SOME PROPERTIES OF SYMMETRIC OPERATOR SPACES

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0. Introduction In this paper, we discuss certain aspects of the theory of rearrangement invariant Banach spaces of measurable operators affiliated with a semi-finite von Neumann algebra, which have been the subject of recent work by the present authors and Ben de Pagter. Such a theory provides a unified approach to the study of trace ideals initiated by Schatten [Sch] and to the study of rearrangement invariant (commutative) Banach function spaces which play a central role in classical real analysis, and derives its motivation from each of these central sources. The first general construction of such spaces, based on real analysis methods and using the theory of non-commutative integration developed by Segal [Se] (see also Dixmier [Dix]), is due to Ovčinnikov [Ov 1,2] and, independently, to Yeadon [Ye1]. More recently, an approach to the construction of symmetric operator spaces has been given in [DDP1,2] at a level of generality that fully reflects the commutative theory. The relation of this construction to many theorems of classical interpolation theory has been given in [DDP3]. In the present survey, our attention will be directed primarily towards the development of a general duality theory and related topological and geometrical properties.

After gathering the necessary preliminaries in Section 1, we outline in Section 2 the principal results concerning Köthe duality obtained in [DDP4]. The results presented in this section find their principal motivation in the well-known theory of Banach function spaces, and considerably extend and refine earlier investigations of Garling [Ga1,2] and Yeadon [Ye2]. The central theme of Section 3 is the study of weakly compact subsets of symmetric operator spaces and its relation to the characterization of several structural and topological properties, such as weak sequential completeness and reflexivity. Full details and proofs of the results of this section will appear in [DDP4,5]. Section 4 combines results

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from [DDP3,4] to give non-commutative extensions of certain results of Lozanovskii [Lo] which are related to the complex method of interpolation. In Section 5, we consider various properties related to uniform convexity in non-commutative spaces, and include a discussion of recent progress made by V.I. Chilin and collaborators in this direction. It is a pleasure to thank V.I. Chilin, A.V. Krygin, F. Sukochev and D.Sz. Goldstein for communicating the preprints and technical reports cited in the bibliography, and it is an equal pleasure to thank Wojtek Chojnacki for his expert assistance in their translation.

1. Preliminaries Throughout this paper, $\mathcal{H}$ will denote a Hilbert space, with inner product $(\, , \,)$, $\mathcal{L}(\mathcal{H})$ the linear space of all bounded linear operators in $\mathcal{H}$ and $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ a von Neumann algebra equipped with a normal, faithful semifinite trace τ. We denote by $\mathcal{M}^p$ the lattice of orthogonal projections in $\mathcal{M}$. A closed densely defined linear operator x in $\mathcal{H}$ is said to be *affiliated with* $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$ then x is called *τ-measurable* if and only if, there exists a number $s \geq 0$ such that $\tau(\chi_{(s,\infty)}(|x|)) < \infty$, where the projection $\chi_{(s,\infty)}(|x|)$ is defined via the usual functional calculus for self-adjoint operators. We denote by $\widetilde{\mathcal{M}}$ the set of all τ-measurable operators. Sum and product in $\widetilde{\mathcal{M}}$ are defined as the respective closures of the algebraic sum and product. For $x \in \widetilde{\mathcal{M}}$, the *decreasing rearrangement* (or *generalized singular value function*) $\mu_.(x)$ of x is defined by

$$\mu_t(x) = \inf\{s \geq 0 : \tau(\chi_{(s,\infty)}(|x|)) \leq t\}, \qquad t \geq 0.$$

It follows simply that $\mu_.(x)$ is a decreasing, right-continuous function on the half-line $[0, \infty)$. For basic properties of decreasing rearrangements of measurable operators, we refer to [FK]. We denote by $\widetilde{\mathcal{M}}_0$ the set of all $x \in \widetilde{\mathcal{M}}$ such that $\mu_t(x) \to 0$ as $t \to \infty$.

The sets $M_{\epsilon,\delta}$ $(\epsilon, \delta > 0)$, consisting of all $x \in \widetilde{\mathcal{M}}$ such that $\mu_\delta(x) < \epsilon$, form a neighbourhood base at 0 for a metrizable Hausdorff topology in $\widetilde{\mathcal{M}}$ called the *measure topology*. It is shown in [Ne] and [Te] that $\widetilde{\mathcal{M}}$ equipped with the measure topology is a complete, Hausdorff, topological *-algebra in which $\mathcal{M}$ is dense.

With the partial ordering defined on $\widetilde{\mathcal{M}}$ by setting $x \geq 0$ if and only if $(x\xi, \xi) \geq 0$ for all ξ in the domain of x, it is shown [DDP3,4] that $(\widetilde{\mathcal{M}}, \leq)$ is an ordered vector space, that the positive cone $\widetilde{\mathcal{M}}_+$ is closed in $\widetilde{\mathcal{M}}$ for the measure topology and that $\widetilde{\mathcal{M}}$ is order complete in the sense that every increasing order bounded net in $\widetilde{\mathcal{M}}_+$ has a supremum in $\widetilde{\mathcal{M}}$. The trace τ can be extended to an additive, positively homogeneous map $\tau : \widetilde{\mathcal{M}}_+ \longrightarrow [0, \infty]$, which is unitarily invariant, and normal in the sense that $x_\alpha \uparrow_\alpha x$ in $\widetilde{\mathcal{M}}_+$ implies $\tau(x) = \sup_\alpha \tau(x_\alpha)$. The crucial link between the order structure in $\widetilde{\mathcal{M}}$ and that in the space $L^0(\mathbb{R}^+)$ of all Lebesgue measurable functions on $[0, \infty)$, is provided by the following result [DDP4].

Theorem 1.1. (a) *If* $\{x_\alpha\}$ *is an increasing net in* $\widetilde{\mathcal{M}}_+$ *and if* $x = \sup x_\alpha$ *holds in* $\widetilde{\mathcal{M}}$, *then* $\mu(x_\alpha) \uparrow_\alpha \mu(x)$ *holds in* $L^0(\mathbb{R}^+)$.

(b) *If* $\{x_\alpha\} \subseteq \widetilde{\mathcal{M}}_0$ *and* $x_\alpha \downarrow_\alpha 0$ *holds in* $\widetilde{\mathcal{M}}$, *then* $\mu(x_\alpha) \downarrow_\alpha 0$ *in* $L^0(\mathbb{R}^+)$.

If $x, y \in \widetilde{\mathcal{M}}$, we say that x is *submajorized by* y, written $x \prec\prec y$, if and only if

$$\int_0^\alpha \mu_t(x)dt \leq \int_0^\alpha \mu_t(y)dt, \qquad \text{for all } \alpha \geq 0.$$

If $x \in \widetilde{\mathcal{M}}$, we set $\Omega(x) = \{y \in \widetilde{\mathcal{M}} \ : \ y \prec\prec x\}$.

A normed linear subspace $E \subseteq \widetilde{\mathcal{M}}$ is called *rearrangement invariant* if and only if $x \in \widetilde{\mathcal{M}}$, $y \in E$ and $\mu(x) \leq \mu(y)$ implies $x \in E$ and $\|x\|_E \leq \|y\|_E$; *symmetric* if and only if $x, y \in E$ and $x \prec\prec y$ implies $\|x\|_E \leq \|y\|_E$; *fully symmetric* if and only if $x \in \widetilde{\mathcal{M}}$, $y \in E$ and $x \prec\prec y$ implies $x \in E$ and $\|x\|_E \leq \|y\|_E$. If $E \subseteq \widetilde{\mathcal{M}}$ is rearrangement invariant, then E is order complete in the sense that, if $\{x_\alpha\} \subseteq E$ and $0 \leq x_\alpha \uparrow_\alpha \leq x \in E$ then also $\sup_\alpha x_\alpha \in E$.

