si]:

$$\nabla_{(i}\nabla_{k)} I\!I_{\ell m} = \nabla_{(\ell}\nabla_{m)} I\!I_{ik} + I\!I_{ik} I\!I^2_{\ell m} - I\!I_{\ell m} I\!I^2_{ik}. \tag{5}$$

This follows by applying the Gauss-Codazzi identities to exchange the various derivatives. The brackets around indices denote symmetrisation.

Substituting this into the evolution equation for the curvature, we find:

$$\frac{\partial}{\partial t}\mathcal{W}_i^j = \dot{F}^\ell_k \nabla^k\nabla_\ell \mathcal{W}_i^j + \ddot{F}(\nabla_i \mathcal{W}, \nabla^j \mathcal{W}) + \mathcal{W}_i^j \dot{F}(\mathcal{W}^2). \tag{6}$$

The nice form of this evolution equation depends very strongly on the degree of homogeneity, since we have used the Euler relation for homogeneous functions: $\dot{F}^k_\ell \mathcal{W}_k^\ell = F$.

We can already deduce several useful things from these evolution equations: The evolution equation for F, for example, has a growth term of the order of F^3. Since the speed is initially strictly positive, this forces the speed to become infinite within a finite time. So we deduce that the time of existence of a solution is finite.

The second thing we can deduce is that a solution which is initially convex remains convex: Consider the evolution of the quotient $\frac{\mathcal{W}}{F}$.

$$\frac{\partial}{\partial t}\left(\frac{\mathcal{W}}{F}\right)=\mathcal{L}\left(\frac{\mathcal{W}}{F}\right)+\frac{2}{F}\dot{F}\left(\nabla F,\nabla\left(\frac{\mathcal{W}}{F}\right)\right)+\frac{1}{F}\ddot{F}(\nabla\mathcal{W},\nabla\mathcal{W}) \tag{7}$$

where $\mathcal{L}$ is the elliptic operator $\dot{F}^{k\ell}\nabla_k\nabla_\ell$. Thus the first term on the right is elliptic, the second is a gradient term, and the third has a definite sign. In the case where F is convex, the last term is positive, and we deduce the the minimum of $\frac{\mathcal{W}}{F}$ increases. If F is concave, then the maximum decreases; further, by taking the trace of this equation we see that the maximum of $\frac{H}{F}$ decreases, where H is the mean curvature. But we have chosen F in such a way that either of these estimates forces the solution to remain convex: In the convex case we have a constant C such that $\lambda_{\min}\geq CF\geq CH\geq C\lambda_{\max}$ at every point, so convexity is preserved. If F is concave and approaches zero on the boundary of the positive cone, then we have $\frac{H}{F}$ uniformly bounded. But this quantity tends to infinity on the boundary of the positive cone, so the solution must stay convex.

In addition to showing that convexity is preserved, this argument gives a pointwise pinching condition for the principal curvatures: There is a constant C depending only on F and φ_0 such that $\frac{\lambda_{\max}(x)}{\lambda_{\min}(x)}\leq C$ for every x in M. This is a rather strong geometric restriction.

Lemma 4. *Suppose $\varphi: M^n\to\mathbb{R}^{n+1}$ is a smooth, strictly convex embedding for which the weingarten curvature $\mathcal{W}$ satisfies a pointwise pinching condition:*

$$\sup_{x\in M^n}\frac{\lambda_{\max}(x)}{\lambda_{\min}(x)}\leq C. \tag{8}$$

Then $\varphi(M^n)$ is contained between concentric spheres of radii ρ_- and ρ_+, where

$$\frac{\rho_+}{\rho_-}\leq C'$$

where C' is a constant depending only on C and n.

