ON SOME TRACE INEQUALITIES

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§1 INTRODUCTION

Let $A \geqq B \geqq 0$ be positive operators on a Hilbert space. It is well-known that this order assumption implies $Tr(f(A)) \geqq Tr(f(B))$, where Tr denotes the usual trace and f is a continuous increasing function on $\mathbb{R}_+$ with $f(0) = 0$. In fact, singular numbers $\{\mu_n(\cdot)\}_{n=1,2,\ldots}$ (see [6], [7] for details) satisfy

$$\mu_n(f(A)) = f(\mu_n(A)) \geqq f(\mu_n(B)) = \mu_n(f(B))$$

because of $\mu_n(A) \geqq \mu_n(B)$ (a consequence of the min-max expression for $\mu_n(\cdot)$). Hence, by summing up over n, one obtains the desired estimate.

The purpose of the present note is to point out two generalizations of the above mentioned trace inequality.

§2 RESULTS

Let A, B be positive operators on a Hilbert space H satisfying $A \geqq B \geqq 0$. By setting $q = 2$ in Furuta's inequality ([5]), we obtain

(1) $$A^{(p+2r)/2} \geqq (A^r B^p A^r)^{1/2}$$

as long as $p, r \geqq 0$ satisfy

(2) $$(1+2r)2 \geqq p + 2r, \quad \text{i.e.,} \quad 2 + 2r \geqq p.$$

Extending the continuous linear map

$$A^{(p+2r)/4}\zeta \in R(A^{(p+2r)/4}) \mapsto (A^r B^p A^r)^{1/4}\zeta \in H$$

1991 *Mathematics Subject Classification* 47A63, 47B65

(well-defined due to (1)), we obtain the contraction a satisfying

$$aA^{(p+2r)/4} = (A^r B^p A^r)^{1/4},$$

$$\text{(3)} \qquad a = 0 \quad \text{on} \quad R(A^{(p+2r)/4})^{\perp}.$$

From the first equality we easily get

$$\text{(4)} \qquad A^r B^p A^r = A^{(p+2r)/4} a^* a A^{(p+2r)/2} a^* a A^{(p+2r)/4}.$$

We claim

$$\text{(5)} \qquad A^{(2r-p)/4} B^p A^{(2r-p)/4} = h A^{(p+2r)/2} h$$

with $h = a^* a$, $0 \leqq h \leqq 1$ (if $2r - p \geqq 0 \cdots$ otherwise we assume the invertibility of A so that the claim trivially follows from (4)). In fact, because the subspace $R(A^{(p+2r)/4}) \oplus \ker A$ is in H, it suffices to check

$$(A^{(2r-p)/4} B^p A^{(2r-p)/4} \xi \mid \xi) = (h A^{(p+2r)/2} h \xi \mid \xi)$$

for a vector $\xi = A^{(p+2r)/4} \zeta + \zeta'$ ($\zeta \in (\ker A)^{\perp}, \zeta' \in \ker A$). However, this follows from straight-forward calculations based on (3) and (4).

THEOREM 1. *Assume $A \geqq B \geqq 0$ and $p > 1, \alpha \geqq max\{-1, -p/2\}$.*

(i) There exists a partial isometry u satisfying

$$A^{\alpha/2} B^p A^{\alpha/2} \leqq u^* A^{p+\alpha} u.$$

(ii) For a continuous increasing function f on $\mathbb{R}_+$ with $f(0) = 0$, we have

$$Tr(f(A^{\alpha/2} B^p A^{\alpha/2})) \leqq Tr(f(A^{p+\alpha})).$$

In the above statements the invertibility of A is assumed when $\alpha < 0$.

PROOF. (i) Let $A^{(p+2r)/4}h = v|A^{(p+2r)/4}h|$ be the polar decomposition. Since

$$|B^{p/2}A^{(2r-p)/4}| = |A^{(p+2r)/4}h| \qquad \text{(by (5))}$$

$$= u^*A^{(p+2r)/4}h \ (= hA^{(p+2r)/4}u),$$

we get

$$A^{(2r-p)/4}B^pA^{(2r-p)/4} = u^*A^{(p+2r)/4}h^2A^{(p+2r)/4}u$$

$$\leqq u^*A^{(p+2r)/2}u$$

(recall $0 \leqq h \leqq 1$). By setting $\alpha = (2r-p)/2$ ($\geqq -1$ by (2), but r cannot be negative), we get (i).

(ii) This follows from

$$\mu_n(A^{\alpha/2}B^pA^{\alpha/2}) \leqq \mu_n(u^*A^{p+\alpha}u) \leqq \mu_n(A^{p+\alpha}),$$

$n = 1, 2, \ldots.$ (Q.E.D.)

It is obvious from the above proof that u in (i) can be chosen to be a unitary when A, B are (finite) matrices. When $0 \leqq p \leqq 1$, we have $A^p \geqq B^p$ (the operator monotonicity of the function λ^p on $\mathbb{R}_+$). Therefore, in this case the above (ii) remains valid for any $\alpha \in \mathbb{R}$. Note that (i) says $B^p \leqq u^*A^pu$, $p > 1$ (although $B^p \leqq A^p$ generally fails). The next fact might also be worth pointing out.

