A SYMMETRIC FUNCTIONAL CALCULUS FOR NONCOMMUTING SYSTEMS OF SECTORIAL OPERATORS

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Abstract. Given a system $A = (A_1, \ldots, A_n)$ of linear operators whose real linear combinations have spectra contained in a fixed sector in $\mathbb{C}$ and satisfy resolvent bounds there, functions $f(A)$ of the system A of operators can be formed for monogenic functions f having decay at zero and infinity in a corresponding sector in $\mathbb{R}^{n+1}$. The paper discusses how the functional calculus $f \mapsto f(A)$ might be extended to a larger class of monogenic functions and its relationship with a functional calculus for analytic functions in a sector of $\mathbb{C}^n$.

1. Introduction

Given a finite system $A = (A_1, \ldots, A_n)$ of bounded linear operators acting on a Banach space X, it has recently been shown how functions $f(A)$ of the n-tuple A can be formed for a large class of functions f, just under the assumption that the spectrum $\sigma(\langle A, \xi\rangle)$ of the operator $\langle A, \xi\rangle := \sum_{j=1}^n A_j\xi_j$ is a subset of $\mathbb{R}$ for every $\xi \in \mathbb{R}^n$ [5]. The operators $A_1, \ldots, A_n$ do not necessarily commute with each other.

A distinguished subset $\gamma(A)$ of $\mathbb{R}^n$ with the property that the bounded linear operator $f(A)$ is defined for any real analytic function $f : U \to \mathbb{C}$ defined in a neighbourhood U of $\gamma(A)$ in $\mathbb{R}^n$ arises in the approach considered in [5]. For a polynomial p in n real variables, $p(A)$ is the operator formed by substituting symmetric products in the bounded linear operators $A_1, \ldots, A_n$ for the monomial expressions in p, that is, we have a *symmetric* functional calculus in the n operators $A_1, \ldots, A_n$. Another way of expressing this symmetry property is that for any $\xi \in \mathbb{R}^n$ and any polynomial $q : \mathbb{C} \to \mathbb{C}$ in one variable, the equality $p(A) = q(\langle A, \xi\rangle)$ holds for the polynomial $p : x \mapsto q(\langle x, \xi\rangle)$, $x \in \mathbb{R}^n$. Moreover, the mapping $f \mapsto f(A)$ is continuous for a certain topology

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defined on the space of functions real analytic in a neighbourhood of $\gamma(A)$ in $\mathbb{R}^n$ [5, Proposition 3.3].

These properties are analogous to the Riesz-Dunford functional calculus of a single bounded linear operator T acting on X, by which a function $f(T) : X \to X$ of T is defined by the Cauchy integral formula

$$f(T) = \frac{1}{2\pi i}\int_C (\lambda I - T)^{-1} f(\lambda)\, d\lambda, \tag{1}$$

for f analytic in a neighbourhood of the spectrum $\sigma(T)$ of T and for C a simple closed contour about $\sigma(T)$. It is by this analogy that the set $\gamma(A)$ mentioned above may be thought of as the "joint spectrum" of the system A, especially if there is no set smaller than $\gamma(A)$ possessing the desirable properties alluded to.

Now for a single operator T, the spectrum $\sigma(T)$ has a simple algebraic definition as the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ is not invertible in the space $\mathcal{L}(X)$ of bounded linear operators acting on X. In the case that A consists of a system of n *commuting*, possibly unbounded, linear operators with real spectra, A. McIntosh and A. Pryde [14, 15] gave a simple algebraic definition of the joint spectrum $\gamma(A)$ of A and used this to obtain operator bounds for solutions of operator equations. Work of A. McIntosh, A. Pryde and W. Ricker [16] established the equivalence of $\gamma(A)$ with other notions of *joint spectrum*.

In the noncommutative case, we cannot expect such a straightforward algebraic definition of the joint spectrum $\gamma(A)$, although such a definition was proposed in [8]. Another example of a symmetric functional calculus is the *Weyl calculus* $\mathcal{W}_A$ considered in [19] for n selfadjoint operators $A = (A_1, \ldots, A_n)$. In the case that the system A consists of bounded selfadjoint operators, it was shown in [4] that $\gamma(A)$ is precisely the support of the operator valued distribution $\mathcal{W}_A$, and E. Nelson characterised this set as the Gelfand spectrum of a certain subalgebra of the Banach algebra of *operants* [17]. Further work along these lines was conducted by E. Albrecht [1].