In order to prove this result, it will be useful to know a little more about some properties of convex regions. In particular, we need to know something about the quantities known as *quermassintegrals* which are associated with convex regions. The kth quermassintegral is given by:

$$V_k(M) = \int_M E_{n-k} d\mu \tag{9}$$

for $k = 0, 1, \ldots, n$. $V_{n+1}(M)$ is just proportional to the enclosed volume $V(M)$:

$$V_{n+1}(M) = (n+1)V(M). \tag{10}$$

These quantities have an simple geometric interpretation: The kth quermassintegral $V_k(M)$ is equal to the k-dimensional volume of the projection of M onto k-planes in $\mathbf{R}^{n+1}$, integrated with respect to the Haar measure on the Grassmannian $G_{n+1,k}$ of such k-planes. In particular, V_n is the surface area, V_{n-1} is the total mean curvature, and V_1 is the mean width of the convex hypersurface.

The *Minkowski inequalities* compare the different quermassintegrals of a convex hypersurface. In the simplest form, these may be stated as follows:

$$V_k^2 \geq V_{k-1} V_{k+1} \tag{11}$$

for $k = 1, \ldots, n$. Since V_0 is just equal to $|S^n|$, a constant, we can deduce further inequalities from these:

$$\frac{V_1}{V_0} \geq \left(\frac{V_2}{V_0}\right)^{\frac{1}{2}} \geq \cdots \geq \left(\frac{V_k}{V_0}\right)^{\frac{1}{k}} \geq \cdots \geq \left(\frac{V_{n+1}}{V_0}\right)^{\frac{1}{n+1}}. \tag{12}$$

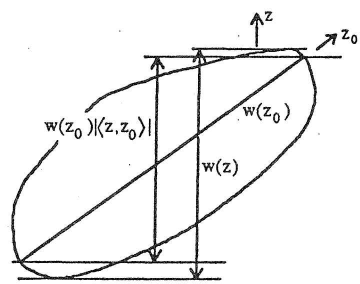
These can be seen as generalisation of the isoperimetric inequality. I will have more to say about these, including a method of proof, in a later section of the talk.

Let us consider the mean width V_1 more closely. An alternative expression for this is as follows:

$$V_1(M) = \frac{1}{2} \int_{S^n} w(z) d\mathcal{H}^n(z), \tag{13}$$

where $w(z)$ is the width of the hypersurface M in the direction z — that is, the distance between the two hyperplanes normal to z which are tangent to the hypersurface. I claim that V_1 bounds the diameter of the hypersurface: First, consider a diameter of the hypersurface — meaning a line of maximum length joining two points of the surface. Let z_0 be the direction of the normal at one of the endpoints of this diameter, so that we have $w(z_0) = \mathrm{diam}(M)$. Then we can estimate the widths in other directions:

$$w(z) \geq w(z_0)\,|\langle z, z_0\rangle|\,.$$



Integrating over the whole of S^n, we obtain the following:

$$\begin{aligned} V_1(M) &\geq \frac{1}{2} w(z_0) \int_{S^n} |\langle z, z_0\rangle|\, d\mathcal{H}^n(z) \\ &= \frac{w(z_0)}{n}. \end{aligned}$$

Hence if we can bound V_1, we have a bound on the diameter of the hypersurface. Even better, if we have a bound on the isoperimetric ratio $\frac{V_1^{n+1}}{V_{n+1}}$, then we have control on the diameter from above, and the enclosed volume from below, as required for the Lemma.

Now, finally, we can apply the pinching estimate obtained above for solutions to the evolution equations: We can obtain a useful estimate simply by applying the Hölder inequality:

$$\begin{aligned}
V_1 &= \int_M E_{n-1}\, d\mu \\
&\le C \int_M E_n^{\frac{n-1}{n}}\, d\mu \\
&\le C \left(\int_M E_n d\mu \right)^{\frac{n-1}{n}} \left(\int_M d\mu \right)^{\frac{1}{n}} \\
&= C V_n^{\frac{1}{n}}.
\end{aligned}$$

