

**ON WEAK SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS
WITH UNBOUNDED COEFFICIENTS**

Beniamin GOLDYS

0. Introduction. We shall consider in this paper weak solutions of the following stochastic evolution equation:

$$\begin{cases} dX=[AX+f(X)]dt+g(X)dW, \\ X(0)=x_0, 0 \leq t \leq 1, \end{cases} \quad (1)$$

where A is a generator of C_0 -semigroup $S(t)$, $t \geq 0$, of bounded operators on a Hilbert space H and W is a cylindrical Wiener process on another Hilbert space K with covariance operator I . It is well-known that if $\dim H < \infty$ then (1) has a global weak solution provided f and g are continuous functions of linear growth. On the other hand, if $\dim H = \infty$ then a solution to (1) need not exist even if $g=0$ and f is uniformly continuous and bounded and hence some additional assumptions are necessary.

There are not many results on weak solutions to (1) in infinite dimension. In those existing two types of conditions appear. Either it is assumed that A is a coercive operator on some Gelfand triple with compact injections or (loosely speaking) some invertibility property is imposed on g in order to allow the use of the Girsanov transformation. This last assumption is quite restrictive. Recently in [7] an existence result for the equation (1) was proved under the more general assumption that A is a generator of a compact semigroup, and f and g are weakly continuous mappings of linear growth defined on H .

The aim of this note is to extend the results of [7] to equations with unbounded coefficients. More precisely it will be assumed that A generates an analytic and compact semigroup and f and g are defined on some interpolation spaces of A . Such equations have been the object of extensive study for some time but the pathwise solutions only were considered, [4], [6].

1.Preliminaries. The following conditions are standing assumptions for the rest of this paper:

- A is a generator of the compact and analytic semigroup $S(t)$, $t \geq 0$, of bounded operators on H . Without loss of generality it can be assumed that for some $M \geq 1$ and positive a

$$\|S(t)\| \leq Me^{-at}.$$

- The drift coefficient f is a measurable mapping from $D_A(\theta, 2)$ to H (see below for the definition of $D_A(\theta, 2)$).
- The diffusion coefficient g is a measurable mapping from $D_A(\theta, 2)$ to the space of Hilbert-Schmidt operators $\mathfrak{L}^2(K, H)$ acting from K to H .

The spaces $D_A(\theta, 2)$ introduced above are real interpolation spaces between $D(A)$ and H defined, for $0 < \theta < 1$ as follows:

$$D_A(\theta, 2) = \left\{ x \in H; \|x\|_{\theta}^2 = \int_0^{\infty} v^{1-2\theta} \|AS(v)\|^2 dv < \infty \right\}.$$

Let us recall the following properties of these spaces which will be useful in future: We have, for $0 < \theta < 1$

$$D(A) \subset D_A(\theta, 2) \subset H$$

with continuous and dense injections. Moreover, if the injection of $D(A)$ in H is compact then the above injections are also compact.

Let us notice that in some important cases we have an explicit representation of the spaces $D_A(\theta, 2)$. Let D be a bounded open domain in \mathbb{R}^n with sufficiently smooth boundary. Let A be a second order strongly elliptic differential operator in D with coefficients which are continuous in the closure of D . This operator, when considered in $H=L^2(D)$ with the domain $D(A)=H^2(D)\cap H_0^1(D)$ (Dirichlet boundary conditions), is a generator of analytic and compact semigroup in H (see for example [1]). It is well-known also that in this case

$$D_A(\theta, 2) = \begin{cases} H^{2\theta}(D) & \text{if } 0 < \theta < \frac{1}{4}, \\ H^{2\theta}(D) \cap H_0^1(D) & \text{if } \frac{1}{4} < \theta < 1. \end{cases}$$

For more details on the interpolation spaces $D_A(\theta, 2)$ see for example [2].

