REGULARITY FOR OBSTACLE PROBLEMS WITH APPLICATIONS TO THE FREE BOUNDARIES IN THE CONSTRAINED LEAST GRADIENT PROBLEM

Graham Williams

The *constrained least gradient problem* involves minimizing

$$\int_\Omega |\nabla u|\, dx$$

amongst all functions u defined on a given bounded set Ω in $\mathbb{R}^n$, satisfying given boundary values ϕ defined on $\partial\Omega$ and a gradient constraint $|\nabla u| \leq 1$ a.e. in Ω. This particular problem was considered by Kohn and Strang ([KS1], [KS2]) and they showed how such problems could arise if one tries to find a bar of constant cross-section which will support a given load and has lightest weight. This type of problem is non-convex and in most cases will not have a solution. However it is possible to convexify the integral leading to the constrained least gradient problem. Minimizing sequences for the original problem will be minimizing sequences for the new one and the infimum of values of the integral will be the same for each problem. The main advantage is that the convexified problem will have a solution.

For applications to the problem above, that is of finding a bar of lightest weight which will support a given load, there are several additional things which can be learnt from solutions to the relaxed problem. Firstly, constructing the solution and evaluating the corresponding integral gives the minimum possible weight. (Usually this weight can't be achieved but at least one can then test a proposed design to see if it is close to the optimal weight.) Secondly, in regions where $|\nabla u| = 0$, and so u is constant, there is no need for material in the rod. Thirdly, in regions where $|\nabla u| = 1$ the stress is at a maximum, the rod will behave plastically and there is no hope of weight reduction. Finally, in the region where $0 < |\nabla u| < 1$ one may hope to reduce weight by using some type of fibred design with the direction of the fibres being related to the solution u (see [KS2]).

Thus two sets concerning the solution which are of particular interest and which have associated free boundaries are $\{ x \in \Omega \mid \nabla u = 0 \}$ and $\{ x \in \Omega \mid |\nabla u| = 1 \}$.

Before addressing the nature of these sets there are some preliminary results about the solution which are needed.

(i) If we assume that the set Ω and the given boundary data ϕ are such that there does exist at least one function v defined on Ω satisfying $v = \phi$ on $\partial\Omega$ and $|\nabla v| \leq 1$ a.e. in Ω, then there will be a solution to the constrained least gradient problem. This follows easily because the integrand is convex and the constraint $|\nabla v| \leq 1$ places bounds on any minimizing sequence which ensure that there is a convergent subsequence.

(ii) The solution given in (i) is unique. This does not seem to be trivial to prove because the function $f(p) = |p|$, while convex, is not strictly convex. It was proved in [SWZ1] using the characterization of the solution u which follows.

(iii) The solution given in (i) is automatically Lipschitz continuous. That is there is a constant K (depending only on the geometry of Ω) such that $|u(x)-u(y)| \leq K|x-y|$ whenever x and y are in Ω. It is not hard to show that even if we take smooth Ω and smooth ϕ we cannot expect better regularity than this. Thus solutions are normally not continuously differentiable. Unlike the results for elliptic problems, the regularity for the solution in the interior of Ω is generally no better than that for the boundary values.

(iv) In [KS1] Kohn and Strang suggested an alternative characterization of the solution in (i). This characterization gives a lot of information about the solution u and the free boundaries mentioned above. The main result of [SWZ1] was to prove the characterization (which we next describe) correct.

For any measurable set E in $\mathbb{R}^n$ we can define a notion of the $(n-1)$-dimensional measure of the boundary ∂E. It is called the *Perimeter* of E and denoted by $P(E)$. A precise definition and many of the properties may be found in the book [G] by Giusti. For sets E with smooth boundaries $P(E)$ returns precisely the value given by more traditional methods of calculating $(n-1)$-dimensional surface area. It is important to note, however, that $P(E)$ is defined for all measurable sets and not just those with smooth boundaries.

