FINITE ELEMENT METHODS FOR IDENTIFICATION OF PARAMETERS IN PARABOLIC PROBLEMS

A.K.Pani and R.S.Anderssen

1.INTRODUCTION

In the present paper, we examine finite element Galerkin methods for the identification of unknown parameters in parabolic partial differential equations, and derive a number of optimal error estimates. In the first part of the paper, we consider the problem of finding, for a homogeneous material, a control parameter $p(t)$, along with the temperature distribution $U(x,t)$, which yields a specified energy trajectory $E(t)$ (see (1.4) below) pescribed on the whole of its spatial domain. The corresponding model gives rise to the following inverse problem: find $p(t)$ and $U(x,t)$ such that

$$U_t - U_{xx} + p(t)U_x = \tilde{f}(x,t), \quad (x,t) \in (0,1) \times (0,T], \tag{1.1}$$

$$U(x,0) = U_0(x), \quad x \in [0,1], \tag{1.2}$$

$$U(0,t) = \tilde{f}_1(t), \qquad U(1,t) = \tilde{f}_2(t), \quad t \in [0,T] \tag{1.3}$$

along with the over–specificed total internal energy condition

$$\int_0^1 U(x,t)\,dx = E(t), \quad t \in [0,T], \tag{1.4}$$

where $\tilde{f}, \tilde{f}_1, \tilde{f}_2, U_0$ and E are given functions of their arguments with $\tilde{f}_2 - \tilde{f}_1 \neq 0$.

In the literature, output error criteria procedures are regularly proposed and analysed (Banks and Kunish [**2**]) for the solution of inverse problems, including ones like that which is formulated above. The basis for such indirect approaches is optimization: the closeness (defined by a given objective function) with which computed estimates can be matched with observations of measurable quantities is minimized with respect to some admissable class of control parameters. Least squares is often used to define the closeness (objective function). The resulting methods are referred to as least squares output error criteria procedures.

Such procedures are not difficult to implement (Anderssen [**1**]), since computationally they reduce to the repeated solution of some underlying forward problem, and to analyse theoretically (Banks and Kunish [**2**]). However, such procedures have their drawbacks which includes:

(i) The essential improperly posedness of the underlying inverse problem is now embedded in the optimization. This is reflected in the fact that the required solution will be the global optimum of a multimodal objective function, with the nature of the multimodality a reflection of the degree of the improperly posedness.

(ii) Even if the objective function is not too wild in its behaviour, some iterative procedure (gradient like search) will be needed to minimize the closeness numerically. This will tend to be time consuming computationally. In addition, there will be no *a priori* criteria for determining when the iteration process should be stopped (e.g. no estimate of the global order of convergence).

(iii) Though each step in the iteration will reduce to the solution of some forward problem (with respect to the current choice for the control parameter), this will involve the numerical solution of some partial differential equation.

Thus, there is a clear need to minimize the number of steps in the iteration as well as reduce the range of the global search. One way this can be achieved is to obtain an independent initial estimate of the structure of the control parameter. This is the point of focus of the present examination; namely, the construction of direct non–iterative numerical procedures. The basic technique used in this paper is to transform the parameter identification problem into a non–classical forward problem, weakly coupled with a functional equation in the control parameter. In most cases, the coupling is such that the forward problem can be solved independent of the unknown parameter.

Existence, uniqueness and regularity results for a more general problem than (1.1)–(1.4), in its non–classical setting can be found in Cannon and Yin [**3**]. Cannon and Yin [4] have discussed a Galerkin method for a more general problem like the one in Section 3, when u_x is replaced by v. Even for the simplest case as in (1.1)–(1.4), their analyses would not establish optimal order of convergence in $L^\infty(L^2)$ for v. Using the auxiliary Ritz

–projection technique of Wheeler [9], it is possible to recover optimality in the $L^\infty(L^2)$ norm for v. But, this does not yield the optimality for u in the $L^\infty(L^2)$ norm. This is where negative norm estimates must be used for v, as well as a superconvergent result for v at the end points.

In the second part of the paper, we consider the more general problem examined by Cannon and Yin [4]. Unlike the previous result, there is now a strong coupling between u and v, so it may not be possible to derive negative norm estimate for v. Therefore, we need a more general formulation, which takes care of the nonlinearity and yields optimal estimates for u in the L^2 norm in a more natural way. It is in this context that we introduce an H^1– Galerkin method. Through out this paper, K is taken to be a generic positive constant, whose dependence can be traced from the proofs.

