WAITING-TIME BEHAVIOUR FOR A FOURTH-ORDER NONLINEAR DIFFUSION EQUATION

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ABSTRACT

The fourth order nonlinear diffusion equation $u_t + (u^n u_{xxx})_x = 0 \quad (n > 0)$ governs a number of important physical processes, such as the flow of a surface tension dominated thin liquid film and the diffusion of dopant in semiconductors. This equation will be analysed using a perturbation scheme in the limit of small n (ie $0 < n << 1$). In this limit, the solution is determined by a system of nonlinear hyperbolic equations. An analysis of the solution shows that if the initial condition is of compact support, the solution does not move outside of its initial domain. Shocks, corresponding to jumps in u_x, can form in the solution. An examination of the shock jump condition shows that a shock cannot propagate outside of the domain of the initial condition. It is concluded that all solutions of $u_t + (u^n u_{xxx})_x = 0$ for $0 < n << 1$ are waiting-time solutions.

1. INTRODUCTION

The fourth order nonlinear diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u^n \frac{\partial^3 u}{\partial x^3}\right) = 0 \tag{1.1}$$

models a number of physical processes, such as the flow of a surface tension dominated thin liquid film, for which $n = 3$ (see Greenspan, 1978; Greenspan and McCay, 1981; Hocking, 1981 and Lacey, 1982) and the diffusion of dopant in a semiconductor (see King, 1986 and Tayler, 1987). The analysis of this equation is difficult due to the nonlinearity and the high order. The solutions of (1.1) which have been found are all similarity solutions, either of the source-type or so-called blow-up solutions which become infinite in finite time (see Smyth and Hill, 1988). Equation (1.1) also possesses waiting-time solutions. These are solutions for which the initial condition is of compact support and which take a finite time to move outside of their initial domain. It is the analysis of these waiting-time solutions which forms a major concern of the present work.

Rather than seek exact solutions of (1.1), asymptotic solutions for $0 < n << 1$ will be found in the present work. The analysis of (1.1) for $0 < n << 1$ is similar to that of Kath and Cohen (1982) for the equivalent second order nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right). \tag{1.2}$$

Their asymptotic analysis showed that the solution of (1.2) for $0 < n << 1$ is governed by a system of nonlinear hyperbolic equations. In general, shocks, corresponding to jumps in u_x, can occur in the perturbation solution. It was found by Kath and Cohen that if the initial condition has

$$u \sim k(x - x_0)^\alpha, \tag{1.3}$$

where $\alpha > 0$, at the front x_0, then the solution exhibits waiting-time behaviour for $n\alpha \geq 2$. For $0 < n\alpha < 2$, the front moves immediately, agreeing with the results of Knerr (1977) and references therein. This immediate movement of the front for $0 < n\alpha < 2$ was shown to be due to the immediate formation of an outward propagating shock at the front. In the present work, it will be shown that for equation (1.1) shocks are inward propagating and so all solutions of (1.1) are waiting-time solutions for $0 < n << 1$.

2. ASYMPTOTIC SOLUTIONS

An asymptotic solution of (1.1) for $0 < n << 1$ will be found in the present section. In this limit, the diffusion coefficient u^n in (1.1) is near 1 for u away from zero and drops rapidly to zero as $u \to 0$. The limit $n \to 0$ is then singular since u^n cannot be expanded uniformly as a series in n uniformly valid for all u. To overcome this, we perform the change of variable

$$v(x, \tau) = u^{\frac{n}{3}} \tag{2.1}$$

$$t = \left(\frac{n}{3}\right)^3 \tau.$$

We note that v depends on the fast time τ.