We identify $L^\infty(\mathbb{R}^+)$ throughout as a commutative von Neumann algebra acting by multiplication on $L^2(\mathbb{R}^+)$ with trace given by integration with respect to Lebesgue measure. A Banach space $E(\mathbb{R}^+)$ of almost everywhere finite, measurable functions on $\mathbb{R}^+$ will be called a *rearrangement invariant (symmetric, fully symmetric) Banach function space* on $\mathbb{R}^+$ if the corresponding conditions above hold with respect to the von Neumann algebra $L^\infty(\mathbb{R}^+)$. While the above terminology differs from that of [KPS], this should cause no confusion in the sequel.

If $E(\mathbb{R}^+)$ is a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$, we denote by $E(\mathcal{M})$ the linear subspace of those $x \in \widetilde{\mathcal{M}}$ for which $\mu(x) \in E(\mathbb{R}^+)$ with norm defined by setting $\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_{E(\mathbb{R}^+)}$, $x \in E(\mathcal{M})$. It is shown in [DDP1,2] (see also [Su]), $E(\mathcal{M})$ is a rearrangement invariant, symmetric Banach space. If $E(\mathbb{R}^+)$ is one of the familiar L^p-spaces, $1 \le p < \infty$, then the spaces $L^p(\mathcal{M})$ given by the preceding construction coincide with those defined by Nelson [Ne]; the equality $L^\infty(\mathcal{M}) = \mathcal{M}$ holds with equality of norms, and it has been shown, for example in [Ov2], [FK] that the equality $(L^1 + L^\infty)(\mathcal{M}) = L^1(\mathcal{M}) + \mathcal{M}$ holds with equality of norms.

2. Köthe duality It is not generally the case that the Banach dual space of a given Banach function space can itself be represented as a Banach function space on the same underlying measure space. An immediate example is provided by the space $L^\infty(\mathbb{R})$. For this reason, a very natural, though more restricted duality, taken in the sense of Köthe, has long played an important role in the classical theory of function spaces. In this section, we indicate how these ideas may be extended to the non-commutative setting at a level of generality which fully includes the known commutative theory. The first steps in this direction may be found, in the case of norm ideals of compact operators, in the work of Schatten [Sch] on symmetric gauge functions and unitarily invariant cross-norms. This theme is further developed in the monograph of Gohberg and Krein [GK] and in the work of Garling [Ga1,2]. In the more general setting of semi-finite von Neumann algebras, the study of non-commutative duality was initiated by Yeadon [Ye].

A Banach space $E \subseteq \widetilde{\mathcal{M}}$ will be called *properly symmetric* if E is rearrangement invariant, symmetric and intermediate for the Banach couple $(L^1(\mathcal{M}), \mathcal{M})$ in the sense that $L^1(\mathcal{M}) \cap \mathcal{M} \subseteq E \subseteq L^1(\mathcal{M}) + \mathcal{M}$ with continuous embeddings. Each rearrangement invariant Banach function space $E(\mathbb{R}^+)$ is necessarily intermediate for the pair $(L^1(\mathbb{R}^+), L^\infty(\mathbb{R}^+))$ [KPS], and so, if in addition $E(\mathbb{R}^+)$ is symmetric, then the operator space $E(\mathcal{M})$ is properly symmetric. On the other hand, if $E \subseteq \widetilde{\mathcal{M}}$ is rearrangement invariant and symmetric, then it need not follow that E is intermediate for the Banach couple $(L^1(\mathcal{M}), \mathcal{M})$, even if $\mathcal{M}$ is commutative, despite the assertion given in [Ov2]. An appropriate example which illustrates this pathology and which shows that this additional assumption is not superfluous, is given in [DDP4].

To facilitate the discussion, we introduce first some properties familiar from classical integration theory. Let $E \subseteq \widetilde{\mathcal{M}}$ be properly symmetric.

(A). The norm $\|\cdot\|_E$ on E is said to be *order continuous* if and only if $0 \leq x_\alpha \downarrow_\alpha 0$ in E implies $\|x_\alpha\|_E \downarrow_\alpha 0$.

(B). E is said to have the *Beppo-Levi property* if and only if $0 \leq x_\alpha \uparrow_\alpha \subseteq E$, $\sup_\alpha \|x_\alpha\|_E < \infty$ implies $\sup_\alpha x_\alpha$ exists in E.

(C). The norm $\|\cdot\|_E$ on E is said to be a *Fatou norm* if and only if $0 \leq x_\alpha \uparrow_\alpha x \in E$ implies $0 \leq \|x_\alpha\|_E \uparrow_\alpha \|x\|_E$.

We remark that each of the preceding properties is a well-known theorem in the special case that $E = L^1(\mathbb{R}^+)$, and this, indeed, is the source of the terminology. We note, however, that the simple, but less suggestive, labels (A), (B), (C) have been adopted by [GS]. As the following theorem shows, each of these properties is inherited by the non-commutative space constructed from a rearrangement invariant symmetric Banach function space on

$\mathbb{R}^+$ having the corresponding property .

Theorem 2.1. *Let $E(\mathbb{R}^+)$ be a rearrangement invariant, symmetric Banach function space on $\mathbb{R}^+$.*

(a) *If $E(\mathbb{R}^+)$ has order continuous norm, then so does $E(\mathcal{M})$.*

(b)*If $E(\mathbb{R}^+)$ has the Beppo-Levi property, then so does $E(\mathcal{M})$.*

(c) *If $E(\mathbb{R}^+)$ has Fatou norm, then so does $E(\mathcal{M})$.*

This theorem is given in [DDP4], and for the case of continuous von Neumann algebras with finite trace is due independently to V.I. Chilin and F.A. Sukochev. See, for example [GS] and the references contained therein. We now introduce the notion of duality which is central to what follows.

If $E \subseteq \widetilde{\mathcal{M}}$ is a properly symmetric Banach space, the *Köthe dual* (or *associate space*) $E^\times$ consists of those $x \in \widetilde{\mathcal{M}}$ such that $xy \in L^1(\mathcal{M})$ for all $y \in E$, with norm defined by setting

$$\|x\|_{E^\times} = \sup\{\tau(|xy|) \ : \ y \in E, \ \|y\|_E \le 1\}, \quad x \in E^\times.$$

Theorem 2.2. *Let $E \subseteq \widetilde{\mathcal{M}}$ be properly symmetric.*

(a) *The Köthe dual $E^\times \subseteq \widetilde{\mathcal{M}}$ is fully symmetric, has Fatou norm and has the Beppo-Levi property. If $x \in \widetilde{\mathcal{M}}$, then $x \in E^\times$ if and only if*

$$sup \ \{\int_{[0,\infty)} \mu_t(x)\mu_t(y)dt \ : \ y \in E, \ \|y\|_E \le 1\} < \infty,$$

in which case,

$$\begin{aligned}\|x\|_{E^\times} &= \sup \ \{|\tau(xy)| \ : y \in E, \ \|y\|_E \le 1\} \\ &= \sup \ \{\int_{[0,\infty)} \mu_t(x)\mu_t(y)dt \ : \ y \in E, \ \|y\|_E \le 1\}.\end{aligned}$$

(b) *There exists a fully symmetric Banach function space $F = F(\mathbb{R}^+)$ with Fatou norm and having the Beppo-Levi property such that $E^\times = F(\mathcal{M})$.*

The commutative version of part (a) of this theorem may be found, for example, in [KPS]. The commutative specialization of the representation theorem of part (b) of the theorem is due to Luxemburg [Lux]. It does not appear to be known if every properly symmetric space $E \subseteq \widetilde{\mathcal{M}}$ admits a representation as the non-commutative space constructed from some properly symmetric Banach function space on $\mathbb{R}^+$. This question has an affirmative answer for fully symmetric spaces [DDP3], for continuous von Neumann algebras with finite trace [CS2],and for trace ideals [MS]. Even in the commutative setting, however, the general situation is not entirely clear. See, for example [Fr2].