If we now pass to unbounded operators, then a similar analysis holds if we retain the spectral reality condition $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for $\xi \in \mathbb{R}^n$, provided that we suitably account for operator domains. However, much of the work [14, 15, 16] on functional calculi just mentioned was motivated by Alan McIntosh's study of the commuting n-tuple $D_\Sigma = (D_1, \ldots, D_n)$ of differentiation operators on a Lipschitz surface Σ in $\mathbb{R}^{n+1}$. In the case that Σ is just the flat surface $\mathbb{R}^n$, the operators $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$, commute with each other and are selfadjoint, otherwise, the unbounded operators D_j, $j = 1, \dots, n$, have spectra $\sigma(D_j)$ contained in a complex sector $S_\omega(\mathbb{C}) = \{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| \leq \omega\}$ with an angle ω depending on the variation of the directions normal to the surface Σ.

Although the existence and properties of the H^∞-functional calculus for the commuting n-tuple D_Σ are now well-understood, see for example [13], the purpose of the present paper is to initiate a study of the symmetric functional calculus for an n-tuple A of unbounded sectorial operators — we do not assume that the operators commute with each other. In particular, the spectral reality condition

$$\sigma(\langle A, \xi \rangle) \subset \mathbb{R}, \quad \text{for all } \xi \in \mathbb{R}^n \tag{2}$$

needs to be relaxed. An alternative approach to forming an H^∞-functional calculus for commuting operators using exponential bounds is given in [11].

Before proceeding with further discussion, we note the definition of the joint spectrum $\gamma(A)$ for a system A satisfying condition (2). The key idea behind [14, 15] in the commuting case and [4, 5, 8, 9] in the noncommuting case, is to produce a higher-dimensional analogue of the Riesz-Dunford formula (1). So what we need is a higher-dimensional analogue of the Cauchy integral formula in complex analysis and then, in the time-honoured fashion of operator theory, substitute an n-tuple of numbers by an n-tuple of unbounded linear operators. But this is easier said than done.

It turns out that Clifford analysis provides a higher dimensional analogue of the Cauchy integral formula especially well-suited to the noncommutative setting. Even for the commuting n-tuple D_Σ of operators mentioned above, it provides the connection between multiplier operators and singular convolution operators for functions defined on a Lipschitz surface. A brief résumé of Clifford analysis [2, 3] and the monogenic functional calculus treated in [5] follows.

Let $\mathbb{C}_{(n)}$ be the *Clifford algebra* generated over the field $\mathbb{C}$ by the standard basis vectors $e_0, e_1, \dots, e_n$ of $\mathbb{R}^{n+1}$ with conjugation $u \mapsto \overline{u}$. The *generalized Cauchy-Riemann operator* is given by $D = \sum_{j=0}^n e_j \frac{\partial}{\partial x_j}$.

Let $U \subset \mathbb{R}^{n+1}$ be an open set. A function $f : U \to \mathbb{C}_{(n)}$ is called *left monogenic* if $Df = 0$ in U and *right monogenic* if $fD = 0$ in U. The *Cauchy kernel* is given by

$$G_x(y) = \frac{1}{\sigma_n} \frac{\overline{x - y}}{|x - y|^{n+1}}, \quad x, y \in \mathbb{R}^{n+1}, x \neq y, \tag{3}$$

with $\sigma_n = 2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$ the volume of unit n-sphere in $\mathbb{R}^{n+1}$. So, given a left monogenic function $f : U \to \mathbb{C}_{(n)}$ defined in an open subset U of $\mathbb{R}^{n+1}$ and an open subset Ω of U such that the closure $\overline{\Omega}$ of Ω is contained in U, and the boundary $\partial\Omega$ of Ω is a smooth oriented n-manifold, then the Cauchy integral formula

$$f(y) = \int_{\partial\Omega} G_x(y)\boldsymbol{n}(x)f(x)\,d\mu(x), \quad y \in \Omega$$

is valid. Here $\boldsymbol{n}(x)$ is the outward unit normal at $x \in \partial\Omega$ and μ is the volume measure of the oriented manifold $\partial\Omega$. An element $x = (x_0, x_1, \ldots, x_n)$ of $\mathbb{R}^{n+1}$ will often be written as $x = x_0e_0 + \vec{x}$ with $\vec{x} = \sum_{j=1}^{n} x_je_j$.

By analogy with formula (1), our aim is to define

$$f(A) = \int_{\partial\Omega} G_x(A)\boldsymbol{n}(x)f(x)\,d\mu(x) \tag{4}$$

for the n-tuple $A = (A_1, \ldots, A_n)$ of bounded linear operators on X. A difficulty occurs in making sense of the Cauchy kernel $x \mapsto G_x(A)$, a function with values in the space $\mathcal{L}(X) \otimes \mathbb{C}_{(n)}$ that should be defined and two-sided monogenic for all x off a nonempty closed subset $\gamma(A)$ of $\mathbb{R}^n$ inside Ω. The set $\partial\Omega$ can be smoothly varied in the region where $x \mapsto G_x(A)$ is right-monogenic. Of course, one would also like $f(A)$ to be the 'correct' operator in the case that f is the unique monogenic extension to $\mathbb{R}^{n+1}$ of a polynomial in n variables.