2. NON–CLASSICAL FORMULATION AND ERROR ESTIMATES

Differentiate $E(t)$ with respect to t and use equations (1.1) and (1.3) to obtain

$$p(t) = \frac{[U_x(1) - U_x(0)] + \int_0^1 \tilde{f}(x,t)\,dx - E'(t)}{\tilde{f}_2 - \tilde{f}_1}, \tag{2.1}$$

where $E'(t) = \frac{d}{dt}E(t)$. Define

$$u(x,t) = U(x,t) - [(1-x)\tilde{f}_1(t) + x\tilde{f}_2(t)],$$

so that $u(x,t) = 0$, at $x = 0$ and 1. Using this transformation, (1.1)–(1.4) can be rewritten in the non–classical form

$$\begin{aligned} u_t - u_{xx} + g(t)(u_x(1) - u_x(0) + h(t))u_x \\ = -(u_x(1) - u_x(0)) - f(x,t), \quad (x,t) \in (0,1) \times [0,T], \end{aligned} \tag{2.2}$$

with

$$u(x,0) = u_0(x),$$

where

$$g(t) = \frac{1}{\tilde{f}_2 - \tilde{f}_1}, \qquad h(t) = \int_0^1 \tilde{f}(x,t)\,dx - E'(t),$$

$$f(x,t) = -\tilde{f} + (1-x)\tilde{f}_1' + x\tilde{f}_2' + h(t), \quad u_0(x) = U_0(x) - (1-x)\tilde{f}_1(0) - x\tilde{f}_2(0).$$

We now introduce the function

$$v(x,t) = u_x(x,t), \quad (x,t) \in (0,1) \times [0,T],$$

where u is a solution to the problem (2.1)–(2.2). In order to construct a weak formulation for v, we form the inner product of (2.2) with w_x, and integrate the first as well as the third terms in the left hand side by parts with respect to x to obtain

$$\begin{aligned}(v_t, w) + (v_x, w_x) + g(t)[v(1) - v(0) + h(t)](v_x, w)\\ = [v(1) - v(0)](w(1) - w(0)) + (f, w_x), \quad t \in (0,T],\end{aligned} \tag{2.3}$$

with

$$(v(0), w) = (u_x(0), w).$$

The Galerkin Procedure. Let S_h be a finite dimensional subspaces of H^1 belonging to the $S_h^{r,1}$ family (for a definition, see Oden and Reddy [8]), and satisfying the following approximation and inverse properties:

(i) For $\phi \in H^m(I)$ and $m \in [1, r+1]$,

$$\inf_{\chi \in S_h} \|\phi - \chi\|_j \leq K h^{m-j} \|\phi\|_m, \quad j = 0, 1.$$

(ii) For $\chi \in S_h$,

$$\|\chi\|_{L^\infty(I)} \leq K h^{-\frac{1}{2}} \|\chi\|_{L^2(I)}.$$

The mapping $v^h : (0,T] \mapsto S_h$ is called a Galerkin approximation of v, if it satisfies

$$\begin{aligned}(v_t^h, \chi) + (v_x^h, \chi_x) + g(t)[v^h(1) - v^h(0) + h(t)](v_x^h, \chi)\\ = [v^h(1) - v^h(0)](\chi(1) - \chi(0)) + (f, \chi_x), \quad \chi \in S_h,\ t \in (0,T],\end{aligned} \tag{2.4}$$

along with the initial condition

$$v^h(0) = P_h u_x(0), \tag{2.5}$$

where P_h is an appropriate projection of $u_x(0)$ on to S_h, to be defined below. Since S_h is finite dimensional, the equations (2.4) and (2.5) yield a system of nonlinear ordinary differential equations. It has a unique local solution. In order to establish the existence of a unique Galerkin approximation v^h in a neighborhood of v, as well as in the domain of v, one can invoke the fixed point analysis given in Pani and Das [5].