We shall use the following definition of a solution to (1):

Definition. A predictable H -valued process X defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is said to be a solution to (1) if

- For $t > 0$ trajectories of X belong to $D_A(\theta, 2)$ a.s. and X is a predictable process with values in $D_A(\theta, 2)$.
- There exists on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ a K -valued cylindrical Wiener process W such that the following equation is satisfied in H for any t a.s.:

$$X(t) = S(t)x_0 + \int_0^t S(t-s)f(X(s))ds + \int_0^t S(t-s)g(X(s))dW(s).$$

It is well-known [3] that if the process X is a solution to (1) then

$$X(t) = x_0 + A \int_0^t X(s)ds + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s). \quad (2)$$

In order to prove existence of solutions to (1) we shall make use of the following infinite-dimensional version of the Riemann-Liouville operator R_α

defined for any $\alpha < 1$:

$$R_\alpha f(t) = \int_0^t (t-s)^{\alpha-1} S(t-s)f(s) ds .$$

The properties of the operator R_α given in the lemma below are crucial for the existence proof in the next section.

Lemma 1. a) For any C_0 -semigroup S , the operator R_α is bounded from $L^p(0,1;H)$ into $C(0,1;H)$ provided $\alpha p > 1$;

b) If the operators $S(t)$ are compact for $t > 0$ then the operator R_α is also compact from $L^p(0,1;H)$ into $C(0,1;H)$ provided $\alpha p > 1$;

c) If S is an analytic semigroup, then R_α is a bounded operator from $L^p(0,1;H)$ to $C^\lambda(0,1;D_A(\theta,2))$ (the space of Hölder-continuous functions with exponent λ) with $\lambda < \alpha - \theta - \frac{1}{p}$ provided $\alpha > \theta + \frac{1}{p}$.

Part a) of this lemma is well-known (see [5]). Part b) was proved in [7]. The last part is an easy consequence of Lemma 2 from [5].

Now let us consider an H -valued stochastic process Z defined as follows:

$$Z(t) = \int_0^t S(t-s)\Phi(s)dW(s).$$

The process Φ is a predictable $\mathfrak{L}^2(K,H)$ -valued process with the property:

$$E \int_0^1 \|\Phi(s)\|_2^2 ds < \infty,$$

where $\mathfrak{L}^2(K,H)$ denotes the space of Hilbert-Schmidt operators with the norm $\|\cdot\|_2$. We shall need the following lemma which was proved in [5].

Lemma 2. Assume that $\alpha < \frac{1}{2}$. Then

$$Z(t) = \frac{\sin \pi \alpha}{\pi} R_{\alpha} Y(t)$$

with

$$Y(t) = \int_0^t (t-s)^{-\alpha} S(t-s) \Phi(s) dW(s).$$

2. Existence result. In this section we shall formulate and prove the main result of this note.

Theorem 1. Assume that the two conditions below are satisfied.

a) Mappings f and g are defined on some $D_A(\theta, 2)$ with $\theta < \frac{1}{2}$ and moreover

$$\|f(x)\| + \|g(x)\|_2 \leq c(1 + \|x\|_{\theta}).$$

b) For any $y \in H$ the mappings $x \rightarrow \langle f(x), y \rangle$ and $x \rightarrow \langle g(x)y, y \rangle$ are continuous on $D_A(\theta, 2)$.

Then there exists a solution to the equation (1). Moreover the process $t^{\theta} X(t)$ is continuous in $D_A(\theta, 2)$, and for $t > 0$, the process X is Hölder continuous in $D_A(\theta, 2)$ with any exponent $\lambda < \frac{1}{2} - \frac{1}{p} - \theta$ provided $\frac{1}{2} - \frac{1}{p} - \theta > 0$.

This theorem generalizes to the nonlipschitz case some results from [4] and [6]. However, the method of the proof does not allow one to consider the limiting case $\theta = \frac{1}{2}$. It is well known that in this case some additional assumptions on g are necessary.

Proof. In order to prove this theorem we start with a definition of Peano-like approximations for the solution of the equation (1). For $n \geq 1$ the sequence of $D_A(\theta, 2)$ -valued processes X_n is defined in the following way:

$$X_n(t) = S(t)x_0 + \int_0^t S(t-s)f_n(s)ds + \int_0^t S(t-s)g_n(s)dW(s)$$

for $t \geq 0$, where $f_n(s) = f(X_n(\xi_n(s)))$, and $g_n(s) = g(X_n(\xi_n(s)))$ with $\xi_n(s) = \frac{k}{2^n}$ for $\frac{k}{2^n} < s \leq \frac{k+1}{2^n}$.

