A FIRST ORDER CORRECTION TO THE THEORY FOR THE ELECTROMAGNETIC RESPONSE OF A THIN CONDUCTING SHEET

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Abstract

The eddy-currents that are induced in a thin conducting sheet by a sinusoidally varying primary magnetic field are investigated in the low frequency limit, when the depth of penetration of the primary field is much greater than the thickness of the sheet, by setting up a perturbation scheme in terms of the small parameter, δ, which is the ratio of the thickness of the sheet to the length scale of the primary field. The terms of zero order in δ give the familiar results that were first obtained by Maxwell. The terms of first order in δ depend on the tangential as well as the normal component of the primary field and give eddy-current distributions that vary quadratically across the sheet. The boundary conditions that determine the accompanying secondary magnetic field are derived. Detailed results are given when the primary field is that due to a dipole.

The results are relevant to problems in ionospheric physics and geophysical exploration.

1. INTRODUCTION

The electromagnetic response of a thin conducting sheet in the low frequency limit when the depth of penetration of the primary magnetic field is much greater than the thickness of the sheet is of interest in a number of branches of geophysics including ionospheric physics and geophysical exploration. It was shown by Maxwell [8] that if the sheet is uniform then to leading order in its thickness the eddy-current density is constant across the sheet and the scalar potential of the secondary magnetic field outside the sheet may be calculated by replacing the sheet by a surface of discontinuity at which a certain boundary condition must be satisfied.

Maxwell's treatment was generalized by Price [9] and Ashour and Price [2] to cases when the sheet is non-uniform laterally and used to calculate the currents induced in the ionosphere by an external magnetic field. Extensions to sheets of finite lateral extent which are of interest in exploration geophysics have been described by Lamontagne and West [7] who considered a rectangular sheet and Siew and Hurley [10]who considered a circular sheet in a non-uniform primary field.

The present paper describes a perturbation scheme in terms of the small parameter δ which is the ratio of the thickness of the sheet to the length scale of the primary field. The case when the sheet is uniform is considered first. The terms of zero order in δ give Maxwell's results. The terms of first order in δ are found to depend on the tangential as well as the normal component of the primary field and give eddy-current distributions that vary quadratically across the sheet, and the boundary conditions that determine the accompanying secondary magnetic field are derived. Detailed results are then given when the primary field is that due to a dipole. The final section gives results when the sheet is non—unifrom laterally. The terms of zero order in δ give Price's boundary condition, and the boundary conditions that determine the first order correction are also derived.

The only other work of which we are aware that takes account of the finite thickness of the sheet is that of Dmitriev which is described in Chapter 2 of Berdichevsky and Zhdanov [3]. His approach is different to that employed herein: he uses analytic continuation to extend in the vertical direction an electro-magnetic field that is supposedly known in a horizontal plane. There are two fundamental differences in his approach and ours. He assumes that inside the sheet the derivatives of the various electromagnetic quantities in both the vertical and horizontal directions are of comparable magnitude whereas we assume that the ratio of the former to the latter are in the ratio of the length scale of the exciting field to the thickness of the sheet. Also we only consider the case when the sheet is surrounded by a vacuum and this leads to boundary conditions at the surfaces of the sheet that are different to those considered by Dmitriev.

2. BASIC EQUATIONS

Consider a thin, infinite plane sheet of material having uniform electrical conductivity σ, magnetic permeability μ and thickness 2d. Let Oxyz be a set of rectangular axes with Oxy lying in the plane of symmetry of the sheet. The sheet is supposed to be surrounded by a vacuum in which there is a sinusoidally varying primary magnetic field $\underset{\sim}{H}^{(p)}\exp(i\omega t)$ whose length scale L is much larger than d. If $\underset{\sim}{H}$ is the total magnetic field with corresponding induction $\underset{\sim}{B}$, then, neglecting displacement currents, the governing equations outside the sheets are

$$\nabla\cdot\underset{\sim}{B} = 0, \quad \text{and} \quad \nabla\times\underset{\sim}{H} = 0 \quad , \tag{2.1}$$

where $\underset{\sim}{H} = \underset{\sim}{H}^{(p)} + \underset{\sim}{H}^{(s)}$ is composed of a primary field $\underset{\sim}{H}^{(p)}$, and the secondary field $\underset{\sim}{H}^{(s)}$ due to currents induced in the conductor and a time variation $\exp(i\omega t)$ is assumed throughout. The above equations imply the existence of a scalar potential Φ such that

$$\underset{\sim}{H}^{(s)} = -\nabla\Phi, \qquad \nabla^2\Phi = 0 . \tag{2.2}$$

Inside the conductor, the governing equations are

$$\nabla \times \underset{\sim}{E} = -i\omega\underset{\sim}{B}, \qquad \nabla\cdot\underset{\sim}{B} = 0, \qquad \nabla \times \underset{\sim}{H} = \sigma\underset{\sim}{E}, \tag{2.3}$$

where $\underset{\sim}{E}$ is the electric field. Eliminating $\underset{\sim}{E}$ from these three equations gives

$$\nabla^2\underset{\sim}{H} = i\mu\sigma\omega\underset{\sim}{H}. \tag{2.4}$$

It is convenient to write $\underset{\sim}{H} = \underset{\sim}{H}^{(p)} + \underset{\sim}{h}$, in which case equation (2.4) becomes

$$\nabla^2\underset{\sim}{h} = i\mu\sigma\omega(\underset{\sim}{H}^{(p)} + \underset{\sim}{h}) . \tag{2.5}$$

Also, the divergence of the magnetic field inside the sheet must vanish so that

$$\nabla\cdot\underset{\sim}{h} = 0. \tag{2.6}$$

Let (H_x, H_y, H_z) and (h_x, h_y, h_z) be the components of the vectors $\underset{\sim}{H}$ and $\underset{\sim}{h}$. The continuity of H_x and H_y and their tangential derivatives at the surfaces of the sheet implies that $\frac{\partial}{\partial z} H_z$ is also continuous there since $\nabla\cdot\underset{\sim}{H}$ vanishes everywhere. Also, the normal component of $\mathbf{B}$ must be continuous. Hence, at $z = \pm d$,

$$H_x^{(s)} = h_x , \qquad H_y^{(s)} = h_y , \tag{2.7}$$

$$\frac{\partial H_z^{(s)}}{\partial z} = \frac{\partial h_z}{\partial z} , \tag{2.8}$$

and

$$\mu_0(H_z^{(p)} + H_z^{(s)}) = \mu(H_z^{(p)} + h_z) \tag{2.9}$$

where μ_0 is the permeability of free space.