For ease of notation, we will write $E^\times(\mathbb{R}^+)$ for the Köthe dual of the properly symmetric Banach function space $E(\mathbb{R}^+)$.

Theorem 2.3. *If $E(\mathbb{R}^+)$ is a properly symmetric Banach function space on $\mathbb{R}^+$, then the equality*

$$E^\times(\mathcal{M}) = E(\mathcal{M})^\times$$

holds in the sense of Banach spaces.

The preceding theorem [DDP4], which is due to Garling [Ga2] in the case of trace ideals, essentially reduces identification of the Köthe duals of properly symmetric spaces to the commutative case. For example, we obtain immediately the Banach space equalities:

$$\left(L^1(\mathcal{M}) + \mathcal{M}\right)^\times = L^1(\mathcal{M}) \cap \mathcal{M}, \quad \left(L^1(\mathcal{M}) \cap \mathcal{M}\right)^\times = L^1(\mathcal{M}) + \mathcal{M},$$

which follow immediately via the preceding theorem and their corresponding commutative specializations, which are standard facts in interpolation theory given, for example, in [KPS]. Similar remarks apply to the identification of the Köthe duals of non-commutative Lorentz and Marcinkiewicz spaces, to which we shall return below.

We now consider the natural embedding of a properly symmetric space into its Köthe bidual. The theorem which follows (see [DDP4]) characterizes those properly symmetric spaces for which the Köthe dual is a norm-determining subspace of the Banach dual. It extends a similar characterization in the commutative setting due to G.G. Lorentz and W.A.J. Luxemburg [Za1].

Theorem 2.4. *Let* $E \subseteq \widetilde{\mathcal{M}}$ *be properly symmetric.*

(a) *The natural embedding of* E *into* $E^{\times\times}$ *is an isometry if and only if* E *has Fatou norm.*

(b) *The natural embedding of* E *into* $E^{\times\times}$ *is a surjective isometry if and only if* E *has Fatou norm and the Beppo-Levi property.*

If $E \subseteq \widetilde{\mathcal{M}}$ is properly symmetric, and if the natural embedding of E into $E^{\times\times}$ is a surjective isometry, then E is called *maximal.*

It is clear that if $E \subseteq \widetilde{\mathcal{M}}$ is properly symmetric, then there is a natural identification of the associate space $E^{\times}$ as a closed subspace of the Banach dual E^*. Those elements of E^* that arise in this way are now identified by the following Radon-Nikodym type theorem.

Theorem 2.5. *If* $E \subseteq \widetilde{\mathcal{M}}$ *is a properly symmetric Banach space and if* $\phi \in E^*$ *then the following are equivalent.*

(i) $x_\alpha \downarrow_\alpha 0 \subseteq E$ *implies* $\phi(x_\alpha) \to_\alpha 0$.

(ii) $e_\alpha \downarrow_\alpha 0 \subseteq \mathcal{M}^p$ *implies* $\max\{|\phi(xe_\alpha)|, |\phi(e_\alpha x)|\} \to_\alpha 0$ *for all* $x \in E$.

(iii) *There exists* $a \in E^{\times}$ *such that* $\phi(x) = \tau(ax)$ *for all* $x \in E$.

When specialized by taking E to be $\mathcal{M}$, $L^1(\mathcal{M})$ respectively, the preceding theorem reduces to the familiar and standard facts that the predual of a semi-finite von Neumann algebra $\mathcal{M}$ may be identified with the space $L^1(\mathcal{M})$, and in turn with the linear subspace of the Banach dual $\mathcal{M}^*$ consisting of all continuous normal (respectively, completely additive) linear functionals on $\mathcal{M}$. Indeed, the proof of the theorem [DDP4] (see also [Ye2]) ultimately reduces to this special case. The scope of the theorem is, however, considerably wider. For example, condition (i) is clearly satisfied by every element ϕ of the Banach dual E^* if E has order continuous norm. Consequently, if E has order continuous norm, then the Banach dual space E^* concides with the Köthe dual $E^\times$. This, in fact, not only characterizes order continuity of the norm but, in conjunction with Theorem 2.3 above, permits ready identification of the Banach dual spaces of those properly symmetric spaces with order continuous norm. By way of example, the norm on each of the spaces $L^p(\mathbb{R}^+)$, $1 \le p < \infty$, is order continuous. Consequently, the well-known identification

$$L^p(\mathcal{M})^* \;=\; L^q(\mathcal{M}), \quad 1 \le p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1$$

follows immediately via Theorems 2.1(a), 2.3 and 2.5 preceding and its well-known commutative specialization. To make some further remarks, we suppose that ψ is a non-decreasing concave function on $\mathbb{R}^+$ with $\psi(0+) = 0$ and $\psi(\infty) = \infty$. Following [KPS] Chapter II.5, we let $\Lambda_\psi(\mathbb{R}^+)$, $M_\psi(\mathbb{R}^+)$ be the Lorentz and Marcinkiewicz spaces with norms defined by setting

$$\|f\|_{\Lambda_\psi(\mathbb{R}^+)} \;=\; \int_{[0,\infty)} \mu_t(f)\, \psi'(t)\, dt, \quad f \in \Lambda_\psi(\mathbb{R}^+),$$

$$\|f\|_{M_\psi(\mathbb{R}^+)} \;=\; \sup_{\alpha > 0} \frac{1}{\psi(\alpha)} \int_0^\alpha \mu_t(f)\, dt.$$

It follows from [KPS] Corollary 1 to Theorem II.5.1 that the Lorentz space $\Lambda_\psi(\mathbb{R}^+)$ has order continuous norm and so the preceding remarks immediately imply that the Banach space dual of the non-commutative Lorentz space $\Lambda_\psi(\mathcal{M})$ is just the non-commutative Marcinkiewicz space $M_\psi(\mathcal{M})$, a result due to L. Ciach [Ci]. Similarly, if $M^0_\psi(\mathbb{R}^+)$ denotes the linear subspace of $M_\psi(\mathbb{R}^+)$ consisting of all $f \in M_\psi(\mathbb{R}^+)$ for which

$$\lim_{\alpha \to 0,\infty} \frac{1}{\psi(\alpha)} \int_0^\alpha \mu_t(f)\, dt \;=\; 0,$$

then it follows via [KPS] Theorem II 5.4 that $M^0_\psi(\mathcal{M})^* = \Lambda_\psi(\mathcal{M})$, provided $\psi(t)/t \to \infty$ as $t \to 0$. Let us remark, for future reference, that if the function ψ satisfies $\psi(t)/t \to 0$ as $t \to \infty$, then the Marcinkiewicz space $M_\psi(\mathcal{M}) \subseteq \widetilde{\mathcal{M}}_0$.