PROPOSITION 2. *For self-adjoint operators A, B with $A \geqq B$, we can find a unitary v satisfying*

$$e^B \leqq v^*e^Av.$$

PROOF. Ando, [1], showed that $A = A^* \geqq B = B^*$ guarantees

$$(0 \leqq) k = e^{-A/2}(e^{A/2}e^Be^{A/2})^{1/2}e^{-A/2} \leqq 1.$$

Let $e^{A/2}k = v \mid e^{A/2}k|$ be the polar decomposition. (Note that v is a unitary, all the involved operators being invertible.) Since $ke^{A}k = e^{B}$, the same argument as in the proof of Theorem 1, (i) shows the desired result. (Q.E.D.)

The next result will be proved based on a majorization argument.

THEOREM 3. *Let A, B be positive operators, and f, g be continuous increasing functions on $\mathbb{R}_+$ vanishing at 0. If $A \geqq B$ (or more generally if $\mu_n(A) \geqq \mu_n(B)$ for $n = 1, 2, \ldots$, i.e., A spectrally dominates B in the sense of for example [2], [3]), then we get*

$$Tr(f(A)g(A)) \geqq Tr(f(A)^{1/2}g(B)f(A)^{1/2}).$$

PROOF. First we further assume $\dim R(B) = m < +\infty$. Let $\beta_1 \geqq \beta_1 \geqq \cdots \geqq \beta_m (> 0)$ be the non-zero eigenvalues of B, and $\xi_1, \xi_2, \ldots, \xi_m$ be corresponding (mutually orthogonal) eigenvectors of length 1. Adding some vectors, we obtain an orthonormal basis $\{\xi_i\}_{i=1,2,3,\ldots}$ for H. For each j we have

$$\sum_{i=1}^{j}(f(A)\xi_i \mid \xi_i) \leqq \sum_{i=1}^{j}\mu_i(f(A)). \tag{6}$$

In fact, the right hand side always majorizes $Tr(pf(A)p)$, where p is a projection satisfying $\dim(pH) \leqq j$ (see [6], [7]). We now compute

$$Tr(f(A)^{1/2}g(B)f(A)^{1/2}) = Tr(g(B)^{1/2}f(A)g(B)^{1/2})$$

$$= \sum_{i=1}^{\infty}(g(B)^{1/2}f(A)g(B)^{1/2}\xi_i \mid \xi_i)$$
$$= \sum_{i=1}^{m} g(\beta_i)(f(A)\xi_i \mid \xi_i)$$
$$= g(\beta_m)\sum_{i=1}^{m}(f(A)\xi_i \mid \xi_i) + \sum_{j=1}^{m-1}(g(\beta_j) - g(\beta_{j+1})) \times (\sum_{i=1}^{j}(f(A)\xi_i \mid \xi_i))$$

$$\leqq g(\beta_m)\sum_{i=1}^{m}\mu_i(A)+\sum_{j=1}^{m-1}(g(\beta_j)-g(\beta_{j+1}))\times(\sum_{i=1}^{j}\mu_i(f(A)))$$

(by (6) and the decreasingness of $\{g(\beta_j)\}$)

$$=\sum_{i=1}^{m}g(\beta_i)\mu_i(f(A))$$

$$\leqq\sum_{i=1}^{m}g(\mu_i(A))\mu_i(f(a)) \text{ (because of } \beta_i=\mu_i(B)\leqq\mu_i(A))$$

$$=\sum_{i=1}^{m}\mu_i(g(A))\mu_i(f(A))$$

$$\leqq Tr(f(A)g(A)).$$

When B is not necessarily of finite rank, we choose an increasing sequence $\{p_i\}$ of finite rank projections tending to the identity operator in the strong operator topology. Notice that each finite rank operator $B_i = p_iBp_i$ is spectrally dominated by A (because of $\mu_n(B_i) \leqq \mu_n(B) \leqq \mu_n(A)$). Thus the first half of the proof says

$$Tr(f(A)^{1/2}g(B_i)f(A)^{1/2}) \leqq Tr(f(A)g(A)).$$

Notice that the sequence $\{f(A)^{1/2}g(B_i)f(A)^{1/2}\}_i$ converges to $f(A)^{1/2}g(B)f(A)^{1/2}$ in the strong operator topology. Therefore, the lower semi-continuity of $Tr(\cdot)$ with respect to this topology shows

$$Tr(f(A)^{1/2}g(B)f(A)^{1/2}) \leqq \liminf_{i\to\infty} Tr(f(A)^{1/2}g(B_i)f(A)^{1/2})$$
$$\leqq Tr(f(A)g(A)). \qquad \text{(Q.E.D.)}$$

All the results in this note remain valid for a semi-finite trace on a von Neumann algebra of type II. (Instead of $\mu_n(\cdot)$, generalized s-numbers in [4] have to be used.)

Acknowledgement This note was written while the author visited the University of New South Wales. He is grateful to Colin Sutherland and the A.R.C. for making this visit possible. He also thanks Jonathan Arazy for useful discussions on the proof of Theorem 3.

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