In the Riesz-Dunford functional calculus for T, the set of singularities of the resolvent $\lambda \mapsto (\lambda I - T)^{-1}$ is precisely the spectrum $\sigma(T)$ of T, so the set $\gamma(A)$ may be interpreted as a higher-dimensional analogue of the spectrum of a single operator. It seems reasonable to call the set $\gamma(A)$ the *monogenic spectrum* of the n-tuple A by analogy with the case of a single operator.

The program was implemented by A. McIntosh and A. Pryde for commuting n-tuples of bounded operators with real spectrum in order to give estimates on the solution of systems of operator equations [14, 15]. In the case that n is odd, we have

$$\gamma(A) = \left\{ \lambda \in \mathbb{R}^n : \sum_{j=1}^{n} (\lambda_jI - A_j)^2 \text{ is invertible in } \mathcal{L}(X) \right\}^c$$

and

$$G_x(A) = \frac{1}{\sigma_n}(\overline{x - A}) \left(x_0^2 I + \sum_{j=1}^{n} (x_jI - A_j)^2 \right)^{-\frac{n+1}{2}},$$

for all $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus (\{0\} \times \gamma(A))$. It turns out that $\gamma(A)$ coincides with the Taylor spectrum for commuting systems of bounded linear operators [16].

If the n-tuple A of bounded linear operators satisfies exponential growth conditions, such as when $A_1, \ldots, A_n$ are selfadjoint, then Weyl's functional calculus $\mathcal{W}_A$ is associated with A and $G_x(A) = \mathcal{W}(G_x)$ is an obvious way to define the Cauchy kernel for all x outside the support of $\mathcal{W}_A$. It is shown in [4] that formula (4) holds. However, in this case, we actually have a symmetric functional calculus defined over $\gamma(A)$ richer than just all real analytic functions.

The Cauchy kernel $G_x(A)$ can also be written as a series expansion like the Neuman series for the resolvent of a single operator if $x \in \mathbb{R}^{n+1}$ lies outside a sufficiently large ball [8, 9], but the expansion does not allow us to identify $\gamma(A)$ as a subset of $\mathbb{R}^n$ in the case that the spectral reality condition (2) holds.

A third way to define the Cauchy kernel $G_x(A)$ for the monogenic functional calculus whenever the spectral reality condition (2) holds, is by the plane wave decomposition for the Cauchy kernel (3) given by F. Sommen [18]. This was investigated by A. M^cIntosh and J. Picton-Warlow soon after the papers [14, 15] appeared. The formula is

$$\begin{aligned} G_x(A) = & \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \operatorname{sgn}(x_0)^{n-1} \\ & \times \int_{S^{n-1}} (e_0 + is)\left(\langle \vec{x}, s\rangle I - \langle A, s\rangle - x_0 s I\right)^{-n} ds \end{aligned} \tag{5}$$

for all $x = x_0 e_0 + \vec{x}$ with x_0 a nonzero real number and $\vec{x} \in \mathbb{R}^n$. Here S^{n-1} is the unit $(n-1)$-sphere in $\mathbb{R}^n$, ds is surface measure and the inverse power $(\langle \vec{x}I - A, s\rangle - x_0 s)^{-n}$ is taken in the Clifford module $\mathcal{L}(X) \otimes C_{(n)}$. The spectral reality condition (2) ensures the invertibility of $(\langle \vec{x}I - A, s\rangle - x_0 s)$ for all $x_0 \neq 0$ and $s \in S^{n-1}$ by the spectral mapping theorem.

Even if A satisfies exponential growth conditions, with the left hand side given by formula (5), the equality $G_x(A) = \mathcal{W}_A(G_x)$ can still be used to good effect. In [6], it was used to geometrically characterise the support of fundamental solution of the symmetric hyperbolic system associated with a pair A of hermitian matrices in the case $n = 2$. Greater understanding of Clifford residue theory would enable a similar treatment in higher dimensions.

In Section 2, it is shown how formula (5) for the Cauchy kernel associated with the system A of sectorial operators still works if the spectral reality condition (2) is replaced by a sectoriality condition with the appropriate resolvent bounds. The system D_Σ of commuting sectorial operators described above is of this type. By this means functions $f(A)$ of the operators A can be formed, provided that f is, say, left monogenic in a sector in $\mathbb{R}^{n+1}$ and satisfies suitable decay estimates at 0 and ∞, in a fashion similar to the case of a single operator of type ω [12]. Because $G_x(A)$ is only defined for x outside a sector in $\mathbb{R}^{n+1}$, the monogenic spectrum $\gamma(A)$ is now contained in that sector in $\mathbb{R}^{n+1}$. Recall that under condition (2), $\gamma(A)$ is a subset of $\mathbb{R}^n$.