The key to the characterization of the solution u is the *co-area formula* ([FR]). This formula says that if u is any Lipschitz continuous function (actually bounded variation will do) and

$$A_t = \{\, x \in \Omega \mid u(x) \geq t \,\}$$

then

$$\int_\Omega |\nabla u|\, dx = \int_{-\infty}^{\infty} P(A_t)\, dt.$$

To minimize the left hand side it seems a good idea to minimize $P(A_t)$ for each t. In the absence of gradient constraints this idea works and has been used by several authors to obtain results about sets of least perimeter (minimal surfaces) rather than functions of least gradient. In our case it is also necessary to build in the gradient constraint.

Consider first the case where Ω is convex. If x and y are two points in Ω then the line between x and y also lies in Ω and $|\nabla u| \leq 1$ along that line. Thus by integrating we have

$$|u(x) - u(y)| \leq |x - y| \quad \text{for } x, y \in \Omega. \tag{1}$$

It is easy to see that the converse also holds. That is if (1) holds then $|\nabla u| \leq 1$ a.e. in Ω.

If Ω is not convex then we must replace the straight line distance $|x-y|$ by distance in Ω, that is, the length of the shortest curve which stays inside Ω and connects x and y. With this replacement the equivalence above, and the results to follow, all hold for non-convex Ω.

An essential idea of Kohn and Strang was to satisfy (1) only when $x \in \partial\Omega$ and $y \in \Omega$. To see how this constrains the level sets A_t we introduce new sets L_t and M_t.

$$L_t = \{ x \in \Omega \mid \exists p \in \partial\Omega, |p - x| \le \phi(p) - t \},$$

$$M_t = \{ x \in \Omega \mid \exists p \in \partial\Omega, |p - x| < t - \phi(p) \}.$$

Suppose v is a Lipschitz continuous function defined on Ω and $A_t = \{ x \in \Omega \mid v(x) \ge t \}$ then

$$v = \phi \text{ on } \partial\Omega \text{ and } |v(x) - v(y)| \le |x - y| \text{ for } x \in \partial\Omega, y \in \Omega$$

$\iff$

$$L_t \subseteq A_t \text{ and } A_t \cap M_t = \emptyset \text{ for } \inf\phi \le t \le \sup\phi.$$

We are thus lead to the following problem

$$\text{(2)} \qquad \text{Minimize} \quad \{ P(E) \mid L_t \subseteq E,\ E \cap M_t = \emptyset \}.$$

It is possible to show, for each t, that this problem has a solution. Indeed in some important cases there is more than one solution. We choose the one of largest volume (it exists and is unique) and call it E_t.

Now define a function u^* on Ω by

$$u^*(x) = \sup\{ t \mid x \in E_t \}.$$

THEOREM. ([SWZ1]) *The function u^*, defined above, is the unique solution to the constrained least gradient problem.*

There is another way of viewing the sets L_t and M_t. If we let

$$K = \{ v \mid v = \phi \text{ on } \partial\Omega,\ |v(x) - v(y)| \le |x - y| \text{ for } x \in \partial\Omega, y \in \Omega \}$$

then K is precisely the set of functions above for which we obtained the equivalent formulation in terms of L_t and M_t. Now let F be the largest function in K and f be the smallest. It can easily be shown that

$$F(x) = \inf\{ \phi(p) + |x - p| \mid p \in \partial\Omega \},$$

$$f(x) = \sup \{ \phi(p) - |x-p| \mid p \in \partial\Omega \},$$

$$M_t = \{ x \in \Omega \mid F(x) < t \},$$

$$L_t = \{ x \in \Omega \mid f(x) \geq t \}.$$

THEOREM. ([SWZ1]) *The solution of the constrained least gradient problem is also the unique solution to*

$$\textit{Minimize} \quad \left\{ \int_\Omega |\nabla v|\, dx \mid v \text{ is Lipschitz continuous and } f(x) \leq v(x) \leq F(x) \right\}.$$

From the above it can be seen that the sets E_t are important and any information about them should be helpful. Although the sets L_t and M_t need not be smooth and may have cusps it turns out, nevertheless, that the solution sets E_t will be smooth.