In the case of trace ideals, the results of the preceding paragraph may be found in [GK] Chapter III.15. Let us indicate that results concerning trace ideals are special cases of the present approach, and not merely analogues. Let Φ be a symmetric norming function, in the sense of Gohberg and Krein [GK] Chapter III, on the space of all finitely non-zero real sequences and denote by $(c_\Phi, \| \|_\Phi)$ the associated symmetrically normed sequence ideal. If $x \in c_\Phi$, then $\mu(x)$ is calculated by identifying x as an element of the commutative von Neumann algebra $l^\infty(\mathbb{N})$ acting by multiplication in the Hilbert space $l^2(\mathbb{N})$, equipped with the natural trace defined by counting measure on $\mathbb{N}$. The decreasing rearrangement $\mu(x)$ is the unique extension of the usual non-increasing rearrangement of the sequence x to a non-increasing, right-continuous function on $[0, \infty)$. The space $(F_\Phi(\mathbb{R}^+), \| \|_\Phi)$ is defined to be the set of all functions $f \in L^0(\mathbb{R}^+)$ for which

$$\|f\|_\Phi = \sup \{ \int_{[0,\infty)} \mu_t(f)\mu_t(x)dt \ : \ x \in c_\Phi, \ \|x\|_\Phi \leq 1\} < \infty,$$

equipped with the norm as indicated. The space $F_\Phi(\mathbb{R}^+)$ is a maximal rearrangement invariant Banach function space on $\mathbb{R}^+$. We define $E_\Phi(\mathbb{R}^+)$ to be the associate space $F_\Phi^\times(\mathbb{R}^+)$. Equipped with the associate norm, the space $E_\Phi(\mathbb{R}^+)$ is also a maximal rearrangement invariant Banach function space on $\mathbb{R}^+$. If we now take $\mathcal{M}$ to be the von Neumann algebra $\mathcal{L}(\mathcal{H})$, equipped with the canonical trace, then it is not difficult, using [GK], section 11 of Chapter III, to see that $E_\Phi(\mathcal{M})$ is precisely the symmetrically normed ideal of compact operators $\mathcal{C}_\Phi$ given in [GK], section 4 of the same chapter.

3. Weak compactness and reflexivity If the usual duality for (commutative) spaces of type L^1, L^∞ is interpreted in the sense of Köthe, then it is well-known that classical criteria for weak compactness in spaces of type L^1 admit natural extensions to very general classes of function spaces, and even more generally to the setting of vector lattices [Fr1], [AB]. These same classical criteria admit natural extensions to the preduals of general von Neumann algebras [Ak], [Ta1]. The central theme of [Ga2], is the study of special properties of weakly compact subsets of trace ideals which reflect the symmetric structure.

Under natural conditions, it was shown by Garling [Ga2] that the rearrangement invariant hull of a weakly compact subset in a trace ideal is again relatively weakly compact by showing that weakly compact subsets in a trace ideal could be characterized in terms of the corresponding set of sequences of singular values. Subsequently, the analogue of Garling's work in a general (commutative) function space setting was given by Fremlin [Fr2], whose work was based on vector lattice techniques. The first result of this section uses the characterization of the Köthe dual of a symmetric operator space given by Theorem 2.5 to unify the work of Garling and Fremlin. Our characterization is of Dunford-Pettis type and for preduals of von Neumann algebras is due, in part, to Akemann [Ak], based on earlier work of Takesaki [Ta 2].

We need some additional terminology. A linear operator T on $L^1(\mathcal{M}) + \mathcal{M}$ is called a *contraction for the pair* $(L^1(\mathcal{M}),\ \mathcal{M})$ if and only if T acts as a contraction in each of $L^1(\mathcal{M})$, $\mathcal{M}$ respectively. We denote by Σ the set of all contractions for the pair $(L^1(\mathcal{M}),\ \mathcal{M})$.

We let $\mathcal{S}_0 = \{f \in L^0(\mathbb{R}^+) \ : \ \mu_t(f) \to 0 \text{ as } t \to \infty\}$.

Our principal result on weak compactness [DDP5] now follows.

Theorem 3.1. *Assume that* $E(\mathbb{R}^+)$ *is maximal and that* $E(\mathbb{R}^+), E(\mathbb{R}^+)^\times \subseteq \mathcal{S}_0$. *If* $K \subseteq E(\mathcal{M})$ *is bounded, then the following statements are equivalent.*

(i) K *is relatively* $\sigma(E(\mathcal{M}), E(\mathcal{M})^\times)$*-compact.*

(ii) *For every system* $\{y_\alpha\} \subseteq E(\mathcal{M})^\times$ *with* $y_\alpha \downarrow_\alpha 0$,

$$\sup\{|\tau(xy_\alpha)| \ : \ x \in K\} \ \longrightarrow_\alpha \ 0.$$

(iii) *For every mutually orthogonal sequence* $\{e_n\} \subseteq \mathcal{M}^p$ *and every* $0 \le y \in E(\mathcal{M})^\times$,

$$\sup\{\max(|\tau(ye_nx)|, |\tau(xe_ny)|) \ : \ x \in K\} \ \longrightarrow_n \ 0.$$

(iv) *For every* $y \in E(\mathcal{M})^\times$ *and* $\{y_n\} \subseteq \Omega(y)$ *with* $y_n \to 0$ *for the measure topology,*

$$\sup\{\int_{[0,\infty)} \mu_t(y_n)\mu_t(x)\,dt \;:\; x \in K\} \longrightarrow_n 0.$$

(v) *For every mutually orthogonal sequence* $\{e_n\} \subseteq \mathcal{M}^p$ *and every* $0 \le y \in E(\mathcal{M})^\times$,

$$\sup\{\int_{[0,\infty)} \mu_t(x)\mu_t(ye_n)\,dt \;:\; x \in K\} \longrightarrow_n 0.$$

(vi) $\{\mu(x) \;:\; x \in K\}$ *is relatively* $\sigma(E(\mathbb{R}^+), E(\mathbb{R}^+)^\times)$*-compact.*

(vii) $\{Tx \;:\; x \in K,\ T \in \Sigma\} \subseteq E(\mathcal{M})$ *is relatively* $\sigma(E(\mathcal{M}), E(\mathcal{M})^\times)$*-compact.*

(viii) $\{\, y \in E(\mathcal{M}) \;: \exists x \in K, y \prec\prec x\} \subseteq E(\mathcal{M})$ *is relatively* $\sigma(E(\mathcal{M}), E(\mathcal{M})^\times)$*-compact.*

The equivalence of statements (i),(ii),(iii) above, in the commutative setting, contains the well-known Dunford-Pettis characterization of weakly compact subsets of spaces of type L^1 and for trace ideals is given in [DL]. The equivalence of statements (i),(v) contains as a special case a characterization of uniform integrability in commutative L^1-spaces in terms of decreasing rearrangements due to Chong [Ch]. The equivalence of statements (i),(vi), (viii) for the case of trace ideals is due to Garling [Ga2] and to Fremlin [Fr2], in the general commutative setting. The equivalence of (vii), (viii) follows from [DDP4]. In the case that $E(\mathbb{R}^+)$ is maximal and has order-continuous norm then the equivalence of statements (i), (viii) is of special interest and asserts that the "rearrangement invariant hull" of a relatively weakly compact set is again relatively weakly compact, subject of course to the additional condition that $E(\mathbb{R}^+)^\times \subseteq \mathcal{S}_0$. These conditions are satisfied, in particular, for the Lorentz spaces given in the preceding section provided the defining function ψ is appropriately restricted at ∞, and for any maximal space in the case that the trace is finite. It follows, in particular, that the rearrangement invariant hull of any relatively weakly compact subset of the predual of a finite von Neumann algebra is again relatively weakly compact. This result fails if the trace is not finite, even in the commutative cas Indeed, if $x = \chi_{[0,1)} \in L^1(\mathbb{R}^+)$, then it is easily seen that the set $\Omega(x) \subseteq L^1(\mathbb{R}^+)$ is not relatively weakly compact. We mention finally that the preceding Theorem has been obtained independently by Goldstein and Sukochev [GS] for von Neumann algebras without atoms and with finite trace.

The preceding characterizations of weak compactness are analogues of similar results that are well-known in the theory of Banach lattices [AB], [Fr1]. This analogy may be developed further by characterizing various topological properties, thereby extending known results in the theory of Banach function spaces. It should be emphasized, however, that arguments based on lattice structure are not applicable in the non-commutative setting, due to lack of even local unconditional structure.