We assume at first that $x_0 \in D_A(\theta, 2)$. Then it can be easily checked that the processes X_n are well-defined and continuous in $D_A(\theta, 2)$. A much stronger property of these approximations is proved below.

Lemma 3. For any $\theta < \frac{1}{2}$ we have

$$\sup_{n \geq 1} \sup_{t \leq 1} E \|X_n\|_{\theta}^p < \infty$$

provided

$$2 < p < \frac{4}{2\theta+1}.$$

Proof. Clearly we have

$$\begin{aligned} E \|X_n\|_{\theta}^p &\leq 3^{p-1} [\|S(t)x_0\|_{\theta}^p + E \left\| \int_0^t S(t-s)f_n(s)ds \right\|_{\theta}^p + \\ &+ E \left\| \int_0^t S(t-s)g_n(s)dW(s) \right\|_{\theta}^p] = 3^{p-1} [I_1 + I_2 + I_3]. \end{aligned}$$

We shall estimate now each of the terms I_1, I_2, I_3 separately. Since $x_0 \in D_A(\theta, 2)$ and the semigroup S is a contraction on $D_A(\theta, 2)$,

$$I_1 = \|S(t)x_0\|_{\theta}^p \leq \|x_0\|_{\theta}^p.$$

Now we consider I_2 :

$$I_2 = E \left\| \int_0^t S(t-s)f_n(s)ds \right\|_{\theta}^p \leq E \left(\int_0^t \|S(t-s)f_n(s)\|_{\theta}^2 ds \right)^{p/2} =$$

$$\begin{aligned}
&= E \left(\int_0^t \int_0^\infty v^{1-2\theta} \| AS(t-s+v)f_n(s) \|^2 dv ds \right)^{p/2} \leq \\
&\leq M^p E \left(\int_0^t \int_0^\infty v^{1-2\theta} (t-s+v)^{-2} \| f_n(s) \|^2 dv ds \right)^{p/2} = \\
&= C E \left(\int_0^t (t-s)^{-\theta} \| f_n(s) \|^2 ds \right)^{p/2} \leq C \int_0^t (t-s)^{-p\theta} E \| f_n(s) \|^p ds,
\end{aligned}$$

where

$$C = M^p \left(\int_0^t u^{1-2\theta} (1+u)^{-2} du \right)^{p/2}.$$

The last estimate is a bit more complicated. Let us denote

$$Z(t) = \int_0^t S(t-s)g_n(s)dW(s).$$

Then we have

$$\begin{aligned}
I_3 &= E \| Z(t) \|^p_\theta = E \left(\int_0^\infty v^{1-2\theta} \| AS(v)Z(t) \|^2 dv \right)^{p/2} \leq \\
&= E \left(\int_0^\infty v^{1-2\theta} \frac{e^{2bv}}{2b} \| AS(v)Z(t) \|^2 (2be^{-2bv}dv) \right)^{p/2},
\end{aligned}$$

where $0 < b < a$. Now it follows from Jensen's inequality that

$$\begin{aligned}
I_3 &\leq E \int_0^\infty v^{p(1-2\theta)/2} (2b)^{-p/2} e^{pbv} \| AS(v)Z(t) \|^p (2be^{-2bv}dv) \leq \\
&\leq C_1 \int_0^\infty v^{p(1-2\theta)/2} e^{pbv} E \| AS(v)Z(t) \|^p dv
\end{aligned}$$

with $C_1 = (2b)^{(2-p)/2}$. Now using known properties of stochastic integrals (see for example [8]) we get