3. THE PERTURBATION SCHEME

We define the small parameter δ by

$$\delta = \frac{d}{L} \tag{3.1}$$

and construct inner and outer expansions, Cole [4] for the limit $\delta \to 0$. In this limit the sheet becomes infinitely thin and may be replaced by a surface having conductance $2\sigma d$. To ensure that the eddy–currents remain finite we require the non–dimensional (induction) number

$$\alpha = \mu\ \sigma\ \omega\ d\ L \tag{3.2}$$

to remain finite in the limit.

3.1 THE INNER SOLUTION

The inner region is the interior of the sheet and here we use the variables

$$x' = \frac{x}{L}\ , \quad y' = \frac{y}{L}\ , \quad z' = \frac{z}{d}\ , \quad \underset{\sim}{h}^{*} = \underset{\sim}{h}/\mathcal{H} \tag{3.3}$$

where $\mathcal{H}$ is a representative value of the primary field.

Dropping the asterisk we assume

$$\underset{\sim}{h} = \underset{\sim}{h}^{(0)} + \delta \underset{\sim}{h}^{(1)} + \delta^{2} \underset{\sim}{h}^{(2)} + \cdots, \tag{3.4}$$

Substitution of equation (3.4) into equations (2.5) and (2.6), and equating powers of δ yield

$$\frac{\partial}{\partial z'} h_{z}^{(0)} = 0, \tag{3.5}$$

$$\frac{\partial^{2} \underset{\sim}{h}^{(0)}}{\partial z'^{2}} = 0, \tag{3.6}$$

$$\frac{\partial^{2}\ \underset{\sim}{h}^{(1)}}{\partial z'^{2}} = i\alpha\left(\underset{\sim}{h}^{(0)} + \underset{\sim}{f}^{(0)}\right), \tag{3.7}$$

$$\frac{\partial h_z^{(1)}}{\partial z'} = -\left[\frac{\partial h_x^{(0)}}{\partial x'} + \frac{\partial h_y^{(0)}}{\partial y'}\right], \tag{3.8}$$

$$\frac{\partial^2 \underset{\sim}{h}^{(2)}}{\partial z'^2} = i\alpha(\underset{\sim}{h}^{(1)} + z' \underset{\sim}{f}^{(1)}) - \left[\frac{\partial^2 \underset{\sim}{h}^{(0)}}{\partial x'^2} + \frac{\partial^2 \underset{\sim}{h}^{(0)}}{\partial y'^2}\right], \tag{3.9}$$

and

$$\frac{\partial h_z^{(2)}}{\partial z'} = -\left[\frac{\partial h_x^{(1)}}{\partial x'} + \frac{\partial h_y^{(1)}}{\partial y'}\right], \tag{3.10}$$

where $\underset{\sim}{f}^{(0)} = \underset{\sim}{H}^{(p)}(x', y', 0)$. Integrating equations (3.6) and (3.7) with respect to z' yield

$$\underset{\sim}{h}^{(0)} = z' \underset{\sim}{C} + \underset{\sim}{D} \tag{3.11}$$

and
$$\underset{\sim}{h}^{(1)} = i\alpha \left\{ \frac{z'^2}{2} \left[\underset{\sim}{f}^{(0)} + \underset{\sim}{D} \right] + \frac{z'^3}{6} \underset{\sim}{C} \right\} + z' \underset{\sim}{F} + \underset{\sim}{G} \tag{3.12}$$

where $\underset{\sim}{C}$, $\underset{\sim}{D}$, $\underset{\sim}{F}$ and $\underset{\sim}{G}$ are vector valued functions of x' and y' (whose components we denote by suffices) which are arbitrary at this stage. Use of equation (3.5) now gives

$$C_z = 0 \tag{3.13}$$

Now, integrating equation (3.9) once and comparing the coefficient of z' in the resulting expression for $\dfrac{\partial h_z^{(2)}}{\partial z'}$ with that obtained from equation (3.10) yields

$$i\alpha G_z = \frac{\partial^2 D_z}{\partial x'^2} + \frac{\partial^2 D_z}{\partial y'^2} - \frac{\partial F_x}{\partial x'} - \frac{\partial F_y}{\partial y'} . \tag{3.14}$$

3.2 THE OUTER SOLUTION

The outer region is that outside the sheet and here we use the variables

$$X = x' = \frac{x}{L}, \quad Y = y' = \frac{y}{L}, \quad Z = \frac{z}{L}, \quad \underset{\sim}{H}^* = \underset{\sim}{H}/\mathcal{H} \tag{3.15}$$

and the asterisk will be dropped in the subsequent analysis. We replace

$\underset{\sim}{H}^{(p)}(X,Y,Z)$ by its Taylor expansion about $Z = 0$:

$$\underset{\sim}{H}^{(p)}(X,Y,Z) = \underset{\sim}{f}^{(0)}(X,Y) + Z\underset{\sim}{f}^{(1)}(X,Y) + \cdots \tag{3.16}$$

where $f^{(m)}(X,Y) = \dfrac{\partial^m \underset{\sim}{H}^{(p)}}{\partial Z^m}(X,Y,0)$.

On the interfaces $Z = \pm\delta$, use of Taylor's expansion gives

$$\underset{\sim}{H}^{(s)}(X,Y,\pm\delta) = -\nabla\Phi_{\pm} \mp \delta\nabla\frac{\partial\Phi_{\pm}}{\partial Z} - \frac{\delta^2}{2}\nabla\frac{\partial^2\Phi_{\pm}}{\partial Z^2} - \cdots \tag{3.17}$$

where the suffices $\pm$ denote limiting values as the plane $Z = 0$ is approached from above or below respectively, and the top and bottom signs go together. We also assume an outer solution given by

$$\Phi = \Phi^{(0)} + \delta\Phi^{(1)} + \delta^2\Phi^{(2)} + \cdots. \tag{3.18}$$

3.3 THE MATCHING CONDITIONS

Equations (2.9), (3.11), (3.12), (3.16) and (3.18) show that the condition that B_z should be continuous at the interfaces gives to zeroth order in δ ,

$$(\lambda-1)f_z^{(0)} - \lambda\frac{\partial\Phi_{\pm}^{(0)}}{\partial Z} = D_z \tag{3.19}$$

where $\lambda = \mu_0/\mu$, and to first order in δ ,

$$-\frac{i\alpha}{2}f_z^{(0)} \pm (\lambda-1)f_z^{(1)} \mp \lambda\frac{\partial^2\Phi_{\pm}^{(0)}}{\partial Z^2} - \lambda\frac{\partial\Phi_{\pm}^{(1)}}{\partial Z} = \frac{i\alpha}{2}D_z + G_z \pm F_z. \tag{3.20}$$