A function $f(D_\Sigma)$ of the system D_Σ has a natural interpretation as a multiplier operator acting on L^p-spaces of functions defined on the Lipschitz surface Σ, as well as a singular convolution operator, so the multiplier f should be a bounded analytic function defined on a sector in $\mathbb{C}^n$ [13], rather than, say, a bounded monogenic function defined in a sector in $\mathbb{R}^{n+1}$. The monogenic functional calculus for a system A of sectorial operators appears to be moving us inexorably in the wrong direction.

The problem arises of establishing a bijection between monogenic functions defined on a sector in $\mathbb{R}^{n+1}$ and analytic functions defined on a sector in $\mathbb{C}^n$, together with the appropriate norms — this is a question of function theory rather than operator theory. The association is via the Cauchy-Kowaleski extension to a sector in $\mathbb{R}^{n+1}$ of the restriction of the analytic function to $\mathbb{R}^n \setminus \{0\}$.

The purpose of this paper is to make some observations about the relationship between monogenic functions defined in a sector in $\mathbb{R}^{n+1}$ and analytic functions defined in the corresponding sector in $\mathbb{C}^n$, with applications to functional calculi of systems of operators firmly in mind. In Section 3, the spectral properties of multiplication operators in $\mathbb{C}_{(n)}$ are examined along the lines of Lecture 1 of [13]. In Section 4, this enables us to uniquely associate a bounded analytic function defined in a sector in $\mathbb{C}^n$ with a suitably decaying monogenic function defined in the corresponding sector in $\mathbb{R}^{n+1}$ via the Cauchy integral formula.

2. The plane wave decomposition

Let $A = (A_1, \dots, A_n)$ be an n-tuple of densely defined linear operators $A_j : \mathcal{D}(A_j) \to X$ acting in X such that $\cap_{j=1}^n \mathcal{D}(A_j)$ is dense in X. The space $\mathcal{L}_{(n)}(X_{(n)})$ of left module homomorphisms of $X_{(n)} = X \otimes \mathbb{C}_{(n)}$ is identified with $\mathcal{L}(X) \otimes \mathbb{C}_{(n)}$ in the natural way and becomes a right Banach module under the uniform operator topology.

If we take formula (5) as the definition of $G_x(A)$, then the convergence of the integral

$$\int_{S^{n-1}} (e_0 + is)\left(\langle \vec{x}I - A, s\rangle - x_0 sI\right)^{-n} ds$$

for particular values of $x = x_0e_0 + \vec{x} \in \mathbb{R}^{n+1}$ is at issue. Now

$$\left(\langle \vec{x}I - A, s\rangle - x_0 sI\right)^{-1} = \left(\langle \vec{x}I - A, s\rangle + x_0 sI\right)\left(\langle \vec{x}I - A, s\rangle^2 + x_0^2 I\right)^{-1}$$

if $0 \notin \sigma\left(\langle \vec{x}I - A, s\rangle^2 + x_0^2\right)$. Thus, we need to ensure the appropriate uniform operator bounds for

$$\left(\langle \vec{x}I - A, s\rangle^2 + x_0^2 I\right)^{-1}, \quad s \in S^{n-1}$$

as $x = x_0e_0 + x$ ranges over a subset of $\mathbb{R}^{n+1}$. In the case that $\sigma(\langle A, s\rangle) \subset \mathbb{R}$ and $(\lambda I - \langle A, s\rangle)^{-1}$ is suitably bounded for all $s \in S^{n-1}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $G_{x_0e_0+\vec{x}}(A)$ is defined for all $x_0 \neq 0$. First, for each $0 < \nu < \pi/2$, set

$$\begin{aligned} S_{\nu+}(\mathbb{C}) &= \{z \in \mathbb{C} : |\arg z| \leq \nu\} \cup \{0\}, \\ S_\nu(\mathbb{C}) &= S_{\nu+}(\mathbb{C}) \cup \underline{i}g(-S_{\nu+}(\mathbb{C})), \\ S^\circ_{\nu+}(\mathbb{C}) &= \{z \in \mathbb{C} : |\arg z| < \nu\}, \\ S^\circ_\nu(\mathbb{C}) &= S^\circ_{\nu+}(\mathbb{C}) \cup \big(-S^\circ_{\nu+}(\mathbb{C})\big). \end{aligned}$$

The $(n-1)$-sphere in $\mathbb{R}^n$ is denoted by S^{n-1}.