Now we can apply the Hölder inequality again to control V_{n+1}:

$$\begin{aligned}
V_1^n &\le C V_n \\
&= \int_M s E_1\, d\mu \\
&\le C \int_M s E_n^{\frac{1}{n}}\, d\mu \\
&\le C \left(\int_M s E_n d\mu \right)^{\frac{1}{n}} \left(\int_M s d\mu \right)^{1-\frac{1}{n}} \\
&= C V_1^{\frac{1}{n}} V_{n+1}^{1-\frac{1}{n}}.
\end{aligned}$$

Rearranging, we obtain the required estimate on the isoperimetric ratio:

$$V_1 \le C V_{n+1}^{\frac{1}{n+1}}, \tag{14}$$

where C depends only on the pinching bound for the principal curvatures.

This controls the shape of the evolving hypersurfaces. Lemma 4 ensures that the hypersurfaces can be written as graphs over a sphere, with height bounded above and below. Writing the evolution equations in this setting, we find that the evolution equation becomes a uniformly parabolic equation for the height function (the coefficients of the elliptic operator are given in terms of the map $\dot{F}$, which is strictly positive and bounded, in view of the pinching estimate).

It follows that we can apply standard regularity estimates, such as those proved by Krylov [K], to deduce uniform estimates on the curvatures and their higher derivatives, after rescaling to keep the volume of the hypersurfaces constant. This in turn ensures that there is a subsequence of times on which the solution converges to a smooth limiting solution after rescaling. Since the pinching ratio improves in time, this limit must be a sphere. From here, it is not difficult to prove that the solution converges in a stronger sense, not just for a subsequence of times.

3. Riemannian background spaces.

In this section I will discuss the application of the ideas of the previous section to the evolution of hypersurfaces in Riemannian manifolds. Here the results really provide something new, and allow us to deduce useful topological information about the hypersurfaces. The basic plan is to make use of the good behaviour of the flows which are homogeneous of degree one, and to see whether the same estimates can be pushed through.

The first work in this direction was by Huisken [Hu2], who considered the mean curvature flow. The mean curvature flow in this setting can be written exactly as for the Euclidean case, as an evolution equation for an immersion φ in to a Riemannian manifold of dimension $n+1$.

Let N be a smooth, complete Riemannian manifold, with metric g^N, Levi-Civita connection ∇^N, and Riemann tensor R^N. We require that N satisfies the following bounds:

$$-K_1 \leq \sigma^N \leq K_2$$
$$\left|\nabla^N R^N\right|_{g^N} \leq L$$

for some non-negative constants K_1, K_2, and L, where σ^N is any sectional curvature of N. The following result is from [Hu2]:

Theorem 5. *Suppose that $\varphi_0 : M^n \to N^{n+1}$ is a smooth immersion, such that every principal curvature λ of φ_0 satisfies the inequality*

$$H\lambda - nK_1 \geq \frac{n^2 L}{H}, \tag{15}$$

where $H = \sum \lambda_i$ is the mean curvature. Then there exists a unique smooth solution to the mean curvature flow with φ_0 as initial condition. This solution converges to a single point p of N in finite time. After rescaling neighbourhoods of p in N to make the volume $|\varphi_t(M)|$ constant, the solution converges to a diffeomorphism φ_∞ of M to the unit sphere S^n in $\mathbf{R}^{n+1}$.

Hence, a hypersurface which satisfies the conditions of the theorem is necessarily the boundary of an immersed disc. In the case of hypersurfaces in a locally symmetric space ($L = 0$), this is a very nice-looking condition, amounting simply to a certain minimum convexity — if the background manifold has non-negative sectional curvatures, then strict convexity is enough. However, if the gradient of the Riemann tensor of the background space does not vanish, the condition is somewhat less satisfying: We must assume that the initial hypersurface has extra convexity to overcome this apparent geometric obstacle.