Now let

$$\Phi^{(0)} = \Phi^{0E} + \Phi^{00},$$

where

$$\Phi^{0E}(X,Y,Z) = \tfrac{1}{2}\{\Phi^{(0)}(X,Y,Z) + \Phi^{(0)}(X,Y,-Z)\},$$

and

$$\Phi^{00}(X,Y,Z) = \tfrac{1}{2}\{\Phi^{(0)}(X,Y,Z) - \Phi^{(0)}(X,Y,-Z)\}.$$

Φ^{0E} is an even function of Z so that $\frac{\partial\Phi^{0E}}{\partial Z}$ is an odd function of Z. Also Φ^{00} is an odd function of Z so that $\frac{\partial\Phi^{00}}{\partial Z}$ is an even function of Z. In terms of these functions the two equations (3.19) can be written as

$$(\lambda-1)f_z^{(0)} - \lambda\left\{\frac{\partial\Phi_+^{0E}}{\partial Z} + \frac{\partial\Phi_+^{00}}{\partial Z}\right\} = D_z$$

and

$$(\lambda-1)f_z^{(0)} - \lambda\left\{\frac{\partial\Phi_-^{0E}}{\partial Z} + \frac{\partial\Phi_-^{00}}{\partial Z}\right\} = D_z .$$

Subtracting these equations gives

$$2\lambda\frac{\partial\Phi_+^{0E}}{\partial Z} = 0.$$

This boundary condition together with the condition that Φ^{0E} tends to zero at large distances like $\left[X^2 + Y^2 + Z^2\right]^{-1/2}$ shows that (see for example Jeffreys and Jeffreys, [6], §6.093) $\Phi^{0E} \equiv 0$ in $Z > 0$. Hence $\Phi^{(0)}$ is an odd function of Z so that by equation (3.11),

$$D_x = D_y = 0, \tag{3.21}$$

and use of equation (2.7) gives

$$C_x = -\frac{\partial\Phi_+^{(0)}}{\partial X} \text{ and } C_y = -\frac{\partial\Phi_+^{(0)}}{\partial Y} . \tag{3.22}$$

Noting that $\frac{\partial}{\partial Z} = \delta\frac{\partial}{\partial z'}$, terms of $O(\delta)$ in equation (2.8) give, using equations (3.12) and (3.16),

$$\frac{\partial^2\Phi_\pm^{(0)}}{\partial Z^2} = \mp i\alpha\,(f_z^{(0)} + D_z) - F_z . \tag{3.23}$$

But $\frac{\partial^2\Phi^{(0)}}{\partial Z^2}$ is an odd function of Z so that

$$F_z = 0 \tag{3.24}$$

and

$$\frac{\partial^2 \Phi_+^{(0)}}{\partial Z^2} = - i\alpha \, (f_z^{(0)} + D_z) \quad . \tag{3.25}$$

Elimination of D_z between equations (3.19) and (3.25) gives

$$\frac{\partial^2 \Phi_+^{(0)}}{\partial Z^2} = - \frac{\partial^2 \Phi_-^{(0)}}{\partial Z^2} = - i\alpha\lambda \, (f_z^{(0)} - \frac{\partial \Phi^{(0)}}{\partial Z}) \ . \tag{3.26}$$

This boundary condition determines $\Phi^{(0)}$ and is equivalent to that obtained by Maxwell (1891). In equation (3.26) $\dfrac{\partial^2 \Phi_+^{(0)}}{\partial Z^2}$ is used on the left when $z > 0$ and $\dfrac{\partial^2 \Phi_-^{(0)}}{\partial Z^2}$ is used when $z < 0$.

Proceeding in a similar way to our treatment of $\Phi^{(0)}$ we take

$$\Phi^{(1)} = \Phi^{1E} + \Phi^{1O} \tag{3.27}$$

where Φ^{1E} is an even function of Z and Φ^{1O} an odd function. Then taking the upper and lower signs in equation (3.20) separately, noting equation (3.24) and adding and subtracting the resulting equations gives

$$\frac{\partial \Phi_+^{1E}}{\partial Z} = - \frac{\partial \Phi_-^{1E}}{\partial Z} = (1 - \frac{1}{\lambda}) \ f_z^{(1)} \tag{3.28}$$

and

$$-\lambda \frac{\partial^2 \Phi_+^{(0)}}{\partial Z^2} - \lambda \frac{\partial \Phi_+^{1O}}{\partial Z} = \frac{i\alpha}{2} \left(f_z^{(0)} + D_z \right) + G_z \ . \tag{3.29}$$

Equation (3.28) is the boundary condition that determines Φ^{1E} .

Now, equations (2.7), (3.11) and (3.12) give

$$F_x = - \frac{\partial \Phi_+^{1O}}{\partial X} - \frac{i\alpha}{6} C_x - \frac{\partial^2 \Phi_+^{(0)}}{\partial X \partial Z}$$

and (3.30)

$$F_y = - \frac{\partial \Phi_+^{1O}}{\partial Y} - \frac{i\alpha}{6} C_y - \frac{\partial^2 \Phi_+^{(0)}}{\partial Y \partial Z} \ .$$

Finally using equations (3.14), (3.19), (3.22) and (3.30) to eliminate D_z and G_z from equation (3.29) gives

$$\frac{\partial^2\Phi_+^{10}}{\partial Z^2} = -\frac{\partial^2\Phi_-^{10}}{\partial Z^2} = -\,i\alpha\,(q - \lambda\,\frac{\partial\Phi^{10}}{\partial Z}) \tag{3.31}$$

where

$$q = (\frac{1}{3} - \lambda)\,\frac{\partial^2\Phi_+^{(0)}}{\partial Z^2} + \frac{1}{\alpha^2}\,(1 - \frac{1}{\lambda})\,\frac{\partial^4\Phi_+^{(0)}}{\partial Z^4}$$

which is the boundary condition that determines Φ^{10} once $\Phi^{(0)}$ has been found. Thus, each of the potentials $\Phi^{(0)}$, Φ^{10}, and Φ^{1E} may be found by solving Laplace's equation subject to prescribed conditions on their normal derivatives at the surface of the conducting sheet.