Definition 2.1. Let $A = (A_1, \dots, A_n)$ be an n-tuple of densely defined linear operators $A_j : \mathcal{D}(A_j) \to X$ acting in X such that $\bigcap_{j=1}^n \mathcal{D}(A_j)$ is dense in X and let $0 \leq \omega < \frac{\pi}{2}$. Then A is said to be *uniformly of type* ω if for every $s \in S^{n-1}$, the operator $\langle A, s\rangle$ is closable with closure $\overline{\langle A, s\rangle}$, the inclusion $\sigma\big(\overline{\langle A, s\rangle}\big) \subset S_\omega(\mathbb{C})$ holds, and for each $\nu > \omega$, there exists $C_\nu > 0$ such that

$$\text{(6)} \qquad \|\big(zI - \overline{\langle A, s\rangle}\big)^{-1}\| \leq C_\nu |z|^{-1}, \quad z \notin S^\circ_\nu(\mathbb{C}),\ s \in S^{n-1}.$$

It follows that $s \mapsto \overline{\langle A, s\rangle}$ is continuous on S^{n-1} in the sense of strong resolvent convergence [7, Theorem VIII.1.5]. Because $\big(zI - \langle A, s\rangle\big)^{-1}$ is densely defined and uniformly bounded in X, the closure symbol will be omitted.

Now suppose that equation (6) is satisfied and let $z = \langle \vec{x}, s\rangle + ix_0$. Then $z \notin S^\circ_\nu(\mathbb{C})$ means that $|\arg z| \geq \nu$ for $-\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}$ or $\pi - \arg z \geq \nu$ for $\frac{\pi}{2} \leq \arg z \leq \pi$ or $\pi + \arg z \geq \nu$ for $-\pi \leq \arg z \leq -\frac{\pi}{2}$. Hence, we have $|x_0| \geq \tan\nu |\langle \vec{x}, s\rangle|$.

First, let

$$\begin{aligned} N_\nu &= \{x \in \mathbb{R}^{n+1} : x = x_0 e_0 + \vec{x},\ |x_0| \geq \tan\nu |\vec{x}|\ \}, \\ S_\nu(\mathbb{R}^{n+1}) &= \{x \in \mathbb{R}^{n+1} : x = x_0 e_0 + \vec{x},\ |x_0| \leq \tan\nu |\vec{x}|\ \}, \\ S^\circ_\nu(\mathbb{R}^{n+1}) &= \{x \in \mathbb{R}^{n+1} : x = x_0 e_0 + \vec{x},\ |x_0| < \tan\nu |\vec{x}|\ \}. \end{aligned}$$

Note that if $x_0 e_0 + x \in N_\nu$, then $z = \langle \vec{x}, s\rangle + i x_0 \notin S^\circ_\nu(\mathbb{C})$ for every $s \in S^{n-1}$, because either $|x_0| \geq \tan\nu |\vec{x}| \geq \tan\nu |\langle \vec{x}, s\rangle|$.

Lemma 2.2. *Let $\omega < \nu < \pi/2$. Suppose that the n-tuple A of linear operators is uniformly of type ω. Then for all $x_0 e_0 + \vec{x} \in N_\nu$, the integral*

$$\int_{S^{n-1}} \left\| (\langle \vec{x} I - A, s\rangle - x_0 s I)^{-n} \right\|_{\mathcal{L}_{(n)}(X_{(n)})} ds$$

converges and satisfies the bound

$$\int_{S^{n-1}} \left\| (\langle \vec{x} I - A, s\rangle - x_0 s I)^{-n} \right\|_{\mathcal{L}_{(n)}(X_{(n)})} ds \leq \frac{C'_\nu}{|x_0|^n}.$$

Proof. For every $x_0 e_0 + \vec{x} \in N_\nu$, we have $z = \langle x, s\rangle \pm i x_0 \notin S_\nu(\mathbb{C})$ so that the operator $(\langle \vec{x}, s\rangle \pm i x_0) I - \langle A, s\rangle$ is invertible and the bound

$$\left\| ((\langle \vec{x}, s\rangle \pm i x_0) I - \langle A, s\rangle)^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{C_\nu}{\sqrt{\langle \vec{x}, s\rangle^2 + x_0^2}}$$

holds. Now

$$\begin{aligned} &(\langle \vec{x} I - A, s\rangle - x_0 s I)^{-1} \\ &\qquad = (\langle \vec{x} I - A, s\rangle + x_0 s I) \left(\langle \vec{x} I - A, s\rangle^2 + x_0^2 I\right)^{-1} \end{aligned}$$

where

$$\begin{aligned} &\left(\langle \vec{x} I - A, s\rangle^2 + x_0^2 I\right)^{-1} \\ &\qquad = ((\langle \vec{x}, s\rangle + i x_0) I - \langle A, s\rangle)^{-1} ((\langle \vec{x}, s\rangle - i x_0) I - \langle A, s\rangle)^{-1}. \end{aligned}$$