THEOREM. ([SWZ2]) *For each t, ∂E_t is $C^{1,1}$ near points of contact with ∂L_t and ∂M_t. Away from points of contact ∂E_t is a minimal surface and so analytic if $n \leq 7$. If $n \geq 8$ the part of ∂E_t away from the points of contact may have singularities. The set of such singularities has dimension at most $n-8$ and away from this set ∂E_t is analytic.*

REMARKS.

(i) If $n = 2$ each ∂E_t, away from ∂L_t and ∂M_t, is a straight line.

(ii) Easy examples quickly show that $C^{1,1}$ is the best that can be expected.

(iii) The last Theorem actually applies not only to E_t, which maximized volume amongst, possibly many, solutions of (2), but to all solutions of (2).

We now turn our attention to some of the free boundaries mentioned earlier. First we look at the nature and location of sets where $\nabla u = 0$.

THEOREM. *Suppose u is a solution to the constrained least gradient problem. Then u is constant, with value t_0 on the open set S (and of course $\nabla u = 0$ on S) if and only if the problem (2) has at least two distinct solutions for $t = t_0$. In this case the largest such set S can be written as the difference of two solutions to (2).*

Finally we make some comments about the set where $|\nabla u| = 1$.

(i) Suppose $u(x) = F(x)$.

By the definition of F there is a point p in $\partial\Omega$ such that $F(x) = \phi(p) + |x - p|$. Then since $|u(x) - u(y)| \leq |x - y|$ for all $y \in \Omega$ and $u(p) = \phi(p)$ we see that if y is any point on the line joining x and p we must have $u(y) = F(y) = \phi(p) + |p - y|$ and $|\nabla u| = 1$.

Similar things happen if $u(x) = f(x)$.

Now define

$$G = \{\, x \in \Omega \mid f(x) < u(x) < F(x) \,\}.$$

Then the above shows that

$$\{\, x \in \Omega \;\big|\; |\nabla u| < 1 \,\} \subseteq G.$$

(ii) On G the constraints are not active and so u is actually a solution to the unconstrained least gradient problem on G.

(iii) It could be expected that $|\nabla u(x)| < 1$ for all x in G, or at least $|u(x)-u(y)| < |x-y|$ when x, y in G. This is certainly the case for the analagous problem when $\int |\nabla u|\, dx$ is replaced by $\int |\nabla u|^2\, dx$. However for the constrained least gradient problem this need not hold. An example is given in [SWZ1] where $|\nabla u| = 1$ in a portion of the set G. On the other hand a good description is obtained in [SWZ1] of such subsets of G. They have to be rectangles (with particular orientation and placement) and u has to be a linear function with slope 1 on the rectangle.

(iv) Note that

$$u(x) = F(x) = t \iff x \in \partial E_t \cap \partial M_t.$$

On the other hand ∂E_t is $C^{1,1}$ at such points of contact and so G must contain all points where ∂M_t or ∂L_t are not $C^{1,1}$. For example points x where there are two distinct points p and q in $\partial\Omega$ with $F(x) = \phi(p) + |x - p| = \phi(q) + |x - q|$, must be in G.

REFERENCES

[FR] Fleming,W.H., Rishel,R., An integral formula for total gradient variation. *Arch. Math.* **11** (1960), 218–222.

[G] Giusti,E., *Minimal Surfaces and Functions of Bounded Variation.* Birkhaüser, 1985.

[KS1] Kohn,R.V., Strang,G., The constrained least gradient problem. *Non-classical Continuum Mechanics,* R.Knops, A.Lacey, eds, Cambridge U. Press 1987, 226–243.

[KS2] Kohn,R.V., Strang,G., Fibred structures in optimal design. *Theory of Ordinary and Partial Differential Equations,* B.D.Sleeman and R,J,Jarvis, eds, Lomgman 1988.

[SWZ1] Sternberg,P., Williams,G., Ziemer,W.P., The constrained least gradient problem in $\mathbb{R}^n$. *Preprint.*

[SWZ2] Sternberg,P., Williams,G., Ziemer,W.P., $C^{1,1}$ -Regularity of constrained area-minimizing hypersurfaces. *Preprint.*

Department of Mathematics
University of Wollongong
PO Box 1144, Wollongong
NSW–2500, Australia.