Set

$$A(u;v,w) = (v_x, w_x) + g(t)[u(1) - u(0) + h(t)](v_x, w), \quad u \in L^\infty(I),\ v,\ w \in H^1.$$

The bilinear form $A(u;\cdot,\cdot)$ satisfies the following properties

$$|A(u;v,w)| \le K\|v\|_1\|w\|_1, \quad u \in L^\infty(I),\ v \text{ and } w \in H^1,$$

and

$$A(u;v,v) \ge M\|v\|_1^2 - \rho\|v\|^2, \quad u \in L^\infty(I),\ v \in H^1(I),$$

where K, M and ρ are positive constants with K and ρ depending on $\|u\|_{L^\infty(I)}$.

Define

$$A_\rho(u;v,w) = A(u;v,w) + \rho(v,w),$$

so that A_ρ is coercive in H^1; that is,

$$A_\rho(u;v,v) \ge M\|v\|_1^2.$$

Following Wheeler [**9**], we define an auxiliary projection $\tilde{v}$ with respect to the form A_ρ; namely,

$$A_\rho(v; v - \tilde{v}, \chi) = 0, \quad \chi \in S_h. \tag{2.6}$$

An application of the Lax–Milgram Lemma now establishes the existence of a unique solution $\tilde{v}$ in S_h .

Let $\eta = v - \tilde{v}$ and $\xi = v^h - \tilde{v}$. With the assistance of the Aubin–Nitsche duality argument, it is not diffucult to prove the following error estimates for η.

Lemma 2.1. *For $t \in [0,T]$, the error η satisfies*

$$\|\eta\|_j + \|\eta_t\|_j \le Kh^{m-j}, \quad -1 \le j \le 1$$

and

$$|\eta(x)| \le Kh^{2(m-1)}, \quad x = 0, 1,$$

for $m \in [1, r+1]$.

From the above estimates, the following result is easily derived

$$\|\tilde{v}\|_{L^\infty(H^1)} \le K_1.$$

Here the dependance of K_1 can be easily traced.

Error Estimates. Let $e = v - v^h = \eta - \xi$, and choose $P_h u_x(0) = \tilde{v}(\cdot, 0)$ so that $\xi(x,0) = 0$. Here we state the following inequalities, which will be used to derive *a priori* error estimates

$$\|\phi\|_{L^\infty(I)} \le \|\phi\|_1$$

and

$$|\phi(x_0)| \le \|\phi\| + \sqrt{2}\|\phi\|^{\frac{1}{2}}\|\phi\|_1^{\frac{1}{2}},$$

for $\phi \in H^1$ and $x_0 = 0$, or 1.

We are now in a position to state a superconvergent result for ξ.

Theorem 2.2. *For* $m \in [2, r+1]$,

$$\|\xi\|_{L^\infty(0,T;L^2(I))} + \|\xi\|_{L^2(0,T;H^1(I))} \le Kh^{m+1}$$

and

$$\|\xi\|_{L^\infty(0,T;L^\infty(I))} + \|\xi\|_{L^\infty(0,T;H^1(I))} \le Kh^m.$$

Proof: Assume that, for $K^* > 2K_1$,

(2.7) $$\|v^h\|_{L^\infty(0,T;L^\infty(I))} \le K^*.$$

From (2.3), (2.4) and (2.6), we obtain

$$\begin{aligned}(\xi_t, \chi) + (\xi_x, \chi_x) + g(t)[v^h(1) - v^h(0) + h(t)](\xi_x, \chi) &= (\eta_t, \chi) - g(t)(\xi(1) - \xi(0))(\tilde{v}_x, \chi) \\ &+ g(t)(\eta(1) - \eta(0))(\tilde{v}_x, \chi) - \rho(\eta, \chi) \\ &+ (\eta(1) - \eta(0))(\xi(1) - \xi(0)) - (\xi(1) - \xi(0))^2.\end{aligned} \tag{2.8}$$

Choose $\chi = \xi$ in (2.8). The boundedness and coercivity of A_ρ, along with the two inequalities introduced above, yield

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \|\xi\|_1^2 \leq K(K^*)(|\eta(0)|^2 + |\eta(1)|^2 + \|\eta_t\|_{-1}^2 + \|\xi\|^2).$$

The first estimates follow on integration with respect to t and using Gronwall's Lemma with the estimates for η given in Lemma 2.1. For the second estimate, we set $\chi = \xi_t$ in (2.8). Using the standard estimates as for the first result, we obtain the second.