Writing $(\langle \vec{x} I - A, s\rangle + x_0 s I) = ((\langle \vec{x}, s\rangle + i x_0) I - \langle A, s\rangle) - i x_0 I + x_0 s I$, we obtain

$$\begin{aligned} (\langle \vec{x} I - A, s\rangle - x_0 s I)^{-1} &= ((\langle \vec{x}, s\rangle - i x_0) I - \langle A, s\rangle)^{-1} \\ &\qquad - i x_0 (e_0 + i s) \left(\langle \vec{x} I - A, s\rangle^2 + x_0^2 I\right)^{-1}, \end{aligned}$$

so that by the estimate (6) we have

$$\begin{aligned} \left\| (\langle \vec{x} I - A, s\rangle - x_0 s I)^{-1} \right\|_{\mathcal{L}_{(n)}(X_{(n)})} &\leq \frac{C_\nu}{\sqrt{\langle \vec{x}, s\rangle^2 + x_0^2}} + \frac{2|x_0| C_\nu^2}{\langle \vec{x}, s\rangle^2 + x_0^2} \\ &\leq \frac{C_\nu + 2C_\nu^2}{|x_0|}, \end{aligned}$$

from which the stated bound follows. □

Thus, if A is uniformly of type ω, then $x_0e_0 + \vec{x} \mapsto G_{x_0e_0+\vec{x}}(A)$ is defined by equation (5) for all $x_0e_0 + \vec{x} \in N_\nu$ with $\omega < \nu < \pi/2$. Standard arguments ensure that $x_0e_0 + \vec{x} \mapsto G_{x_0e_0+\vec{x}}(A)$ is both left and right monogenic as an element of $\mathcal{L}(X) \otimes \mathbb{C}_{(n)}$. If we denote by $\gamma(A) \subset \mathbb{R}^{n+1}$ the set of all singularities of the function $x_0e_0 + \vec{x} \mapsto G_{x_0e_0+\vec{x}}(A)$, then

$$\gamma(A) \subseteq S_\omega(\mathbb{R}^{n+1}).$$

Suppose that $\omega < \nu < \pi/2$, $0 < s < n$ and f is a left monogenic function defined on $S_\nu^\circ(\mathbb{R}^{n+1})$ such that for every $0 < \theta < \nu$ there exists $C_\theta > 0$ such that

$$(7) \qquad |f(x)| \le C_\theta \frac{|x|^s}{(1+|x|^{2s})}, \quad x \in S_\theta^\circ(\mathbb{R}^{n+1}).$$

According to Lemma 2.2, for every $\omega < \nu' < \theta < \nu$, we have

$$\|G_x(A)\|.|f(x)| \le C_{\theta,\nu'} \frac{|x|^s}{|x_0|^n(1+|x|^{2s})}, \quad x = x_0e_0 + \vec{x}$$

for all $x \in S_\theta^\circ(\mathbb{R}^{n+1}) \cap N_{\nu'}$.

Now if $\omega < \theta < \nu$ and

$$(8) \quad H_\theta = \{x \in \mathbb{R}^{n+1} : x = x_0e_0 + \vec{x},\ |x_0|/|x| = \tan\theta\} \subset S_\nu^\circ(\mathbb{R}^{n+1}).$$

it follows that $\|G_x(A)\|.|f(x)| = O(1/|x|^{n-s})$ as $x \to 0$ in H_θ. Hence, $x \mapsto G_x(A)\boldsymbol{n}(x)f(x)$ is locally integrable at zero with respect to n-dimensional surface measure on H_θ. Similarly, $\|G_x(A)\|.|f(x)| = O(1/|x|^{n+s})$ as $|x| \to \infty$ in H_θ, so $x \mapsto G_x(A)\boldsymbol{n}(x)f(x)$ is integrable with respect to n-dimensional surface measure on H_θ.

Therefore, we define

$$(9) \qquad f(A) = \int_{H_\theta} G_x(A)\boldsymbol{n}(x)f(x)\,d\mu(x).$$

If $\psi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{C}$ has a two-sided monogenic extension $\tilde{\psi}$ to $S_\nu^\circ(\mathbb{R}^{n+1})$ that satisfies the bound (7) for all $0 < \theta < \nu$, then $\tilde{\psi}(A)$ is written just as $\psi(A)$.