4. THE EDDY CURRENTS

In the conductor $\underset{\sim}{H} = \underset{\sim}{H}^{(p)} + \underset{\sim}{h}$ and $\nabla \times \underset{\sim}{H}^{(p)} = 0$ so that, using the last of equations (2.3), the eddy-current density is

$$\underset{\sim}{j} = \sigma\underset{\sim}{E} = \nabla \times \underset{\sim}{h}\ . \tag{4.1}$$

Let the non-dimensional current density be

$$\underset{\sim}{j}^{*} = \underset{\sim}{j}d/\mathcal{H} = \underset{\sim}{j}^{(0)} + \delta\underset{\sim}{j}^{(1)} + \delta^2\underset{\sim}{j}^{(2)} + \cdots. \tag{4.2}$$

Then equations (3.11), (3.12), (3.22) and (4.1) give

$$\underset{\sim}{j}^{(0)} = \left(\frac{\partial\Phi_+^{(0)}}{\partial Y}\ ,\ -\frac{\partial\Phi_+^{(0)}}{\partial X}\ ,\ 0\right) \tag{4.3}$$

and

$$\left.\begin{aligned} \underset{\sim}{j}^{(1)} &= \left(j_x^{(1)},\ j_y^{(1)},\ 0 \right) \\ j_x^{(1)} &= \frac{\partial}{\partial y'}D_z - F_y - i\alpha\left(z' f_y^{(0)} + \frac{z'^2}{2}\,C_y\right) \\ j_y^{(1)} &= -\frac{\partial}{\partial x'}D_z + F_x + i\alpha\left(z' f_x^{(0)} + \frac{z'^2}{2}\,C_x\right). \end{aligned}\right\} \tag{4.4}$$

Equations (3.19), (3.22), (3.30) and (4.4) may now be used to obtain

$j_x^{(1)}$ and $j_y^{(1)}$ in terms of the outer solution:

$$j_x^{(1)} = (1-\lambda)\frac{\partial^2\Phi_+^{(0)}}{\partial Y\partial Z} - (1-\lambda)\frac{\partial}{\partial Y}f_z^{(0)} - i\alpha z' f_y^{(0)}$$

$$+ i\alpha\left(\frac{z'^2}{2} - \frac{1}{6}\right)\frac{\partial\Phi_+^{(0)}}{\partial Y} + \frac{\partial\Phi^{10}}{\partial Y}, \qquad (4.5)$$

and

$$j_y^{(1)} = -(1-\lambda)\frac{\partial^2\Phi_+^{(0)}}{\partial X\partial Z} + (1-\lambda)\frac{\partial}{\partial X}f_z^{(0)} + i\alpha z' f_x^{(0)}$$

$$- i\alpha\left(\frac{z'^2}{2} - \frac{1}{6}\right)\frac{\partial\Phi_+^{(0)}}{\partial X} - \frac{\partial\Phi^{10}}{\partial X}. \qquad (4.6)$$

Equation (4.3) shows that $j^{(0)}$ is uniform through the thickness of the sheet and equation (4.4) that the first order correction $j^{(1)}$ is quadratic in z'. It also shows the non-uniformity becomes significant when $\delta\alpha$ is not negligible. Now

$$\delta\alpha = \sigma\mu\omega d^2 = 2(d/D_s)^2 \qquad (4.7)$$

where $D_s = \sqrt{2/(\sigma\mu\omega)}$ is the skin depth, so that the condition for the non-uniformity to be significant is that d/D_s should not be negligible, in other words, when the sheet is "inductively thick" as described by Lamontagne and West [7].

In terms of the integrated current density

$$\underset{\sim}{I} = \int_{-1}^{1} \underset{\sim}{j}\; dz'$$

across the sheet we have from equations (4.3), (4.5) and (4.6) that

$$\underset{\sim}{I}^{(0)} = 2\left(\frac{\partial\Phi_+^{(0)}}{\partial Y}, -\frac{\partial\Phi_+^{(0)}}{\partial X}, 0\right) \qquad (4.8)$$

and

$$\begin{aligned} I_x^{(1)} &= 2(1-\lambda)\frac{\partial^2\Phi_+^{(0)}}{\partial Y\partial Z} - 2(1-\lambda)\frac{\partial}{\partial Y}f_z^{(0)} + 2\frac{\partial\Phi^{10}}{\partial Y}, \\ I_y^{(1)} &= -2(1-\lambda)\frac{\partial^2\Phi_+^{(0)}}{\partial X\partial Z} + 2(1-\lambda)\frac{\partial}{\partial X}f_z^{(0)} - 2\frac{\partial\Phi^{10}}{\partial X}. \end{aligned} \qquad (4.9)$$

Putting $\lambda = 1$ in equation (4.9) and using the last relation in (2.3) we obtain the following jump discontinuity relations for the horizontal components of the electric and magnetic fields.

$$\sigma\left[E_{x+} - E_{x-}\right] = - 2i\alpha\delta\, f_y^{(0)} + O(\delta^2), \qquad (4.10)$$

$$\sigma\left[E_{y+} - E_{y-}\right] = 2i\alpha\delta\, f_x^{(0)} + O(\delta^2), \qquad (4.11)$$

$$h_x(x',y',1) - h_x(x',y',-1) = H_{x+} - H_{X-}$$

$$= I_y^{(0)} + \delta\left(I_y^{(1)} - 2\,\frac{\partial^2\Phi_+^{(0)}}{\partial x\partial z}\right) + O(\delta^2), \qquad (4.12)$$

and

$$h_y(x',y',1) - h_y(x',y',-1) = H_{y+} - H_{y-}$$

$$= - I_x^{(0)} - \delta\left(I_x^{(1)} + 2\,\frac{\partial^2\Phi_+^{(0)}}{\partial Y\partial Z}\right) + O(\delta^2). \qquad (4.13)$$

These results may be compared with those of Dmitriev given on page 24 of Berdichevsky and Zhdanov [3]. Bearing in mind that in our case the sheet has thickness 2d and our time variation is $e^{i\omega t}$ the results given by equations (4.12) and (4.13) agree with his except that his do not contain the terms $\delta I_y^{(1)}$ and $\delta I_X^{(1)}$.

The results for the distribution within the sheet of the components of the fields and the eddy–currents are quite different. According to equations (3.11), (3.12) and (3.13) the horizontal components of the magnetic field are cubic in z′ and the vertical component quadratic. However, equations (3.18) and (3.20) of Berdichevsky and Zhdanov (with p=0) give linear expressions for each of the components. The differences arise because Dmitriev assumes that the derivatives of the various electromagnetic quantities in both the vertical and horizontal directions are of comparable magnitude whereas we assume that the former are $1/\delta$ times the latter.

5. A VERTICAL DIPOLE ABOVE AN INFINITE CONDUCTING SHEET.

We report here the case of a dipole source above an infinite conducting sheet. A circular conductor of finite extent can also be treated and will be the subject of another paper.

Consider a unit dipole at (0,0,h) with its axis in the direction of Oz so that

$$H_Z^{(p)} = \frac{2(Z-1)^2 - r^2}{4\pi[(Z-1)^2 + r^2]^{5/2}} \tag{5.1}$$

where $r^2 = X^2 + Y^2$ and in equation (3.15) we have replaced L by h .