Equation (1.1) becomes, in terms of the variables x, τ and v

$$v_\tau + \left(1 - \frac{n}{3}\right)\left(1 - \frac{2n}{3}\right) v_x^4 + n\left(1 - \frac{n}{3}\right)\left(2 - \frac{n}{3}\right) v v_x^2 v_{xx}$$
$$+ \frac{1}{3}n^2\left(1 - \frac{n}{3}\right) v^2 v_{xx}^2 + \frac{n^2}{9}\left(4 - \frac{n}{3}\right) v^2 v_x v_{xxx} + \frac{n^3}{27} v^3 v_{xxxx} = 0. \tag{2.2}$$

This equation can be solved using the perturbation series

$$v(x,\tau) = v_0(x,\tau) + n v_1(x,\tau) + n^2 v_2(x,\tau) + \ldots . \tag{2.3}$$

The solution for u is then found by inverting (2.1) to give

$$u = v_0^{\frac{3}{n}} e^{\frac{3v_1}{v_0}} \left[1 + 3n\left(\frac{v_2}{v_0} - \frac{v_1^2}{2v_0^2}\right) + \ldots\right]. \tag{2.4}$$

It can be seen from (2.4) that to determine the solution for u to $0(1)$, the solution for v must be found to $0(n)$.

Substituting the series (2.3) into (2.2), we find at $0(1)$,

$$v_{o\tau} + v_{0x}^4 = 0 \tag{2.5}$$

and at $0(n)$

$$v_{1\tau} + 4v_{0x}^3 v_{1x} = -v_0 v_{0x}^2 v_{0xx} + v_{0x}^4. \tag{2.6}$$

These equations may be solved by the method of characteristics (see Whitham, 1974) to give

$$v_0 = 3f'^4(\zeta)\tau + f(\zeta) \tag{2.7}$$

$$v_1 = \frac{3}{4}f'^4(\zeta)\tau - \frac{1}{12}\left[f(\zeta) - \frac{f'^2(\zeta)}{4f''(\zeta)}\right] X \tag{2.8}$$

$$\log(1 + 12f'^2(\zeta)f''(\zeta)\tau) + f_1(\zeta)$$

$$x = 4f'^3(\zeta)\tau + \zeta, \tag{2.9}$$

where $v_0 = f(\zeta)$, $v_1 = f_1(\zeta)$, $x = \zeta$ when $\tau = 0$.

The initial value of v_0, f, will in general be some positive function of compact support with $f' \neq 0$ at the fronts. If $f' \neq 0$ at the fronts, then we see from (2.9) that there will be a gap between the inward sloping characteristics at the fronts and the characteristics of zero slope outside of the domain in which f is non-zero. This gap is filled with centred simple waves originating at the fronts. For convenience, let us denote by f_r and f_r' the values of f and f' at the right hand front x_r and by f_l and f_l' the values of f and f' at the left hand front x_l. Then from (2.5) and (2.6), it can be found that the solutions in the centred simple wave at the right hand front are

$$\left.\begin{aligned} v_0 &= 3\tau\left(\frac{x_r - x}{4\tau}\right)^{\frac{4}{3}} \\ v_1 &= \frac{3}{4}\tau\left(\frac{x_r - x}{4\tau}\right)^{\frac{4}{3}} \end{aligned}\right\} \qquad f_r'^3 \leq \frac{x - x_r}{4\tau} \leq 0 \tag{2.10}$$

and the solutions in the centred simple wave at the left hand front are

$$\left.\begin{aligned} v_0 &= 3\tau\left(\frac{x - x_l}{4\tau}\right)^{\frac{4}{3}} \\ v_1 &= \frac{3}{4}\tau\left(\frac{x - x_l}{4\tau}\right)^{\frac{4}{3}} \end{aligned}\right\} \qquad 0 \leq \frac{x - x_l}{4\tau} \leq f_l'^3. \tag{2.11}$$

It can be seen that the solution given by (2.7) to (2.11) does not move outside its initial domain, and thus constitutes a waiting-time solution with infinite waiting-time. However, since (2.5) is a nonlinear hyperbolic equation, shocks can form and these shocks could modify the waiting-time behaviour, as was the case for the second order equation (1.2)(see Kath and Cohen, 1982). For this second order equation, outward propagating shocks could form immediately at the fronts (see (1.3)), so that waiting-time behaviour will not occur in this case. The occurrence and effect of shocks will be discussed in the next section.