As far as (2.7) is concerned, using inverse property we note that for sufficiently small h,

$$\begin{aligned} \|v^h\|_{L^\infty(0,T;L^\infty(I))} &\leq \|\xi\|_{L^\infty(0,T;L^\infty(I))} + \|\tilde{v}\|_{L^\infty(0,T;L^\infty(I))} \\ &\leq K(K^*)h^m + K_1 \leq 2K_1 < K^*. \end{aligned}$$

Therefore (2.7) holds for sufficiently small h and this completes the proof.

We define the Galerkin approximations u^h and p^h of u and p by

$$u_x^h = v^h, \quad (x,t) \in I \times [0,T],$$

and

$$p^h(t) = \frac{[v^h(1) - v^h(0)] + \int_0^1 \tilde{f}(x,t)\,dx - E'(t)}{\tilde{f}_2 - \tilde{f}_1}, \quad t \in [0,T],$$

respectively.

The final error estimates are stated in the following Theorem.

Theorem 2.3. *For* $m \in [2, r+1]$,

$$\|u - u^h\|_{L^\infty(0,T;L^2(I))} + \|e\|_{L^\infty(0,T;H^{-1}(I))} \leq Kh^{m+1}$$

and

$$\|p - p^h\|_{L^\infty(0,T)} \leq Kh^m.$$

Remark 2.1. From Theorem 2.2 and Lemma 2.1, it follows that optimal estimates hold for e in the $L^\infty(L^\infty)$ and the $L^\infty(H^1)$. A slightly more refined analysis shows that the inverse property can be circumvented.

Fully Discrete Schemes. Let us define a fully discrete Galerkin procedure, based on the backward differencing in time. Let $k > 0, (\frac{N}{k}) \in \mathcal{Z}$ and $t^n = nk$, for $n = 0, 1, \dots N$. Further, let $\phi^n = \phi(x, t^n)$ and $d_t\phi^n = \frac{(\phi^{n+1}-\phi^n)}{k}$. Denote the time discretization of v^h by $V : \{t^0, t^1, \dots, t^N\} \mapsto S_h$ and V^{n+1} as the solution of the following discrete equation

$$(d_tV^n, \chi) + (V_x^{n+1}, \chi_x) + g^{n+1}[V^{n+1}(1) - V^{n+1}(0) + h^{n+1}](V_x^{n+1}, \chi)$$

$$= [V^{n+1}(1) - V^{n+1}(0)](\chi(1) - \chi(0)) + (f^{n+1}, \chi_x), \quad \chi \in S_h, \; n \geq 1, \tag{2.9}$$

with

$$V^0 = P_h u_x(0).$$

The error estimates for the fully discrete scheme (2.9) as well as for the parameter at $t = t^n$ are given in form of a Theorem.

Theorem 2.4. *For $m \in [2, r+1]$ and for $k = o(h)$,*

$$\|u(t^n) - Z^n\| + \|v(t^n) - V^n\|_{H^{-1}(I)} \leq K(h^{m+1} + k)$$

and

$$|p(t^n) - P^n| \leq K(h^m + k), \quad n \geq 1,$$

where

$$Z_x^n = V^n, \quad x \in I$$

and

$$P^n = \frac{[V^h(1) - V^h(0)] + \int_0^1 \tilde{f}(x, t^n)\, dx - E'(t^n)}{\tilde{f}_2(t^n) - \tilde{f}_1(t^n)}, \quad n \geq 0.$$

3. ERROR ESTIMATES FOR MORE GENERAL PROBLEMS

In this section, we consider the following generalization of (2.1), (2.2) and (1.2)

$$u_t - a(x, t, u, u_x; u_x(x_0, t))u_{xx} = b(x, t, u, u_x; u_x(x_0, t)), \quad (x, t) \in I \times (0, T], \tag{3.1}$$

$$u(x, 0) = u_0(x), \quad x \in I, \tag{3.2}$$

$$u(0, t) = 0, \quad \text{and} \quad u(1, t) = 0, \quad t \in [0, T], \tag{3.3}$$

where $x_0 \in [0, 1]$ is a fixed point.

The framework of (3.1)–(3.3) includes the nonclassical formulation of a large class of parabolic inverse problems (see, for example, Cannon and Yin[**3**]). For most of these problems, the unknown parameters depend on the gradient of the solution at certain fixed points in their functional equation settings. Therefore, it is important to recover the superconvergent results for the gradient in order to achieve optimal rates of convergence for the unknown parameters.