A Banach space $(X, \| \|_X)$ is said to have *property* (u) if and only if whenever $\{x_n\} \subseteq X$ is a $\sigma(X, X^*)$-Cauchy sequence, it follows that there exists a sequence $\{y_n\} \subseteq X$ such that the series $\sum_{n=1}^{\infty} y_n$ is weakly unconditionally convergent (that is $\sum_{n=1}^{\infty} |y^*(y_n)| < \infty$ for all $y^* \in X^*$) and such that the sequence $\{x_n - \sum_{i=1}^{n} y_i\}$ is weakly convergent to zero. The preceding property was introduced by A. Pelczynski, who showed that property (u) is hereditary in the sense that each closed subspace of a Banach space with property (u) has itself property (u). See, for example [AB] or [LT2]. We now characterize those symmetric operator spaces with order continuous norm.

Theorem 3.2. *Let $E \subseteq \widetilde{\mathcal{M}}$ be properly symmetric and consider the following statements.*

(i) *E has order continuous norm.*

(ii) *E has property* (u).

(iii) *E contains no isomorphic copy of l^∞.*

(iv) *E contains no positive copy of l^∞.*

(v) *For all $x \in E$, the set $\Omega(x)$ is $\sigma(E, E^*)$-compact.*

The implications (ii)$\Rightarrow$(iii)$\Rightarrow$(iv), (v)$\Rightarrow$(i) *always hold. If $E^{\times\times} \subseteq \widetilde{\mathcal{M}}_0$ then* (i) $\Rightarrow$(ii), *if E is fully symmetric, then* (iv)$\Rightarrow$(i) *and if $E^\times \subseteq \widetilde{\mathcal{M}}_0$ then* (i)$\Rightarrow$(v).

We mention first that the first four conditions are equivalent if E is maximal, and in this case, the theorem is a non-commutative analogue of a well-known result for Dedekind complete Banach lattices due to Lozanovski and Mekler [LT2], [AB], [Za 2]. The equivalence of statements (i), (v) contains as special cases results of Ryff [Ry] and Luxemburg [Lux].

We turn now to weak sequential completeness.

Theorem 3.3. *Let $E \subseteq \widetilde{\mathcal{M}}$ be properly symmetric. The following statements are equivalent.*

(i) *E is $\sigma(E, E^*)$-sequentially complete .*

(ii) *E has the Beppo-Levi property and has order continuous norm.*

(iii) *Each norm-bounded increasing sequence in E is convergent.*

(iv) *E contains no isomorphic copy of c_0 .*

If, in addition, E is fully symmetric, then each of the preceding statements is equivalent to

(v) *E contains no positive copy of c_0 .*

Condition (ii) is easily seen to be satisfied if E a non-commutative Lorentz space $\Lambda_\psi(\mathcal{M})$. Condition (ii) therefore shows that such spaces are weakly sequentially complete. This contains as a special case the well-known fact [Ta] that the predual of any semi-finite von Neumann algebra is weakly sequentially complete. It is well known that statements (i), (iv) are equivalent in any Banach lattice, but in general are not equivalent in arbitrary Banach spaces.

Theorem 3.4. *Let $E \subseteq \widetilde{\mathcal{M}}$ be properly symmetric. If E has order continuous norm, and if $E^{\times\times} \subseteq \widetilde{\mathcal{M}}_0$ then the following statements are equivalent.*

(i) *$E^\times$ $(= E^*)$ has order continuous norm.*

(ii) *The unit ball of E is conditionally $\sigma(E, E^*)$- sequentially compact.*

(iii) *E contains no isomorphic copy of l^1* .

(iv) *E contains no positive copy of l^1* .

The preceding theorem is an analogue of a well-known Banach lattice characterization of dual norms which are order continuous. The equivalence of statements (ii), (iii) is, of course, an immediate consequence of Rosenthal's l^1-theorem, although an appeal to this theorem is not necessary in the proof of Theorem 3.4.

Theorems 3.3, 3.4 preceding imediately yield the following characterization of reflexivity.

Theorem 3.5. *Let $E \subseteq \widetilde{\mathcal{M}}$ be properly symmetric. The following statements are equivalent.*

(i) *E is reflexive.*

(ii) *E contains no isomorphic copy of c_0 or l^1* .

(iii) *E contains no positive copy of c_0 or l^1* .

(iv) *E has the Beppo-Levi property and the norms on E and E^* are order continuous.*

In the commutative case, the preceding equivalence of statements (i), (iv) specialize to a well-known reflexivity criterion due to T. Ogasawara, and the equivalence of statements (i), (iv) in the Banach lattice setting is due to G. Ja. Lozanovski. It is also worth mentioning that Theorems 3.3, 3.4, 3.5, in the commutative setting, find their roots in the work of R.C. James on Banach spaces with unconditional bases. For more detailed information, the reader is referred to [AB], [LT1,2], [Za].

The final result of this section is obtained by combining Theorems 3.5, 2.1.

Theorem 3.6. *Let $E(\mathbb{R}^+)$ be a rearrangement invariant, symmetric Banach function space on $\mathbb{R}^+$. If $E(\mathbb{R}^+)$ is reflexive, then so is $E(\mathcal{M})$.*

We remark that each of Theorems 3.2-3.6 have been obtained independently by Goldstein and Sukochev [GS], for the case of finite von Neumann algebras without atoms.

4. Duality and the Calderón-Lozanovskii construction If $0 < \theta < 1$, we denote by $[\cdot,\cdot]_\theta$ the (first) complex interpolation method given by Calderón [Cal]. Let E_0, E_1 be Banach function spaces on $\mathbb{R}^+$ and let $0 < \theta < 1$. Following [Cal] (see also [KPS], section 11 of Chapter 4) the space $E_0^{1-\theta}\, E_1^\theta$ is defined to be the space of those measurable functions f such that there exists $\lambda \in \mathbb{R}^+, f_i \in E_i$ with $\|f\|_{E_i} \leq 1$, $i = 0, 1$, such that $|f| \leq \lambda |f_0|^{1-\theta} |f_1|^\theta$. This space is equipped with the norm given by the greatest lower bound of such numbers λ taken over all possible such representations. If either E_0 or E_1 has order continuous norm then it was shown by Calderón [Cal] (see also [KPS], Theorem 4.1.14) that the equality

$$E_0^{1-\theta}\, E_1^\theta \;=\; [E_0,\; E_1]_\theta$$

holds with equality of norms. It follows, in particular, that if E_0, E_1 are fully symmetric Banach function spaces on $\mathbb{R}^+$, at least one of which has order continuous norm, then the space $E_0^{1-\theta}\, E_1^\theta$ is again a fully symmetric Banach function space on $\mathbb{R}^+$, and has order continuous norm, as noted in the remark following [KPS] Theorem 4.1.14.

It was proved by Lozanovski [Lo], that if E_0, E_1 are Banach function spaces on $\mathbb{R}^+$ and if $0 < \theta < 1$, then

$$\left(E_0^{1-\theta}\, E_1^\theta\right)^\times \;=\; \left(E_0^\times\right)^{1-\theta}\, \left(E_1^\times\right)^\theta.$$

A proof of this basic result is given in Reisner [Re]. In this section, we indicate non-commutative extensions of several theorems due to G.Ya. Lozanovskii [Lo].

Theorem 4.1. *Let* E_0, E_1 *be fully symmetric Banach function spaces on* $\mathbb{R}^+$ *and let* $0 < \theta < 1$. *If at least one of the spaces* E_0, E_1 *has order continuous norm, and if at least one of the spaces* $E_0^\times, E_1^\times$ *has order continuous norm, then*

$$[E_0(\mathcal{M}),\ E_1(\mathcal{M})]_\theta^* \ = \ [E_0(\mathcal{M}),\ E_1(\mathcal{M})]_\theta^\times \ = \ [E_0^\times(\mathcal{M}),\ E_1^\times(\mathcal{M})]_\theta .$$

The proof is a synthesis of the results summarized in the remarks preceding the theorem, combined with [DDP3] Theorem 3.2. A rearrangement invariant space E is called a KB-*space* if E has order continuous norm and the Beppo-Levi property.