What happens when we consider flows with other speeds? It turns out that carefully chosen speeds can improve the results significantly. In the following result, an optimal conclusion is obtained for a class of evolution equations which does not include the mean curvature, or the nth root of the Gauss curvature or the square root of the scalar curvature. A good example of a flow which does behave well is the flow by the harmonic mean curvature H_{-1}.

Theorem 6 [A3]. *Suppose φ_0 is a smooth immersion of M to N, such that every principal curvature λ of φ_0 satisfies*

$$\lambda > \sqrt{K_1}. \tag{16}$$

Then there exists a unique smooth solution $\varphi : M\times[0,T) \to N$ to the following equation:

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(x,t) &= -\left(\frac{1}{n}\sum_{i=1}^{n}\left(\lambda_i - \sqrt{K_1}\right)^{-1}\right)^{-1}\nu(x,t); \\ \varphi(x,0) &= \varphi_0(x). \end{aligned} \tag{17}$$

φ converges to a point in finite time, becoming spherical in the sense of Theorem 5.

The convexity condition (16) is still sharp, and does not depend on the size of the gradient of the background tensor.

I will sketch the proof for the case of non-negative sectional curvatures ($K_1 = 0$). As in the Euclidean case, the problem simplifies greatly once a pointwise curvature pinching estimate is obtained. The evolution of metric, normal and curvature changes slightly in the setting of a Riemannian background space:

Lemma 7. *The metric and normal evolve as in Lemma 3. The curvature $\mathcal{W}$ evolves as follows:*

$$\frac{\partial}{\partial t}\mathcal{W}(u) = \nabla_u\nabla F + F\mathcal{W}^2(u) + FR^N(\nu,u,\nu). \tag{18}$$

Hence the speed F evolves according to the following equation:

$$\frac{\partial}{\partial t}F = \mathcal{L}F + F\dot{F}(\mathcal{W}^2) + F\dot{F}(R^N(\nu,.,\nu)), \tag{19}$$

where $\mathcal{L}$ is the elliptic operator $\dot{F}^{k\ell}\nabla_k\nabla_\ell$.

As before, there is a Simons' identity which can be applied to obtain a good evolution equation for the curvature. The resulting equation is as follows:

$$
\begin{aligned}
(20)\qquad \frac{\partial}{\partial t}\mathcal{W}_i^j = {} & \mathcal{L}\mathcal{W}_i^j + \ddot{F}(\nabla_i\mathcal{W},\nabla^j\mathcal{W}) + \dot{F}(\mathcal{W}^2)\mathcal{W}_i^j + \dot{F}(R^N(\nu,.,\nu))\mathcal{W}_i^j \\
& + 2\dot{F}^{pr}\mathcal{W}_r^q(R^N)_{piq}{}^j - \dot{F}^{pq}\mathcal{W}_k^j(R^N)_{piq}{}^k - \dot{F}^{pq}\mathcal{W}_i^k(R^N)_{pkq}{}^j \\
& + \dot{F}^{pq}\nabla_i^N(R^N)_{0pq}{}^j - \dot{F}^{pq}\nabla_p^N(R^N)_{iq0}{}^j
\end{aligned}
$$

Here the subscript 0 represents the normal direction.

Now a straightforward calculation gives the evolution of the quantity $\frac{\mathcal{W}}{F}$ which controls the ratios of principal curvatures:

(21)

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\mathcal{W}_i^j}{F}\right) = {} & \mathcal{L}\left(\frac{\mathcal{W}_i^j}{F}\right) + \frac{1}{F}\ddot{F}(\nabla_i\mathcal{W},\nabla^j\mathcal{W}) + \frac{2}{F}\dot{F}^{k\ell}\nabla_k F\,\nabla_\ell\left(\frac{\mathcal{W}_i^j}{F}\right) \\
& + \frac{1}{F}\left(2\dot{F}^{pr}\mathcal{W}_r^q(R^N)_{piq}{}^j - \dot{F}^{pq}\mathcal{W}_k^j(R^N)_{piq}{}^k - \dot{F}^{pq}\mathcal{W}_i^k(R^N)_{pkq}{}^j\right) \\
& + \frac{\dot{F}}{F} * \nabla^N R^N
\end{aligned}
$$