Then by taking Hankel transforms we find that the harmonic function, that vanishes at infinity and satisfies equation (3.26) is

$$\Phi^{(0)} = - \frac{i\alpha\lambda}{4\pi} \int_0^\infty \frac{s\ e^{-s(Z+1)}}{s + i\alpha\lambda} J_0(sr)\ ds, \quad z > 0, \tag{5.2a}$$

and

$$\Phi^{(0)} = \frac{i\alpha\lambda}{4\pi} \int_0^\infty \frac{s\ e^{s(Z-1)}}{s + i\alpha\lambda} J_0(sr)\ ds, \quad z < 0, \tag{5.2b}$$

where J_0 is the Bessel function of order zero, of the first kind. Similarly the function that satisfies equation (3.28) is

$$\Phi^{1E} = \begin{cases} -\frac{1}{4\pi}\left(1 - \frac{1}{\lambda}\right) \dfrac{2(Z+1)^2 - r^2}{\left[r^2 + (Z+1)^2\right]^{5/2}}, & z > 0 \\[2ex] -\frac{1}{4\pi}\left(1 - \frac{1}{\lambda}\right) \dfrac{2(1-Z)^2 - r^2}{\left[r^2 + (1-Z)^2\right]^{5/2}}, & z < 0 \end{cases} \tag{5.3}$$

and the one that satisfies equation (3.31) is

$$\Phi^{10} = -(\mathrm{sgn}\ z)\frac{\alpha^2\lambda}{4\pi} \int_0^\infty \frac{s^2\ e^{-s(|Z|+1)}}{(s + i\alpha\lambda)^2} J_0(sr) \left(\tfrac{1}{3} - \lambda + \frac{1}{\alpha^2} [1 - \tfrac{1}{\lambda}] s^2 \right) ds, \tag{5.4}$$

where sgn is the sign function.

Equation (5.3) shows that in $z > 0$, Φ^{1E} is the potential of a quadrupole of strength $(1 - \frac{1}{\lambda})$ situated at $(0,0,-1)$, with a corresponding interpretation for the case $z < 0$; a result that may be deduced by inspection of equations (3.28) and (5.1).

5.1 THE EDDY CURRENTS.

Since all quantities are independent of the azimuthal angle θ, equations (4.2) to (4.6) show that the eddy-currents are in the direction of θ increasing and have magnitude

$$j_\theta = -\frac{\partial \Phi_+^{(0)}}{\partial r} - \delta\left(\frac{\partial p}{dr} + i\alpha z' T\right) + O(\delta^2), \qquad (5.5)$$

where $p(r) = (1 - \lambda)\left(\frac{\partial \Phi_+^{(0)}}{\partial z} - f_z^{(0)}\right) + i\alpha\left(\frac{z'^2}{2} - \frac{1}{6}\right)\Phi_+^{(0)} + \Phi^{10}$,

and

$$T = \left|(f_x^{(0)}, f_y^{(0)}, 0)\right| \qquad (5.6)$$

is the magnitude of the tangential component of the primary field.

For the case $\lambda = 1$, and $\delta = 0.1$, Figure 1 gives $j_\theta^{(0)}$ as a function of r for $\alpha = 1$ and 5 and Figures 2 and 3 show the variation of j_θ across the sheet, as given by equation (5.5) for $\alpha = 1$ and 5 respectively and three values of r.

The term in equation (5.5) that involves T (which is real) is the contribution to the eddy-current distribution from the tangential component of the primary field which we see from the above equation only influence the quadrature component of the eddy–current distribution. This may be explained by considering Faraday's law

$$\int_C \underset{\sim}{E}.d\underset{\sim}{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\underset{\sim}{\mathbf{n}}\, dS$$

where $\underset{\sim}{\mathbf{n}}$ is a unit vector perpendicular to the surface S that is bounded by C. We take C to be a small rectangle within the sheet having vertical and horizontal sides and lying in a plane whose normal is in the radial

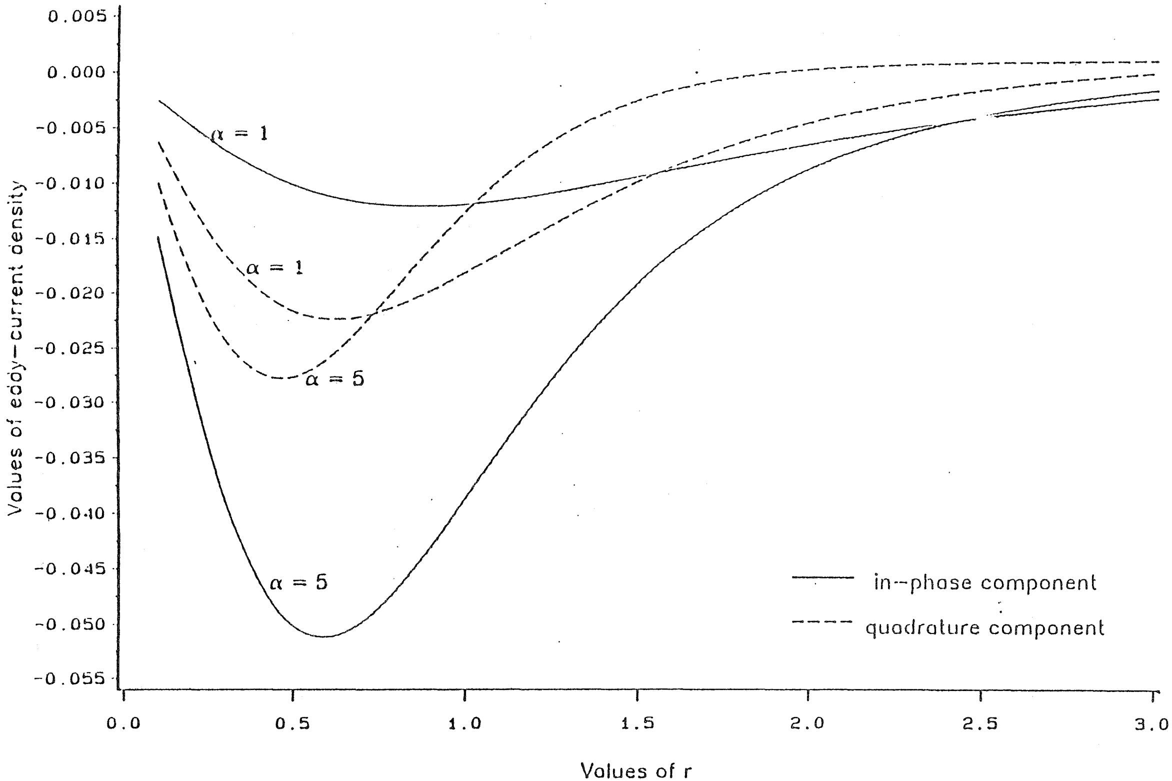

Figure 1. Zeroth order current density, $j_{\theta}^{(0)}$, as a function of r for $\alpha = 1$ and 5 .