Theorem 4.2. *Suppose that* $E_0,\ E_1$ *are fully symmetric Banach function spaces on* $\mathbb{R}^+$. *If at least one of* $E_0,\ E_1$ *is a KB-space, and if at least one of* $E_0^\times,\ E_1^\times$ *is a KB-space, then* $[E_0(\mathcal{M}),\ E_1(\mathcal{M})]_\theta,\ 0 < \theta < 1$, *is a reflexive Banach space.*

Under the given assumptions, the space $E_0^{1-\theta}\ E_1^\theta$ is reflexive ([Lo], Theorem 3) and consequently, so is the non-commutative space $\left(E_0^{1-\theta}\ E_1^\theta\right)(\mathcal{M}) \ = \ [E_0(\mathcal{M}),\ E_1(\mathcal{M})]_\theta$, by Theorem 3.6. □

We remark, in the case that $E_0 = E_1$ the preceding Theorem reduces to a special case of the non-commutative extension of Ogasawara's theorem given in Theorem 3.5. We comment further on the special case obtained by setting $E_0 = L^\infty(\mathbb{R}^+)$. If $1 < p < \infty$ and if E is any Banach function space on $\mathbb{R}^+$ then, setting $\theta = 1/p$, it is easily seen that $E^\theta\ L^\infty(\mathbb{R}^+)^{1-\theta} \ = \ E^{(p)}$, where $E^{(p)}$ is the space of those functions $f \in L^0(\mathbb{R}^+)$ for which $|f|^p \in E$ with norm given by setting $\|f\|_{E^{(p)}} \ = \ \|\ |f|^p\ \|_E^{\frac{1}{p}}$. The space $E^{(p)}$ may be identified with the (so-called) $p-$convexification of E (see [LT2], 1d).

Theorem 4.3. *If* E *is a fully symmetric KB-space on* $\mathbb{R}^+$ *and if* $1 < p < \infty$ *then* $E^{(p)}(\mathcal{M})$ *is reflexive. Conversely, if* E *is a rearrangement invariant, symmetric Banach function space on* $\mathbb{R}^+$, *and if* $E^{(p)}(\mathcal{M})$ *is reflexive then* $E(\mathcal{M})$ *has order continuous norm and has the Beppo-Levi property.*

We note that the preceding Theorem reduces to [Lo], Theorem 4, in the commutative setting.

Theorem 4.4. *Let* E *be a fully symmetric Banach function space on* $\mathbb{R}^+$. *If either* E *or* $E^\times$ *has order continuous norm, then*

$$[E(\mathcal{M}),\ E^\times(\mathcal{M})]_{\frac{1}{2}} \;=\; L^2(\mathcal{M}).$$

The Theorem follows immediately from its commutative specialization, which is a special case of [Lo] Theorem 5, and [DDP3] Theorem 3.2.

The final result of this section is again due to Lozanovskii ([Lo], Theorem 6) in the commutative case, and does not depend on interpolation.

Theorem 4.5. *Let* E *be a rearrangement invariant symmetric Banach function space on* $\mathbb{R}^+$. *If* $0 \le x \in L^1(\mathcal{M})$ *and if* $\epsilon > 0$ *is given then there exist* $0 \le y \in E(\mathcal{M})$, $0 \le z \in E^\times(\mathcal{M}) = E(\mathcal{M})^\times$ *with* $yz = zy$ *such that* $x = yz$ *and such that* $\|y\|_{E(\mathcal{M})}\ \|z\|_{E^\times(\mathcal{M})} \le (1+\epsilon)\|x\|_1$. *If* E *is maximal, then* ϵ *may be taken to be* 0.

5. Convexity properties We consider first a non-commutative extension of an interpolation theorem of Pisier [Pi]. We recall some necessary terminology [LT2]. Assume that X is a Banach lattice and let $1 \le p \le q \le \infty$.

(i) X is called $p-$*convex* if there exists a constant M such that: $\forall n, \forall\{x_1, \ldots, x_n\} \subseteq X$

$$\left\|\Big(\sum_1^n |x_i|^p\Big)^{1/p}\right\| \;\le\; M\Big(\sum_1^n \|x_i\|^p\Big)^{1/p}.$$

The smallest such M is denoted $M^{(p)}(X)$ and is called the *modulus of* $p-$*convexity* of X.

(ii) X is called $q-$*concave* if there exists a constant M such that: $\forall n, \forall\{x_1, \ldots, x_n\} \subseteq X$

$$\left(\sum_1^n \|x_i\|^q\right)^{1/q} \leq M\left\|(\sum_1^n |x_i|^q)^{1/q}\right\|.$$

The smallest such value of M is denoted $M_{(q)}(X)$, and is called the *modulus of $q-$concavity* of X. We adhere to the usual convention if $p, q = \infty$. We remark that if $1 < p \leq 2 \leq q < \infty$ and if $(E, \|\cdot\|_E)$ is a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$ which is p-convex and q-concave, then it follows from the proof of [LT2] 1.d.8 that there exists on E an equivalent norm $\|\cdot\|'_E$ such that $(E, \|\cdot\|'_E)$ is a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$ with moduli of $p-$convexity and $q-$concavity both equal to 1.

Theorem 5.1. *Let E be a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$, and let $1 < p \leq 2 \leq q < \infty$. If E is p-convex and q-concave with $M^{(p)}(E) = 1 = M_{(q)}(E)$ then there exist a fully symmetric Banach function space E_0 with the Beppo-Levi property and having a Fatou norm such that*

$$E(\mathcal{M}) = [E_0(\mathcal{M}), L^s(\mathcal{M})]_\theta,$$

where θ satisfies

$$1/p = \theta/s + (1-\theta)/1, \qquad 1/q = \theta/s.$$

We remark that the given conditions on E imply that E is reflexive, and hence fully symmetric. The proof of Theorem 5.1 is then an immediate consequence of its commutative specialization, which is a minor adaptation of [Pi] Theorem 2.3 and Remark 2.6, and [DDP3] Theorem 3.2.

Let us make some further remarks concerning Theorem 5.1. If $q = p' = p/(p-1)$, then we obtain

$$E(\mathcal{M}) = [E_0(\mathcal{M}), L^2(\mathcal{M})]_\theta,$$

with θ defined by $1/p = \theta/2 + (1-\theta)/1$. Since $L^2(\mathcal{M})$ is a Hilbert space, it follows that the non-commutative space $E(\mathcal{M})$ is θ-Hilbertian in the terminology of [Pi], and some implications of this are pointed out explicitly in section 4 of [Pi]. Theorem 5.1 yields some immediate information concerning convexity properties of non-commutative spaces. We recall first the necessary definitions and terminology.