where the $*$ represents some linear combination of contractions. Remarkably, all the terms in this equation, except for the final one involving the derivatives of the Riemann tensor of N, are good: The first term is just an elliptic operator, and the second a gradient term; the concavity term is negative; and the terms involving the background curvature can be written in terms of the sectional curvatures, so that the positivity of the sectional curvatures yields a negative term. The last term is uniformly bounded, since $\dot{F}$ is uniformly bounded above (this would not be the case if we has chosen the nth root of the Gauss curvature, or the square root of the scalar curvature, for example), and F is uniformly bounded below (by the evolution equation (19)). Therefore $\lambda_{\max} \leq C(1+t)F$ by the maximum principle. But equation (19) shows that the time of existence is finite, so $\lambda_{\max} \leq CF$. Since $F = H_{-1} \leq \lambda_{\min}$, this implies $\lambda_{\max} \leq C\lambda_{\min}$.

Thus a pinching estimate holds as in the Euclidean case. Of course, it takes much more work to prove from this that the evolution equation behaves well and converges nicely, since there are no such convenient quantities as the quermassintegrals. However, careful application of the regularity estimates of Krylov shows that there is a subsequence of times t_k, and a corresponding sequence of points x_k in M, such that after rescaling about the points x_k to make the curvature there equal to 1, the hypersurfaces converge to a complete, smooth, strictly convex hypersurface in Euclidean space with pinched principal curvatures. A result of Hamilton [Ha1] ensures that such hypersurfaces must be compact, and the proof proceeds as for the Euclidean case.

There are some useful applications of this result: For example, we obtain a new proof of the 1/4-pinching sphere theorem of Klingenberg, Berger, and Rauch. The idea of the proof as is follows:

We start with a Riemannian manifold N of dimension $n+1$, which is strictly 1/4-pinched: There is some $\varepsilon > 0$ such that all the sectional curvatures σ^N of N satisfy $\frac{1}{(2-\varepsilon)^2} \le \sigma^N \le 1$. Now choose a point p in N, and consider geodesic spheres about p. These are a family of immersions φ of the sphere S^n into N, with all principal curvatures becoming infinite as the distance s from p approaches zero. As s increases, the immersions evolve by moving outward at constant speed:

$$\frac{\partial}{\partial t}\varphi(x,s) = \nu(x,s). \tag{22}$$

The evolution equation for the curvature is given by Lemma 7:

$$\frac{\partial}{\partial s}\mathcal{W}_i^j = -\left(\mathcal{W}^2\right)_i^j - \left(R^N\right)_{0i0}{}^j. \tag{23}$$

This gives:

$$\begin{aligned}
\frac{\partial}{\partial s}\lambda_{\max} &\le -\lambda_{\max}^2 - \frac{1}{(2-\varepsilon)^2};\\
\frac{\partial}{\partial s}\lambda_{\min} &\ge -\lambda_{\max}^2 - 1.
\end{aligned}$$

Hence the curvatures of the geodesic sphere at distance s satisfy:

$$\frac{1}{2-\varepsilon}\cot\left(\frac{s}{2-\varepsilon}\right) \ge \lambda_{\max} \ge \lambda_{\min} \ge \cot(s).$$

Hence for $s_0 \in \left((1-\frac{1}{2}\varepsilon)\pi, \pi\right)$ we have $0 > \lambda_{\max}(s_0) \geq \lambda_{\min}(s_0) > -\infty$, so that the geodesic sphere has bounded curvatures, and is strictly convex in the outward direction from p. The evolution equations for the metric and for the higher derivatives of the curvature imply that the hypersurface remains smooth as long as the curvatures remain finite; in particular, φ_{s_0} is a smooth immersion.