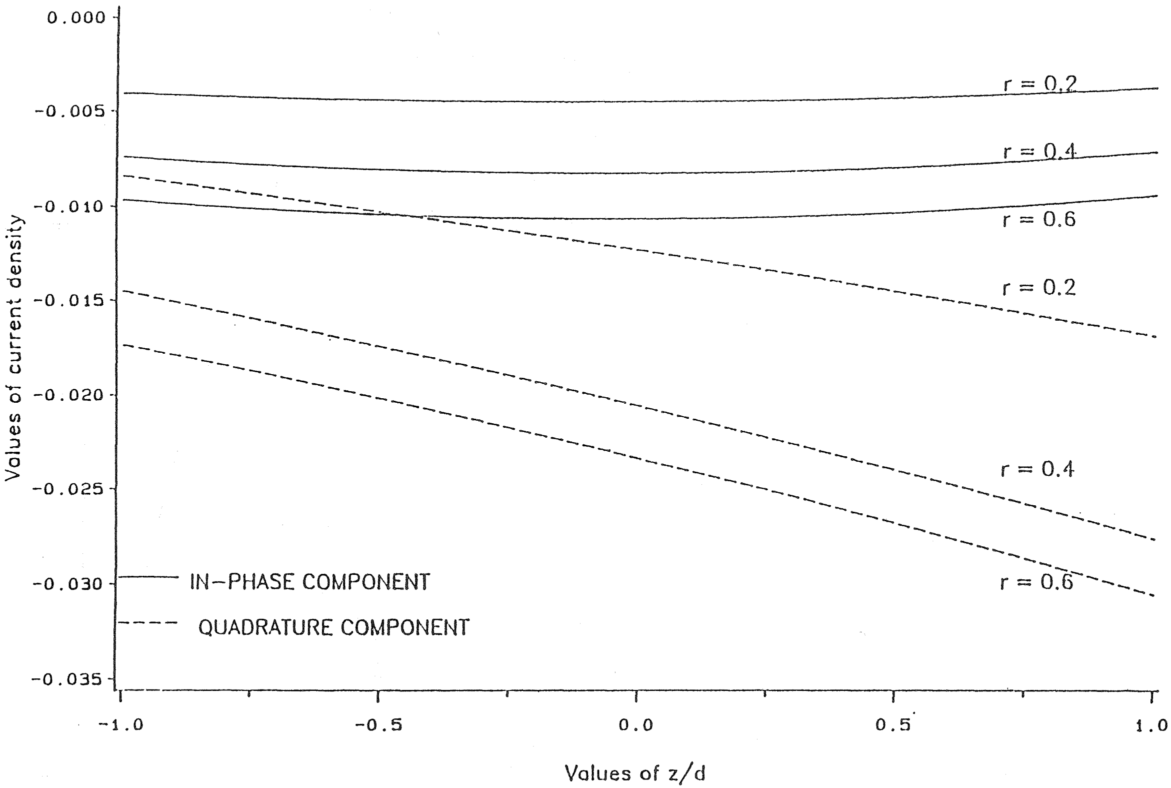

Figure 2. Current density variation across the thickness of the sheet at three different values of r. $\alpha = 1$, $\lambda = 1$ and $\delta = 0.1$.

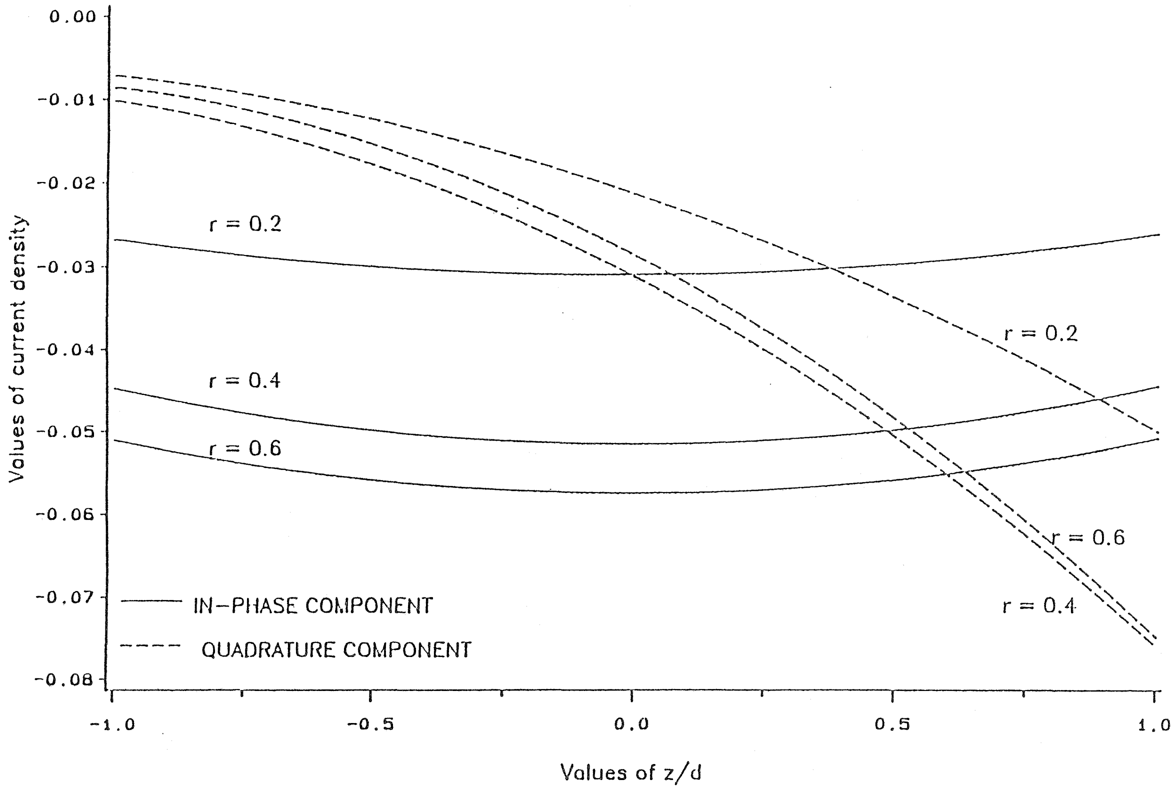

Figure 3. Current density variation across the thickness of the sheet with $\alpha = 5$, $\lambda = 1$ and $\delta = 0.1$.

direction. The right hand side is pure imaginary. Also only the horizontal parts of C will contribute to the integral on the left since $\underset{\sim}{E} = \underset{\sim}{j}/\sigma$ is in the azimuthal direction. Thus only the quadrature component of $\underset{\sim}{E}$ and $\underset{\sim}{j}$ will be affected. This effect is evident in Figures 2 and 3 and is seen to be most marked for the larger values of r and α as expected.