$$\begin{aligned}
I_3 &\leq C_1 \int_0^\infty v^{p(1-2\theta)/2} e^{pbv} \left[E \int_0^t \|AS(t-s+v)g_n(s)\|_2^2 ds \right]^{p/2} dv \leq \\
&\leq C_1 \int_0^\infty v^{p(1-2\theta)/2} e^{pbv} E \int_0^t \|AS(t-s+v)g_n(s)\|_2^p ds dv \leq \\
&\leq C_1 C_p \int_0^\infty v^{p(1-2\theta)/2} e^{pbv} E \int_0^t \|AS(t-s+v)\|^p \|g_n(s)\|_2^p ds dv \leq \\
&\leq C_1 C_p \int_0^\infty v^{p(1-2\theta)/2} e^{pbv} \int_0^t M^p e^{-pa(t-s+v)} (t-s+v)^{-p} E \|g_n(s)\|_2^p ds dv \leq \\
&\leq C_1 C_p M^p \int_0^\infty v^{p(1-2\theta)/2} \int_0^t (t-s+v)^{-p} E \|g_n(s)\|_2^p ds dv = \\
&= C_1 C_p C_\theta \int_0^t (t-s)^{1-p(2\theta+1)/2} E \|g_n(s)\|_2^p ds
\end{aligned}$$

with

$$C_\theta = \int_0^\infty \left(\frac{u^{(1-2\theta)/2}}{1+u} \right)^p du.$$

Now taking into account all the above estimates and using the assumptions of the theorem, we get, for some constant D independent of n,

$$\begin{aligned}
E \|X_n(t)\|_\theta^p &\leq D \left[1 + \int_0^t (t-s)^{-p\theta} E \|X_n(\xi_n(s))\|_\theta^p ds + \right. \\
&\quad \left. + \int_0^t (t-s)^{1-p(2\theta+1)/2} E \|X_n(\xi_n(s))\|_\theta^p ds. \right]
\end{aligned}$$

Let us define now the function $F_n(t) = \sup_{s \leq t} E \|X_n(s)\|_\theta^p$. It can be easily seen

that

$$E \|X_n(t)\|_\theta^p \leq D \left[1 + \int_0^t (t-s)^{\beta-1} F_n(s) ds \right],$$

where

$$\beta = 2 - \frac{p(2\theta+1)}{2}.$$

We need now the following simple

Lemma 4. For a nonnegative nondecreasing function u , let v be defined as follows:

$$v(t) = \int_0^t (t-s)^{a-1} s^{b-1} u(s) ds$$

with $a, b > 0$, $a+b-1 > 0$. Then v is nondecreasing.

It follows immediately from this lemma that

$$F_n(t) \leq D \left[1 + \int_0^t (t-s)^{\beta-1} F_n(s) ds \right].$$

Lemma 7.1.2. from [9] implies that

$$\sup_{n \geq 1} \sup_{t \leq 1} F_n(t) < \infty.$$

Lemma 3 follows.

Let us introduce now, for $\alpha < \frac{1}{2}$ the sequence of processes Y_n by the formula

$$Y_n(t) = \int_0^t (t-s)^{-\alpha} S(t-s) g_n(X_n, s) dW(s).$$

Then

$$\begin{aligned} \int_0^1 E \|Y_n(t)\|^p dt &\leq C_p \int_0^1 E \left[\int_0^t (t-s)^{-2\alpha} \|S(t-s) g_n(s)\|_2^2 ds \right]^{p/2} dt \leq \\ &\leq C_p M^p \int_0^1 E \left[\int_0^t (t-s)^{-2\alpha} 2c^2 (1 + \|X_n(\xi_n(s))\|_\theta^2) ds \right]^{p/2} dt \leq \\ &\leq A + BE \int_0^1 \left[\int_0^t (t-s)^{-2\alpha} \|X_n(\xi_n(s))\|_\theta^2 ds \right] dt \end{aligned}$$

for some constants A and B. Now Young's inequality yields

$$\int_0^1 E \|Y_n(t)\|_{\Theta}^p dt \leq A + B_1 \int_0^1 E \|X_n(\xi_n(s))\|_{\Theta}^p ds \leq A + B_1 \int_0^1 F_n(s) ds$$

with

$$B_1 = B \left(\int_0^1 s^{-2\alpha} ds \right)^{p/2}.$$

Hence the processes Y_n are uniformly bounded in L^p for p satisfying the assumptions of Lemma 3. It follows from Lemma 2 that