We shall now state our main assumptions on the coefficients and also on the solution and call them collectively condition R.

Condition R.

(i) For $(x,t) \in I \times [0,T]$, $|u| \leq K$, $p \in R^1$, and $|q| \leq K$,

$$a(x,t,u,p;q) \geq a_0(K),$$

where a_0 is a positive constant depending on K.

(ii) The functions a and b are such that $a \in C^3$ and $b \in C^1$. Furthermore, they are bounded along with their respective derivatives by a common constant $C(K)$ with K as in (i).

(iii) There exists a smooth unique solution u to the problem (3.1)–(3.3) in $I \times [0,T]$.

The existence, uniqueness and regularity conditions given below are based on those of Cannon and Yin [**3**].

Regularity Conditions. For $r \geq 1$,

$$u \in W^{1,2}(0,T;H^{r+1}(I)) \cap W^{1,\infty}(0,T;W^{2,\infty}(I)).$$

As in the previous section, it may not be possible to obtain a negative norm estimate for v in order to achieve an optimal error estimate in $L^\infty(L^2)$ for u. Therefore, we apply an H^1–Galerkin method in order to tackle the nonlinearity and to obtain the required optimality. The only result available in the literature is due to Pani and Das [**6**] regarding an H^1–Galerkin method for a quasilinear parabolic equation with the coefficients depending on u. Hence, the present work has an added significance.

To construct a weak formulation, we multiply both sides of (3.1) by w_{xx} and integrate only the first term on the left hand side with respect to x to obtain

$$(3.4) \qquad (u_{tx}, w_x) + (a(u, u_x; u_x(x_0,t))u_{xx}, w_{xx}) + (b(u, u_x; u_x(x_0,t)), w_{xx}) = 0,$$

for $w \in H^2 \cap H_0^1$.

Let S_h be a finite dimensional subspace of $H^2 \cap H_0^1$ with the following approximation and inverse properties:

(i) For $\phi \in H^m(I) \cap H_0^1$ and $m \in [2, r+1]$,

$$\inf_{\chi \in S_h} \|\phi - \chi\|_j \leq K h^{m-j} \|\phi\|_m, \quad j = 0, 1, 2.$$

(ii) For $\chi \in S_h$,

$$\|\chi\|_{W^{2,\infty}(I)} \leq K h^{-\frac{1}{2}} \|\chi\|_{H^2(I)}.$$

The mapping $u^h : (0, T] \mapsto S_h$ is called an H^1- Galerkin approximation of u, if it satisfies

$$(3.5) \quad (u^h_{tx}, \chi_x) + (a(u^h, u^h_x; u^h_x(x_0,t))u^h_{xx}, \chi_{xx}) + (b(u^h, u^h_x; u^h_x(x_0,t)), \chi_{xx}) = 0, \quad \chi \in S_h$$

along with

$$u^h(x, 0) = Q_h u_0,$$

where Q_h is an appropriate projection to be defined below.

For the global existence of a Galerkin approximation u^h in [0,T], we refer to Pani *et al.* [7]. If there is no confusion, we simply write $a(u), b(u)$ to represent the coefficients in (3.1).

A priori Error Estimates. We now introduce the following form

$$A(u; v, w) = (a(u)v_{xx} - (a_p(u)u_{xx} + b_p(u))v_x - (a_u(u)u_{xx} + b_u(u))v, w_{xx}),$$

for $u \in W^{2,\infty}$ and $v, w \in H^2 \cap H_0^1$. Here $a_p(u) = \frac{\partial}{\partial p} a(x, t, u, p, q)$.

It is an easy matter to show that

$$|A(u;v,w)| \le K\|v\|_2\|w\|_2$$

and

$$A(u;v,v) \ge \alpha\|v\|_2^2 - \rho\|v\|_1^2,$$

for $u \in W^{2,\infty}$, v and $w \in H^2 \cap H_0^1$. Here, the constants K, ρ may depend on $\|u\|_{W^{2,\infty}}$. Furthermore, define

$$A_\rho(u;v,w) = A(u;v,w) + \rho(v_x, w_x).$$

Note that A_ρ is coercive in $H^2 \cap H_0^1$.