Formula (9) does just what we would expect in the noncommuting situation. For example, let p be a polynomial of degree n with $p(0) = 0$ and $b_\lambda(z) = p(z)(\lambda - z)^{-n-1}$ for some $\lambda \notin S_\nu^\circ(\mathbb{C})$. Let $\xi \in \mathbb{R}^n$ and set $\phi_{\lambda,\xi}(x) = b_\lambda(\langle x, \xi\rangle)$ for all $x \in \mathbb{R}^n$. Denote the two-sided monogenic extension of $\phi_{\lambda,\xi}$ to $S_\nu^\circ(\mathbb{R}^{n+1})$ by $\tilde{\phi}_{\lambda,\xi}$. Then $\tilde{\phi}_{\lambda,\xi}$ has decay at zero and infinity and we have $\phi_{\lambda,\xi}(A) = \tilde{\phi}_{\lambda,\xi}(A) = p(\langle A, \xi\rangle)(\lambda I - \langle A, \xi\rangle)^{-n-1}$ is a bounded linear operator.

In order to form functions $f(A)$ of the system A of operators for a class of monogenic functions f larger than those which satisfy a bound like (7), a greater understanding of function theory in the sector $S_\omega(\mathbb{R}^{n+1})$ is needed. To this end, the simple system $A = \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ of multiplication operators in the algebra $\mathbb{C}_{(n)}$ is considered in the next section.

3. Joint spectral theory in the algebra $\mathbb{C}_{(n)}$

Let $\zeta = (\zeta_1, \dots, \zeta_n)$ be a vector belonging to $\mathbb{C}^n$. The complex spectrum $\sigma(i\zeta)$ of the element $i\zeta = i(\zeta_1 e_1 + \cdots + \zeta_n e_n)$ of the algebra $\mathbb{C}_{(n)}$ is

$$\sigma(i\zeta) = \{\lambda \in \mathbb{C} : (\lambda e_0 - i\zeta) \text{ does not have an inverse in } \mathbb{C}_{(n)}\}.$$

Following [13, Section 5.2], we check that

$$(\lambda e_0 + i\zeta)(\lambda e_0 - i\zeta) = \lambda^2 e_0 - i^2\zeta^2 = (\lambda^2 - |\zeta|_{\mathbb{C}}^2)e_0,$$

where $|\zeta|_{\mathbb{C}}^2 = \sum_{j=1}^n \zeta_j^2$. So, for all $\lambda \in \mathbb{C}$ for which, $\lambda \neq \pm|\zeta|_{\mathbb{C}}$, the element $(\lambda e_0 - i\zeta)$ of the algebra $\mathbb{C}_{(n)}$ is invertible and

$$(\lambda e_0 - i\zeta)^{-1} = \frac{\lambda e_0 + i\zeta}{\lambda^2 - |\zeta|_{\mathbb{C}}^2}.$$

If $|\zeta|_{\mathbb{C}}^2 \neq 0$, the two square roots of $|\zeta|_{\mathbb{C}}^2$ are written as $\pm|\zeta|_{\mathbb{C}}$ and $|\zeta|_{\mathbb{C}} = 0$ for $|\zeta|_{\mathbb{C}}^2 = 0$. Hence $\sigma(i\zeta) = \{\pm|\zeta|_{\mathbb{C}}\}$. When $|\zeta|_{\mathbb{C}}^2 \neq 0$, the spectral projections

$$\chi_\pm(\zeta) = \frac{1}{2}\left(e_0 + \frac{i\zeta}{\pm|\zeta|_{\mathbb{C}}}\right)$$

are associated with each element $\pm|\zeta|_{\mathbb{C}}$ of the spectrum $\sigma(i\zeta)$ and $i\zeta$ has the spectral representation $i\zeta = |\zeta|_{\mathbb{C}}\chi_+(\zeta) + (-|\zeta|_{\mathbb{C}})\chi_-(\zeta)$. Henceforth, we use the symbol $|\zeta|_{\mathbb{C}}$ to denote the positive square root of $|\zeta|_{\mathbb{C}}^2$ in the case that $|\zeta|_{\mathbb{C}}^2 \notin (-\in fty, 0]$.

On the other hand, according to the point of view mentioned in the Introduction, the *monogenic spectrum* $\gamma(\zeta)$ of $\zeta \in \mathbb{C}^n$ should be the set of singularities of the Cauchy kernel $x \mapsto G_x(\zeta)$ in the algebra $\mathbb{C}_{(n)}$. Although $G_x(\zeta)$ is defined by formula (3) only for $\zeta \in \mathbb{R}^n$ and $x \neq \zeta$, a natural choice for the Cauchy kernel for $\zeta \in \mathbb{C}^n$ is to take the maximal analytic extension $\zeta \mapsto G_x(\zeta)$ of formula (3) for $\zeta \in \mathbb{C}^n$, that is,