$$X_n(t) = S(t)x_0 + R_1 f_n(t) + R_{\alpha} Y_n(t).$$

Processes Y_n and f_n have their laws concentrated on $L^p(0,1;H)$ and in fact it follows from Lemma 3 that those laws form a tight family on $L^p(0,1;H)$. Now Lemma 2 implies that the family of measures corresponding to the processes X_n is tight on $C(0,1;H)$. But in view of Lemma 3 and Lemma 1 an appropriate version of Dubinsky's Lemma (see for example [10]) implies that this family of measures is also tight on $L^p(0,1;D_A(\theta,2))$. Skorochod's theorem implies that we can, eventually changing the probability space, pick up a subsequence of X_n which is convergent to a certain process X in $L^p(0,1;D_A(\theta,2))$ and $C(0,1;H)$ simultaneously. Standard arguments, see [7], show that X is a solution to (1).

Let us assume now that the initial condition x_0 lies in H . Take a sequence of initial conditions $x_n \in D_A(\theta,2)$ and such that x_n converges to x in H . It follows from the first part of the proof that for each x_n we can choose a solution X_n to equation (1) starting from x_n (in fact this choice can be made measurable). Our aim now is to find uniform bounds for X_n similar to those obtained in Lemma 3. We start from an obvious inequality

$$E \|X_n\|_{\Theta}^p \leq 3^{p-1} [\|S(t)x_n\|_{\Theta}^p + E \left\| \int_0^t S(t-s)f(X_n(s)) ds \right\|_{\Theta}^p +$$

$$+ E \left\| \int_0^t S(t-s)g(X_n(s))dW(s) \right\|_{\theta}^p] = 3^{p-1} [I_1 + I_2 + I_3].$$

We can estimate I_2 and I_3 in the same way as in the proof of Lemma 3 and obtain

$$I_2 \leq Cc^p \int_0^t (t-s)^{-p\theta} (1+E\|X_n(s)\|_{\theta}^p) ds,$$

$$I_3 \leq C_1 C_p C_{\theta} c^p \int_0^t (t-s)^{1-p(2\theta+1)/2} (1+E\|X_n(s)\|_{\theta}^p) ds.$$

The first term has to be estimated differently:

$$\begin{aligned} I_1 &= \|S(t)x_n\|_{\theta}^p = \left(\int_0^t v^{1-2\theta} \|AS(t+v)x_n\|^2 dv \right)^{p/2} \leq \\ &\leq M^p \left(\int_0^t v^{1-2\theta} (t+v)^{-2} dv \right)^{p/2} \|x_n\|^p = \\ &= M^p \left(\int_0^t u^{1-2\theta} (1+u)^{-2} du \right)^{p/2} t^{-p\theta} \|x_n\|^p = C t^{-p\theta} \|x_n\|^p. \end{aligned}$$

Taking those estimates together we get

$$\begin{aligned} E\|X_n(t)\|_{\theta}^p &\leq D[t^{-p\theta} + \int_0^t (t-s)^{-p\theta} E\|X_n(s)\|_{\theta}^p ds + \\ &+ \int_0^t (t-s)^{1-p(2\theta+1)/2} E\|X_n(s)\|_{\theta}^p ds. \end{aligned}$$

Let us define now the function

$$G_n(t) = \sup_{s \leq t} (s^{p\theta} E\|X_n(s)\|_{\theta}^p).$$

Then in view of Lemma 4 we have

$$G_n(t) \leq D \left[1 + \int_0^t (t-s)^{\beta-1} s^{-p\theta} G_n(s) ds \right].$$

All constants in the above inequalities have the same meaning as in the first part of the proof. Once more Lemma 7.1.2. of [9] implies that

$$\sup_{n \geq 1} \sup_{t \leq 1} G_n(t) \leq \infty.$$

Given the above estimate the remaining part of the proof is exactly the same as for $x_0 \in D_A(\theta, 2)$.

Regularity properties of a solution follow easily from its continuity in H and Lemmas 1 and 2.

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Benjamin GOLDYS

The University of New South Wales

School of Mathematics, Department of Statistics

P.O. Box 1, Kensington, NSW 2033, Australia