Let $\tilde{u}$ be an approximation to u with respect to the form A_ρ; i.e.,

$$A_\rho(u; u - \tilde{u}, \chi) = 0, \quad \chi \in S_h.$$

We obtain the following error estimate for $\eta = u - \tilde{u}$ (see Pani *et al.* [7]).

Lemma 3.1. *For $t \in [0,T]$ and $m \in [2, r+1]$,*

$$\|\eta\|_j + \|\eta_t\|_j \le Kh^{m-j}, \quad 0 \le j \le 2,$$

and for any fixed $x_0 \in [0,1]$,

$$|\eta_x(x_0)| \le Kh^{2(m-2)}.$$

Further,

$$\|\tilde{u}\|_{L^\infty(W^{2,\infty})} \le K.$$

Set $Q_h u_0$ as the solution of

$$A_\rho(u_0; u_0 - Q_h u_0, \chi) = 0, \quad \chi \in S_h.$$

Clearly, $u^h(x,0) = \tilde{u}(x,0)$.

Let $\xi = u^h - \tilde{u}$ and $e = u - u^h = \eta - \xi$. We now state the following superconvergent results for ξ.

Theorem 3.1. *For* $m \in [4, r+1]$,

$$\|\xi\|_{L^\infty(H^1)} + \|\xi\|_{L^2(H^2)} \leq Kh^m$$

and

$$\|\xi\|_{L^\infty(H^2)} \leq Kh^{m-1}.$$

For a detailed analysis, see Pani *et al.* [7]. The resulting error estimates are stated in the following Theorem.

Theorem 3.2. *Let* u *satisfy the regularity conditions* R. *Further, let* u^h *be the unique approximate solution of (3.5). Then the following estimates hold for* $m \in [4, r+1]$

$$\|e\|_{L^\infty(H^j)} \leq Kh^{m-j}, \quad j = 0, 1, 2,$$

and

$$\|e\|_{L^\infty(L^\infty)} + \|e_x(x_0, .)\|_{L^\infty(0,T)} \leq Kh^m.$$

Remark 3.1. The functional equations in the unknown parameters usually involve the gradient at some fixed points. But the superconvergent result in Theorem 3.2 allows one to obtain optimal estimates for the parameters.

Let us define a fully discrete Galerkin procedure based on a linearised modification of the backward differencing in time which is only first order accuracy in temporal variable. Denote the time discretization of u^h by the mapping $Z : \{t^0, t^1, \ldots, t^N\} \mapsto S_h$ and Z^{n+1} as the solution of the following discrete equation

$$(3.6) \qquad (d_t Z_x^n, \chi_x) + (a(Z^n) Z_{xx}^{n+1}, \chi_{xx}) = -(b(Z^n), \chi_{xx}), \quad \chi \in S_h,$$

with $Z^0 = Q_h u_0$.

We note that higher regularity is needed for the solution u in order to achieve optimal error estimates. Below we shall only state the final error estimates for the fully discrete Galerkin scheme.

Theorem 3.3. *For $m \in [4, r+1]$ and $k = o(h)$,*

$$\sup_n \|Z^n - u(t^n)\|_j \leq K(h^{m-j} + k),$$

and

$$\sup_n \{\|Z^n - u(t^n)\|_{L^\infty} + |Z_x^n(x_0) - u_x(t^n, x_0)|\} \leq K(h^m + k).$$

Remark 3.2. For optimal error estimates, C^1 cubic splines or better are needed. Since the evaluation of gradient plays an important role, we may consider both u and its gradient $v = u_x$ of (3.1)–(3.3) as primary variables and apply an H^1–Galerkin method to the system as follows: find a pair $\{u, v\} \in H_0^1 \times H^1$ such that

$$(u_x, w_x) = (v, w_x), \quad w \in H_0^1,$$

and

$$(v_t, z) + (a(u, v; v(x_0))v_x, z_x) + (b(u, v; v(x_0)), z_x) = 0, \quad z \in H^1.$$

A finite element error analysis for the above system, along with numerical results, will be pursued elsewhere.

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A. K. Pani
Department of Mathematics
Indian Institute of Technology, Powai
Bombay–400076, India.

R. S. Anderssen
CSIRO Division of Mathematics and Statistics
GPO Box 1965, Canberra
ACT–2601,
and
Center for Mathematics and its Applications
The Australian National University
GPO Box 4, Canberra
ACT–2601, Australia.