3. SHOCKS

The family of characteristics (2.9) have an envelope given by

$$\tau = -[12f'^2(\zeta)f''(\zeta)]^{-1}. \tag{3.1}$$

Since $\tau \geq 0$, shocks will thus form in the solution if $f''(\zeta) < 0$ for some ζ. To determine the jump condition for a shock, we express (2.5) in conservation form

$$\frac{\partial}{\partial\tau}(v_{0x}) + \frac{\partial}{\partial x}(v_{0x}^4) = 0. \tag{3.2}$$

The conserved density is v_{ox} and so shocks will occur as jumps in v_{ox} with v_0 continuous at a shock. It is expected that shocks will occur in the derivative rather than the function as the original equation (1.1) is parabolic. From (3.2), the jump condition is

$$U = \frac{[v_{0x}^4]}{[v_{0x}]} = v_{0x1}^3 + v_{0x1}^2 v_{0x2} + v_{0x1} v_{0x2}^2 + v_{0x2}^3, \tag{3.3}$$

where [ ] denotes a jump in a quantity, U is the shock velocity and 1 and 2 indicate values ahead of and behind the shock respectively. Since (1.1) is a parabolic equation, solutions which have discontinuous derivatives are unacceptable. To overcome this, the shocks in v_{0x} could be smoothed out using a corner layer, as was done by Kath and Cohen (1982) for (1.2). The evaluation of this corner layer solution will not be attempted here as it has no major effect on the solution.

To investigate the effect of a shock on waiting-time behaviour, let us assume that a shock forms at the right hand front of the initial distribution of v_0, f. Then $v_{0x1} = 0$ and $v_{0x2} < 0$, so that $U < 0$ and the shock propagates into the domain of f. Similarly, it can be found from (3.3) that if a shock forms at the left hand front of f, it propagates inwards. Hence in contrast to the behaviour of the solutions of (1.2), the solutions of (1.1) for $0 < n << 1$ exhibit waiting-time behaviour even if a shock forms immediately at a front of f.

It was found in section 2 that if a shock does not form in the solution for v_0, then the solution exhibits infinite waiting-time behaviour. This solution shows furthermore that if there is no shock at the front of the solution, then it has infinite waiting-time as no characteristics propagate outside of the initial domain of f (see (2.9)). Let us now suppose that the solution for v_0 has a shock at its right hand front. Then $v_{0x1} = 0$ and $v_{0x2} < 0$, so that $U < 0$. In a similar manner, a shock at the left hand front has $U > 0$. Taken together, we thus have that a shock cannot propagate outside of the initial domain of f. Our analysis then shows that the solutions of (1.1) have infinite waiting-time when $0 < n << 1$. This is a surprising result as it is expected that all solutions which initially have compact support will approach the similarity source solution of (1.1) as $t \to \infty$, as was proved for (1.2) by Kamenomostskaya (1973). Smyth and Hill (1988) found an exact similarity source solution of (1.1) for $n = 1$ and, using a Frobenius expansion, found the behaviour of the front of the similarity source solution for $\frac{3}{2} < n < 3$. The existence of a similarity source solution of (1.1) with outward propagating fronts for small n is then an open question. Presumably the nature of the singularity at the fronts of the similarity source solution of (1.1) for small n is more complicated than the algebraic behaviour assumed by Smyth and Hill. A possible reason why the asymptotic solution of section 2 does not match into the similarity source solution as $t \to \infty$ is that the expansion (2.3) is not valid for large time, as was found by Kath and Cohen (1982) for (1.2). It is thus expected that all solutions of (1.1) will exhibit waiting-time behaviour for $0 < n << 1$, but not the infinite waiting-time behaviour predicted by the asymptotic solution.

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