5.2 THE SECONDARY MAGNETIC FIELD.

Suppose there is a small receiving coil coincident with the transmitting dipole and whose axis is in the Oz direction. Its output due to the presence of the sheet will be proportional to $H_z^{(s)}$, which by equations (5.2) to (5.4) is found to be (see Gradshteyn and Ryzhik, [5] §3.356(1) & (2); and Abramowitz and Stegun, [1] p.232).

$$H_z^{(s)}(0,0,1) = -\frac{\partial\Phi^{(0)}}{\partial z} - \delta\left(\frac{\partial\Phi^{10}}{\partial z} + \frac{\partial\Phi^{1E}}{\partial z}\right) + O(\delta^2), \qquad (5.7)$$

$$= -\frac{1}{4\pi}(X + iY) + 0\,(\delta^2)$$

where

$$X = \frac{\beta^2}{2} - \beta^3 f(\gamma) + \delta\left[\frac{3}{8}\left(1 - \frac{1}{\lambda}\right) + X_{10}\right],$$

$$Y = \frac{\beta}{4} - \beta^3 g(\gamma) + \delta\, Y_{10},$$

$$X_{10} = \frac{\beta^2}{\lambda}\left(\frac{1}{3} - \lambda\right)\left\{\frac{1}{4} - 3\beta^2 g(\gamma) + \beta^2(1 - \gamma f(\gamma))\right\}$$
$$+ (\lambda - 1)\left\{\frac{3}{8} - \frac{3}{4}\beta^2 + 5\beta^4 g(\gamma) - \beta^4(1 - \gamma f(\gamma))\right\},$$

$$Y_{10} = -\frac{2\beta^3}{\lambda}\left(\frac{1}{3} - \lambda\right)\left(\frac{1}{2} - 2\beta f(\gamma) + \frac{1}{2}\beta[f(\gamma) + \gamma g(\gamma)]\right)$$
$$-2\beta(\lambda - 1)\left(\frac{1}{4} - \beta^2 + 3\beta^3 f(\gamma) - \frac{\beta^3}{2}[f(\gamma) + \gamma g(\gamma)]\right),$$

$$f(\gamma) = \mathrm{ci}\,\gamma \sin\gamma - \mathrm{si}\,\gamma\cos\gamma,$$

$$g(\gamma) = - \text{ci}\ \gamma \cos\gamma - \text{si}\ \gamma \sin\gamma ,$$

$$\beta = \alpha\lambda \text{ and } \gamma = 2\beta .$$

Figure 4 gives values of $X + iY$ against β, and shows the effect of the first order correction to the leading term, for the case $\lambda = 1$. δ is again taken to be 0.1. The increase in the magnitude of the in-phase component is evident only for values of β greater than about 2 which corresponds to $d/D_s = 0.3$. Figure 5 gives values of $X + iY$ for various values of λ.

6. THE NON-UNIFORM SHEET

We now suppose the conductivity σ of the sheet depends on x and y but is independent of z. Since the development of the theory closely resembles the case when the sheet is uniform only the main results will be given.

Outside the sheet the governing equations are unchanged. Inside the sheet the governing equation is found to be

$$\nabla^2\underset{\sim}{H} + \left(0,\ 0,\ \frac{1}{\sigma}\frac{\partial\sigma}{\partial x}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) - \frac{1}{\sigma}\frac{\partial\sigma}{\partial y}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\right) = i\mu\sigma\omega\underset{\sim}{H} \qquad (6.1)$$

We suppose that the length scale for variations in σ is the same as the length scale L of the primary field and use a similar perturbation scheme to that already described.

Define

$$\sigma^* = \frac{\sigma}{\Sigma} \qquad (6.2)$$

where Σ is a representative value of σ and

$$\alpha = \mu\Sigma\omega dL \qquad (6.3)$$

and drop the asterisk.

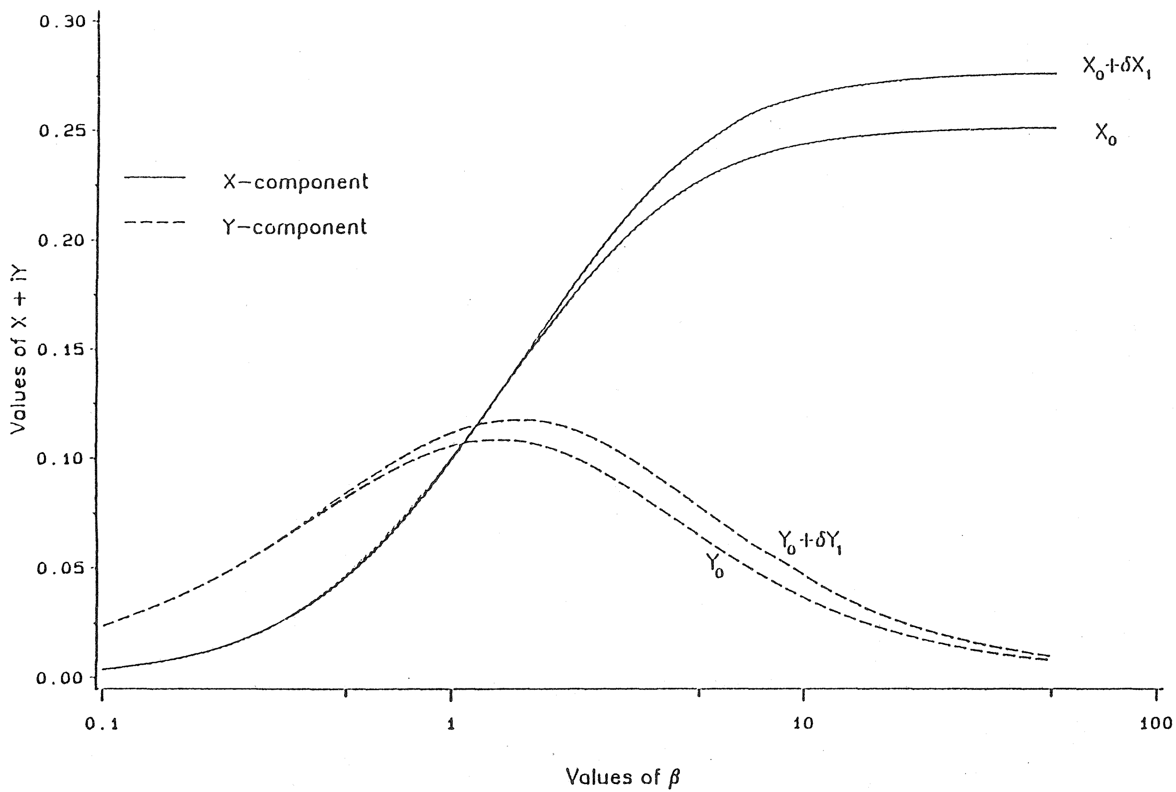

Figure 4. Plot of $X + iY$ against β with $\delta = 0.1$ and $\lambda = 1$.