A Banach space $(X, \|\cdot\|_X)$ is called *uniformly convex* if $x_n, y_n \in X$, $\|x_n\|_X \leq 1$, $\|y_n\|_X \leq 1$, $n = 1, 2, \ldots$, $\|x_n + y_n\|_X \to 2$ imply that $\|x_n - y_n\|_X \to 0$. The Banach space X is said to be *uniformly convexifiable* if X admits an equivalent uniformly convex norm. For every Banach space X, the *modulus of convexity* $\delta_X(\epsilon)$, $\epsilon \in (0, 2]$, is defined by

$$\delta_X(\epsilon) = \inf\{1 - \|x+y\|_X/2 : \ x, y \in X, \ \|x\|_X = 1 = \|y\|_X, \ \|x - y\|_X = \epsilon\}$$

and the *modulus of smoothness* $\rho_X(\alpha)$, $\alpha > 0$ is defined by

$$\rho_X(\alpha) = \sup\{(\|x+y\|_X + \|x-y\|_X)/2 - 1 : \ \ x, y \in X, \ \|x\|_X = 1, \|y\|_X = \alpha\}.$$

It is clear that X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for every $\epsilon > 0$. X is said to be *uniformly smooth* if and only if $\lim_{\alpha \to 0} \rho_X(\alpha)/\alpha = 0$. A uniformly convex (respectively, uniformly smooth) space X is said to have *modulus of convexity* (respectively, *smoothness*) *of power type p* (respectively q) if and only if $\delta_X(\epsilon) \geq K\epsilon^p$ (respectively, $\rho_X(\alpha) \leq K\alpha^q$), for some $K > 0$. It is shown in [Be] Chapter IV, that if $(X, \|\cdot\|_X)$ is uniformly convex, then there is an equivalent norm $\|\cdot\|'$ on X and $p \in (1,2)$, $q \in (2, \infty)$ such that $(X, \|\cdot\|')$ has modulus of convexity of power type q and modulus of smoothness of power type p. This in turn implies ([LT2], 1.e.16) that X has type p and cotype q, where, following [LT2] 1.e.12, a Banach space X is said to be of *type p* for some $1 < p \leq 2$, respectively, of *cotype q* for some $q \geq 2$, if there exists a (finite) constant $M > 0$ such that, for every finite set of vectors $\{x_j\}_{j=1}^n$ in X, we have

$$\int_0^1 \|\Sigma_1^n r_j(t) x_j\| \, dt \leq M \left(\Sigma_1^n \|x_j\|^p\right)^{1/p},$$

respectively,

$$\int_0^1 \|\Sigma_1^n r_j(t)x_j\|\,dt \geq M\left(\Sigma_1^n \|x_j\|^q\right)^{1/q},$$

where $\{r_n(\cdot)\}$ denotes the usual Rademacher sequence on $[0,1]$. If $1 < r < \infty$ and if X is a Banach lattice of type r (respectively, of cotype r), then it is well-known ([LT2] 1.f.9) that X is $p-$convex (respectively, $q-$concave) for every $1 < p < r < q$. It follows from these remarks that if a Banach lattice $(X, \|\cdot\|_X)$ is uniformly convex, then there exists $p \in (1,2)$ and $q \in (2,\infty)$ such that $(X, \|\cdot\|_X)$ is $p'-$convex and $q'-$concave, for all $p' \in (1,p)$ and $q' \in (q,\infty)$. We may now state the following result, which for the case of trace ideals, is due to Arazy [Ar] Proposition 2.2.

Theorem 5.2. *Let E be a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$. If E is uniformly convexifiable, then so is $E(\mathcal{M})$.*

Indeed, when combined with the remarks of section 4 of [Pi], Theorem 5.1 shows immediately that if $1 < p \leq 2 \leq q < \infty$ and if E is p-convex and q-concave with moduli of p-convexity and q-concavity respectively equal to 1, then the inequality

$$\left(\frac{\|x+y\|_{E(\mathcal{M})}^{r'} + \|x-y\|_{E(\mathcal{M})}^{r'}}{2}\right)^{\frac{1}{r'}} \leq \left(\|x\|_{E(\mathcal{M})}^{r} + \|y\|_{E(\mathcal{M})}^{r}\right)^{\frac{1}{r}}$$

holds for all $x, y \in E(\mathcal{M})$, where $r = \min(p, q')$, $q' = q/(q-1)$ and $r' = r/(r-1)$. It follows that the space $E(\mathcal{M})$ is uniformly convex with modulus of convexity of power type r' and uniformly smooth of power type r. For the case of trace ideals, these remarks have been made in [GT]. The following sharpening of this result is due to Q. Xu [Xu].

Theorem 5.3. *Let E be a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$, and suppose that $1 < p \leq 2 \leq q < \infty$. If $M^{(p)}(E) = 1 = M_{(q)}(E)$, then $E(\mathcal{M})$ is uniformly smooth of power type p and uniformly convex of power type q; further there exists a constant $C > 0$ independent of p and q such that*

$$\rho_{E(\mathcal{M})}(\epsilon) \leq C\epsilon^p, \qquad \delta_{E(\mathcal{M})}(\epsilon) \geq C^{-1}\epsilon^q, \qquad (0 < \epsilon \leq 1).$$

In the case of trace ideals, the preceding theorem is due to Tomczak-Jaegermann [T-J]. The method of [Xu] is an extension of that of [T-J], and is based on the approach of Fack [Fa].

The theorem which follows is due to Chilin, Krygin and Sukochev [CKS], and solves, in general setting, a problem raised by Arazy [Ar] for the case of trace ideals. We recall first that a Banach space $(X, \|\cdot\|_X)$ is called *locally uniformly convex* if $x_n,\ x \in X,\ n = 1, 2, \ldots,\ \|x_n\|_X \to \|x\|_X,\ \|x_n + x\|_X \to 2\|x\|_X$ imply $\|x_n - x\|_X \to 0$. It is shown in [DGL] that a Banach lattice X has order continuous norm if and only if X has an order equivalent locally uniformly convex norm.

Theorem 5.4. *Let E be a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$. If E is locally uniformly convex (respectively, uniformly convex) then $E(\mathcal{M})$ is locally uniformly convex (respectively, uniformly convex).*

The proof of the preceding theorem given in [CKS] depends on certain characterizations of norm convergence in terms of convergence for the measure topology and is of independent interest. In fact, one key element in their argument may be isolated as follows.

Theorem 5.5. *If $x_n, x \in L^1(\mathcal{M}) + \mathcal{M}$, if $\mu(x_n) = \mu(x)$, $n = 1, 2, \ldots$, and if $\mu(x) \in S_0$, then the following statements are equivalent.*

(i) *$x_n \to x$ for the measure topology.*

(ii) $\int_0^s \mu_t(x_n + x)dt \ \to\ 2\int_0^s \mu_t(x)dt, \qquad \forall s > 0.$

We outline the proof. Assume that (i) is satisfied. It then follows also that $\frac{1}{2}(x_n + x) \to x$ in measure, and consequently $\mu\left(\frac{1}{2}(x_n + x)\right) \to \mu(x)$ a.e. From

$$\mu\left(\frac{1}{2}(x_n + x)\right) \prec\prec \frac{1}{2}\left(\mu(x_n) + \mu(x)\right) = \mu(x),$$

it follows from the commutative version of Theorem 3.1 that the sequence $\{\mu\left(\frac{1}{2}(x_n + x)\right)\}$ is a relatively weakly compact subset of $L^1[0, s)$, and so from the well-known theorem of Vitali, it follows that $\int_0^s \left|\mu_t\left(\frac{1}{2}(x_n+x)\right) - \mu_t(x)\right| dt \rightarrow 0$. In particular, it follows that $\int_0^s \mu_t\left((x_n+x)\right) dt \rightarrow 2\int_0^s \mu_t(x) dt$, for all $s > 0$. The converse implication, (ii) implies (i), is the non-trivial implication and is essentially the argument of [CKS], Theorem 2.1. To see that their proof can be adapted to the present situation, we make several remarks. It may be assumed that $\mathcal{M}$ has no atoms. We next observe that the condition (ii) implies that $\mu_t\left(\frac{1}{2}(x_n+x)\right) \rightarrow \mu_t(x)$ at every point of continuity of $\mu(x)$. This is shown in [Fr2], Proposition 40. Consequently, if for each $\lambda > 0$, we set $q_n(\lambda) = \chi_{(\lambda,\infty)}(|x_n|)$, $p_n(\lambda) = \chi_{(\lambda,\infty)}\left(\frac{1}{2}|x_n+x|\right)$ and $p(\lambda) = \chi_{(\lambda,\infty)}(|x|)$ and noting that $\tau(p_n(\lambda)) = m\left(\left\{\mu\left(\frac{1}{2}(x_n+x)\right) > \lambda\right\}\right)$, and

$$\tau(p(\lambda)) = m(\{\mu(x) > \lambda\}) = m(\{\mu(x_n) > \lambda\}), \quad n = 1, 2, \ldots,$$

then it follows from [CKS], Lemma 1.5 that

$$\tau(q_n(\lambda)) = \tau(p(\lambda)) \; \forall n, \quad \tau(p_n(\lambda)) \rightarrow \tau(p(\lambda)).$$