Now φ_{s_0} can be used as an initial hypersurface for Theorem 6. This gives a family of hypersurfaces which converge to a point, showing that the exterior of this geodesic sphere (with respect to p) is a disk. Thus N can be written as a union of two discs, and is homeomorphic to a sphere.

4. Other degrees of homogeneity.

I have considered in detail the very strong results which have been obtained for solutions to flows where the speed is homogeneous of degree one in the curvatures. Now I want to consider some more general flows. One flow which has received considerable attention is the Gauss curvature flow. This flow actually predates the mean curvature flow, having been considered by Firey [F] in 1974, as a model for the wearing of pebbles on a beach. The derivation of this equation for such a process is quite simple: Consider a pebble, which we think of as a convex region in space $\mathbb{R}^{n+1}$. I do not wish to speculate about whether such higher-dimensional pebbles exist. Suppose that the wearing down of this pebble is caused by the continual impact of other pebbles; these impacts are supposed to come from every direction with equal frequency. Now make the simplifying assumption that these other pebbles are much larger than the one being modelled, so that they can reasonably be approximated as half-planes. The rate of collisions in some small region of the pebble is just proportional to the measure of the set of hyperplanes which have tangential contact with the pebble in this region. This is precisely the measure of the Gauss image of the region, or locally the Gauss curvature times the surface area element. The rate at which the surface wears away is just proportional to the rate of collisions per unit area, which is just the Gauss curvature.

Firey was able to show that any initial region which is centrally symmetric (invariant under the involution $z \to -z$ of $\mathbb{R}^{n+1}$) shrinks to a point, and becomes spherical in the process (actually, the problems of existence and regularity of solutions were not addressed by Firey; the arguments required for this were provided later by Tso [T]). Firey conjectured that the conclusion should hold without the assumption of central symmetry. Tso showed that the solution always converges to a point, but the harder problem of determining whether the solution becomes round remains unresolved despite considerable attention.

The problem arises because there does not seem to be any good estimate on the curvatures — certainly nothing as simple and elegant as the pinching estimate for the homogeneous degree one flows. This problem is shared by the myriad other flows which have a degree of homogeneity different from one.

However, even without knowing that the rescaled solutions converge, useful information can be obtained: I will next describe an interesting application of the Gauss curvature flow to prove some of the Minkowski inequalities (12) between Quermassintegrals.

Lemma 8. *Under the Gauss curvature flow (equation (2) with $F = K$), the following inequality holds:*

$$\frac{\partial}{\partial t}\left(V_k^{\frac{n+1}{k}} - V_{n+1}V_0^{\frac{n+1-k}{k}}\right) \leq 0, \tag{25}$$

with equality if and only if M_t is a sphere.

This is easy to prove: First, V_{n+1} evolves as follows:

$$\frac{\partial}{\partial t}V_{n+1} = -(n+1)\int_M K\,d\mu = -(n+1)V_0.$$

Next we compute how V_k evolves:

$$\begin{aligned}\frac{\partial}{\partial t}V_k &= -k\int_M KE_{n+1-k}d\mu\\ &\leq -k\int_M K\left(\frac{E_{n-k}}{K}\right)^{-\frac{n+1-k}{k}}d\mu\\ &\leq -k\left(\int_M K\left(\frac{E_{n-k}}{K}\right)d\mu\right)^{-\frac{n+1-k}{k}}\left(\int_M Kd\mu\right)^{\frac{n+1}{k}}\\ &= -kV_k^{-\frac{n+1-k}{k}}V_0^{\frac{n+1}{k}},\end{aligned}$$

where we have used the Newton inequalities $E_{n+1-k} \geq K^{\frac{n+1-k}{n}}$ and $E_{n-k} \geq K^{\frac{n-k}{n}}$, and applied the Hölder inequality. Now we can combine these:

$$\begin{aligned}\frac{\partial}{\partial t}\left(V_k^{\frac{n+1}{k}} - V_{n+1}V_0^{\frac{n+1-k}{k}}\right) &\leq \frac{n+1}{k}V_k^{\frac{n+1-k}{k}}\left(-kV_k^{-\frac{n+1-k}{k}}V_0^{\frac{n+1}{k}}\right)\\ &\quad - V_0^{\frac{n+1-k}{k}}\left(-(n+1)V_0\right)\\ &= 0,\end{aligned}$$

as stated in the Lemma.