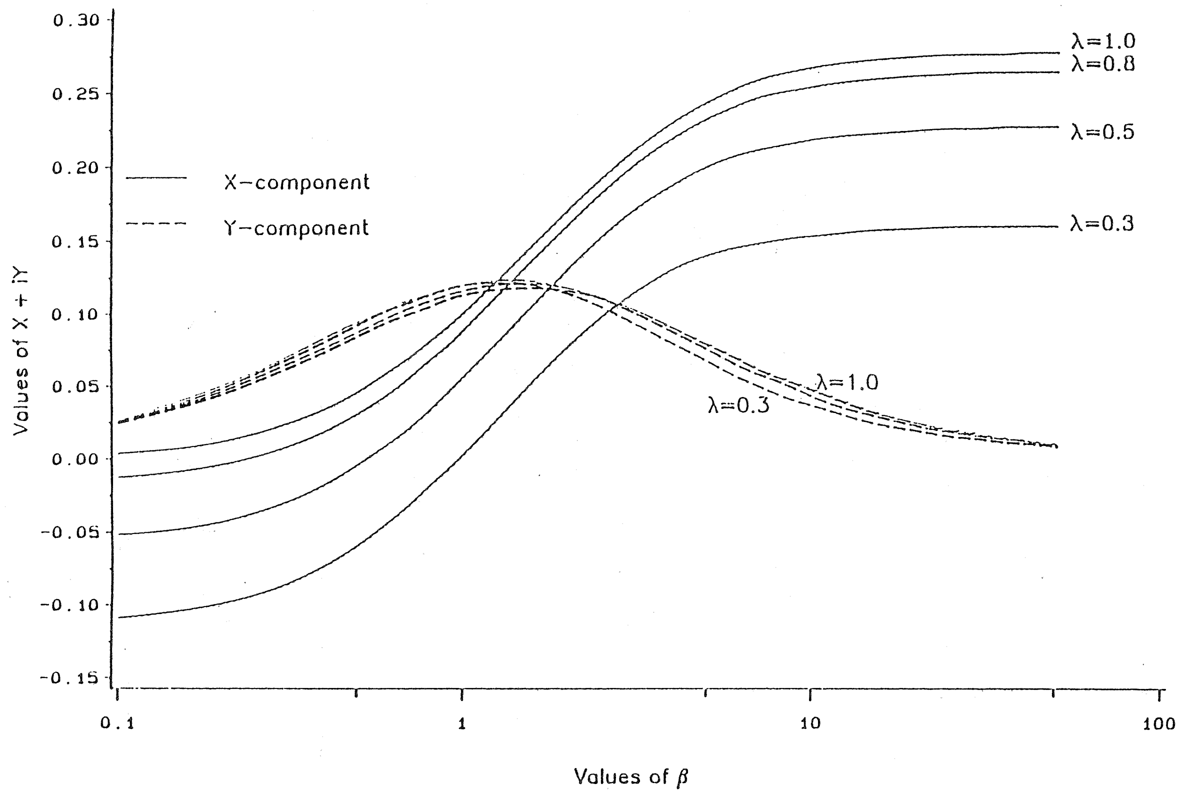

Figure 5. Plot of $X + iY$ against β with $\delta = 0.1$ and various values of λ.

Equations (3.11) and (3.12) stil hold except that the term $\frac{-z'^2}{2\sigma}\left\{c_x \frac{\partial\sigma}{\partial x'} + c_y \frac{\partial\sigma}{\partial y'}\right\}$ must be added to the righthand side of the z–component of equation (3.12).

We again assume that in the outer region

$$\phi = \phi^{(0)} + \delta\phi^{(1)} + \ldots \tag{6.4}$$

and using the same matching conditions we find that the boundary condition for $\phi^{(0)}$ is

$$\frac{\partial^2\phi_+^{(0)}}{\partial Z^2} = -\frac{\partial^2\phi_-^{(0)}}{\partial Z^2} = -i\sigma\lambda\left(f_z^{(0)} - \frac{\partial\phi^{(0)}}{\partial Z}\right) - \frac{1}{\sigma}\nabla_2\sigma.\nabla_2\phi_+^{(0)} \tag{6.5}$$

where ∇_2 is the vector operator $\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right)$.
Equation (6.5) is the boundary condition obtained by Price [9] .

For the first order correction we again take

$$\phi^{(1)} = \phi^{1E} + \phi^{10} . \tag{6.6}$$

ϕ^{1E} is found to satisfy the same boundary condition as before, namely

$$\frac{\partial\phi_+^{1E}}{\partial Z} = \left(1 - \frac{1}{\lambda}\right)f_z^{(1)} = -\frac{\partial\phi_-^{1E}}{\partial Z} \tag{6.7}$$

and ϕ^{10} the boundary condition

$$\frac{\partial^2\phi_+^{10}}{\partial Z^2} - i\lambda\alpha\sigma\frac{\partial\phi^{10}}{\partial Z} + \frac{1}{\sigma}\nabla_2\sigma.\nabla_2\phi_+^{10}$$

$$= -(\lambda - 1)\left(f_z^{(2)} - \frac{\partial^3\phi^{(0)}}{\partial Z^3}\right) + i\alpha\sigma\left(\lambda - \frac{1}{3}\right)\frac{\partial^2\phi_+^{(0)}}{\partial Z^2}$$

$$+ \frac{(\lambda - 1)}{\sigma}\nabla_2\sigma.\nabla_2\left(\frac{\partial\phi^{(0)}}{\partial Z} - f_z^{(0)}\right) \qquad (6.8)$$

We note that equation (6.8) is similar to equation (6.5) so that little additional effort would be needed to calculate the first order covection to Price's results.

For the uniform sheet, we note from equations (4.5) to (4.9) that while the horizontal components of the primary field affect the first order correction to the induced current density, only the vertical component is of importance in the integrated current density. We expect this to be true also for the case of non-uniform conductivities. In fact equations (6.5) to (6.8) indicate that, to first order, the outer solution is completely determined by the vertical component of the primary magnetic field.

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