Here m denotes Lebesgue measure on $\mathbb{R}^+$. We next observe that condition (ii) implies that the sequence $\left\{\mu\left(\frac{1}{2}(x_n+x)\right)\right\}$ is equi-integrable in the sense that, setting $x_0 = x$, and given $\epsilon > 0$, there exists $\delta > 0$ such that $\sup_n \int_0^\delta \mu_t\left(\frac{1}{2}(x_n+x)\right) dt < \epsilon$. It follows, in particular, that

$$\int_0^{\tau(p_n(\lambda))} \mu_t\left((x_n+x)/2\right) dt \rightarrow \int_0^{\tau(p(\lambda))} \mu_t(x) dt, \quad \forall \lambda > 0.$$

The argument of [CKS], Theorem 2.1 now shows that

$$\|p_n(\lambda) - p(\lambda)\|_1 \rightarrow 0, \quad \|q_n(\lambda) - p(\lambda)\|_1 \rightarrow 0.$$

By spectral approximation, it follows that $|x_n| \rightarrow |x|$, $\left|\frac{1}{2}(x_n+x)\right| \rightarrow |x|$ for the measure topology. By [CKS] Lemma 1.1 , this implies that $x_n \rightarrow x$ for the measure topology.

To indicate the relation of Theorem 5.5 to characterizations of norm convergence, we mention first the following result, due to Chilin and Sukochev [CS2].

Theorem 5.6. *If E is a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$ with order continuous norm, and if $x, x_n \in E(\mathcal{M})$, $n = 1, 2, \ldots$, then the following statements are equivalent.*

(i) $\|x_n - x\|_{E(\mathcal{M})} \to 0$.

(ii) $x_n \to x$ *in measure and* $\|\mu(x_n) - \mu(x)\|_{E(\mathbb{R}^+)} \to 0$.

We may now state the following theorem.

Theorem 5.7. *If E is a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$ with order continuous norm, and if $x, x_n \in E(\mathcal{M})$, $n = 1, 2, \ldots$, then the following statements are equivalent.*

(i) $\|x_n - x\|_{E(\mathcal{M})} \to 0$.

(ii) $\|\mu(x_n) - \mu(x)\|_{E(\mathbb{R}^+)} \to 0$ *and* $\int_0^s \mu_t(x_n + x)dt \to 2\int_0^s \mu_t(x)dt$, *for all* $s > 0$.

The implication (i) implies (ii) is a simple consequence of the Markus inequality [DDP1,2]. To show the implication (ii) implies (i), it suffices, via Theorem 5.6, to show that condition (ii) implies that $x_n \to x$ for the measure topology. It may be assumed that $\mathcal{M}$ is atomless. Using the first assertion of condition (ii), and the fact that $\mathcal{M}$ has no atoms, it may be shown [CKS] that there exists a sequence $\{x'_n\} \subseteq E(\mathcal{M})$ such that $\mu(x'_n) = \mu(x)$, $n = 1, 2, \ldots$ and such that $\|x'_n - x\|_{E(\mathcal{M})} \to 0$. Observe that the Markus inequality implies that, for all $s > 0$,

$$\left|\int_0^s (\mu_t(x + x'_n) - \mu_t(x + x_n))\, dt\right| \leq \int_0^s \mu_t(x_n - x'_n)dt \leq \|x_n - x'_n\|_{E(\mathbb{R}^+)} \|\chi_{[0,s)}\|_{E(\mathbb{R}^+)^\times}.$$

Consequently, it follows from Theorem 5.5 that $x'_n \to x$ for the measure topology. From this, it follows also that $x_n \to x$ for the measure topology, and this suffices to complete the proof.

We remark that if $\mathcal{M}$ is commutative, then the preceding Theorem 5.7 reduces to

[DGL], Proposition 1.1 in the special case that $E(\mathbb{R}^+) = L^1(\mathbb{R}^+)$. In fact, via Vitali's theorem, the same argument as in the proof of the implication (i) implies (ii) of Theorem 5.5 shows that, in this special case, the first condition in (ii) above is equivalent to the condition

$$\int_0^s \mu_t(x_n)dt \rightarrow \int_0^s \mu_t(x)dt, \qquad \forall s > 0.$$

As is shown in [CKS], condition (ii) of Theorem 5.7 is satisfied if the norm on $E(\mathbb{R}^+)$ is locally uniformly convex, if $x, x_n \in E(\mathcal{M})$, $n = 1, 2, \ldots$, if $\|x_n\|_{E(\mathcal{M})} \rightarrow \|x\|_{E(\mathcal{M})}$, and if $\|(x_n + x)\|_{E(\mathcal{M})} \rightarrow 2\|x\|_{E(\mathcal{M})}$. Condition (i) then implies that $E(\mathcal{M})$ is locally uniformly convex. On the other hand, condition (ii) of Theorem 5.7 is also satisfied if $E(\mathbb{R}^+)$ has order continuous norm, if $x, x_n \in E(\mathcal{M})$, $n = 1, 2, \ldots$, if $x_n \rightarrow x$ $\sigma(E(\mathcal{M}), E(\mathcal{M})^*)$ and if $\|\mu(x_n) - \mu(x)\|_{E(\mathbb{R}^+)} \rightarrow 0$. To see this, observe that this latter condition readily implies that $\int_0^s \mu_t(x)dt \rightarrow \int_0^s \mu_t(x)dt$ for all $s > 0$. This implies, for all $s > 0$ that

$$\|x_n\|_{L^1(\mathcal{M})+s\mathcal{M}} \rightarrow \|x\|_{L^1(\mathcal{M})+s\mathcal{M}}.$$

The condition that $x_n \rightarrow x$ $\sigma(E(\mathcal{M}), E(\mathcal{M})^*)$ implies that $x_n \rightarrow x$ pointwise on $L^1(\mathcal{M}) \cap (s\mathcal{M})$, which is a norming subset of the Banach dual $(L^1(\mathcal{M}) + s\mathcal{M})^*$. Consequently,

$$\|(x + x_n)\|_{L^1(\mathcal{M})+s\mathcal{M}} \rightarrow 2\|x\|_{L^1(\mathcal{M})+s\mathcal{M}},$$

and this is just the assertion that $\int_0^s \mu_t(x_n + x)dt \rightarrow 2\int_0^s \mu_t(x)dt$ for all $s > 0$. We obtain, therefore, the following theorem.

Theorem 5.8 *If $E(\mathbb{R}^+)$ has order continuous norm, and if $x, x_n \in E(\mathcal{M})$, $n = 1, 2, \ldots$, then the following statements are equivalent.*

(i) $\|x_n - x\|_{E(\mathcal{M})} \rightarrow 0$.

(ii) $x_n \rightarrow x$ $\sigma(E(\mathcal{M}), E(\mathcal{M})^*)$ *and* $\|\mu(x_n) - \mu(x)\|_{E(\mathbb{R}^+)} \rightarrow 0$.

The preceding theorem is essentially due to Chilin and Sukochev and is stated in [CS1] under the assumption that $\mathcal{M}$ has no atoms. However, in the stronger form given above, the theorem immediately specializes to trace ideals and this special case is due to Arazy [Ar]. We remark that the commutative version of this theorem has been proved by B. de Pagter [dP].

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