Corollary 9.

$$V_k^{\frac{n+1}{k}} - V_{n+1}V_0^{\frac{n+1-k}{k}} \geq 0 \tag{26}$$

for $k = 1,\ldots,n$, with equality only for spheres.

This follows since the solution to the Gauss curvature flow converges to a point. Thus at the final time, the claim (26) holds with equality. By the lemma, the left-hand side is decreasing in time, so it must have been non-negative initially (and strictly positive unless M_t is a sphere).

By using other flows as well as the Gauss curvature flow, one can in fact prove all of the Minkowski inequalities, and even the more general Aleksandrov-Fenchel inequalities [A4].

The family of Gauss curvature flows, with speeds given by K^α for any positive α, has received a lot of attention: Chow [Ch1] showed that each of these flows contracts convex hypersurfaces to points; he also proved Harnack inequalities [Ch3], and in the special case $\alpha = 1$, he found a decreasing, scaling-invariant integral (known as the entropy). Hamilton has shown that for the Gauss curvature flow [Ha2], the entropy estimate gives control on the isoperimetric ratio, and the Harnack inequality gives a bound on the Gauss curvature of the rescaled solutions.

Many of these results which have been proved for the case $\alpha = 1$ can be proved also for other α. The following Theorem summarises various partial results:

Theorem 10. *Let φ_0 be a smooth, strictly convex initial immersion of a compact manifold M into $\mathbf{R}^{n+1}$, and let φ be the solution to the flow* (2) *with $F = K^\alpha$, $\alpha > 0$. Then φ converges in finite time to a single point* [T, Ch1]. *If $\alpha = \frac{1}{n+2}$, then the rescaled solution converges to a diffeomorphism φ_∞ from M to an ellipsoid in $\mathbf{R}^{n+1}$* [A6]. *If $\frac{1}{n} \geq \alpha > \frac{1}{n+2}$, then the rescaled solutions converge to a limiting solution which evolves homothetically* [A8]; *if $\alpha = \frac{1}{n}$ then this limiting solution is a sphere* [Ch1]. *If $\alpha > \frac{1}{n}$ then the rescaled solution has bounded isoperimetric ratio, and bounded Gauss curvature; if it has a smooth limit, then this must be homothetic* [Ha2,A8].

The general picture seems to be that the solutions will converge (probably to spheres) for $\alpha > \frac{1}{n+2}$, and will generically diverge for $\alpha < \frac{1}{n+2}$: In the case $n = 1$ of curves in the plane, it can be shown [A7] that for $\alpha < \frac{1}{3}$ there are initial conditions for which the rescaled solutions have no smooth limit at the final time; the hypersurfaces collapse onto a one-dimensional line. In higher dimensions I expect that similar behaviour occurs, but I can prove so far only that there are examples which do not converge to spheres [A5]. In a sense, in these cases the speed does not "react" quickly enough to large differences in curvature between different points of the hypersurface.

Also in the case of curves, more can be said for higher homogeneities: In fact, for this case everything behaves perfectly: For any $\alpha > \frac{1}{3}$, the solutions converge to a round circle after rescaling [A7].

The Gauss curvature flows are at present the only family of flows for which any reasonable results are known. It would be nice to have techniques which were more robust, with less dependence on the precise details of the speed function (just as for the homogeneous-degree-one flows the techniques worked for a wide class of speeds F).

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