(10)
$$G_x(\zeta) = \frac{1}{\sigma_n}\frac{\overline{x} + \zeta}{|x - \zeta|_{\mathbb{C}}^{n+1}}, \quad x \in \mathbb{R}^{n+1}, \quad \begin{cases} |x-\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0], & n \text{ even} \\ |x-\zeta|_{\mathbb{C}}^2 \neq 0, & n \text{ odd} \end{cases}$$

Here $|x - \zeta|_{\mathbb{C}}^2 = x_0^2 + \sum_{j=1}^n (x_j - \zeta_j)^2$ and $|x - \zeta|_{\mathbb{C}}$ is the positive square root of $|x - \zeta|_{\mathbb{C}}^2$, coinciding with the analytic extension of the modulus function $\xi \mapsto |x - \xi|$, $\xi \in \mathbb{R}^n \setminus \{x\}$.

The analogous reasoning for multiplication by $x \in \mathbb{R}^{n+1}$ in the algebra $\mathbb{C}_{(n)}$ just gives us the Cauchy kernel (3), so that $\gamma(x) = \{x\}$, as expected.

Remark 3.1. If $\zeta = (\zeta_1, \dots, \zeta_n)$ satisfies the conditions of Definition 2.1, then there exists $\theta \in [-\omega, \omega]$ and $x \in \mathbb{R}^n$ such that $\zeta = e^{i\theta}x$. To see this, write $\zeta = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}^n$. If $|\langle \beta, \xi\rangle| \le |\langle \alpha, \xi\rangle| \tan\omega$ for all $\xi \in \mathbb{R}^n$, then $\alpha^\perp \subset \beta^\perp$, so that $\beta \in \mathrm{span}\{\alpha\}$.

In this case, the plane wave formula (5) with $A = \zeta$ and equation (10) agree by analytic continuation, at least for $x \in N_\nu$ with $\nu > |\theta|$.

Given $\zeta \in \mathbb{C}^n$, if singularities of (10) occur at $x \in \mathbb{R}^{n+1}$, then $|x - \zeta|^2_{\mathbb{C}} \in (-\infty, 0]$, otherwise we can simply take the positive square root of $|x - \zeta|^2_{\mathbb{C}}$ in formula (10) to obtain a monogenic function of x. To determine this set, write $\zeta = \xi + i\eta$ for $\xi, \eta \in \mathbb{R}^n$ and $x = x_0e_0 + \vec{x}$ for $x_0 \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$. Then

$$\begin{aligned} |x-\zeta|^2_{\mathbb{C}} &= x_0^2 + \sum_{j=1}^n (x_j - \zeta_j)^2 \\ &= x_0^2 + \sum_{j=1}^n (x_j - \xi_j - i\eta_j)^2 \\ &= x_0^2 + |\vec{x} - \xi|^2 - |\eta|^2 - 2i\langle \vec{x} - \xi, \eta\rangle. \qquad (11) \end{aligned}$$

Thus, $|x-\zeta|^2_{\mathbb{C}}$ belongs to $(-\infty, 0]$ if and only if x lies in the intersection of the hyperplane $\langle \vec{x} - \xi, \eta\rangle = 0$ passing through ξ and with normal η, and the ball $x_0^2 + |\vec{x} - \xi|^2 \le |\eta|^2$ centred at ξ with radius $|\eta|$. If n is even, then

$$\gamma(\zeta) = \{x = x_0e_0 + \vec{x} \in \mathbb{R}^{n+1} : \langle \vec{x} - \xi, \eta\rangle = 0,\ x_0^2 + |\vec{x} - \xi|^2 \le |\eta|^2\}. \qquad (12)$$

and if n is odd, then

$$\gamma(\zeta) = \{x = x_0e_0 + \vec{x} \in \mathbb{R}^{n+1} : \langle \vec{x} - \xi, \eta\rangle = 0,\ x_0^2 + |\vec{x} - \xi|^2 = |\eta|^2\}. \qquad (13)$$

In particular, if $\Im(\zeta) = 0$, then $\gamma(\zeta) = \{\zeta\} \subset \mathbb{R}^n$.

Remark 3.2. The distinction between n odd and even is reminiscent of the support of the fundamental solution of the wave equation in even and odd dimensions.

Off $\gamma(\zeta)$, the function $x \mapsto G_x(\zeta)$ is clearly two-sided monogenic, so the Cauchy integral formula gives

$$\tilde{f}(\zeta) = \int_{\partial\Omega} G_x(\zeta)\boldsymbol{n}(x) f(x)\, d\mu(x) \qquad (14)$$

for a bounded open neighbourhood Ω of $\gamma(\zeta)$ with smooth oriented boundary $\partial\Omega$, outward unit normal $\boldsymbol{n}(x)$ at $x \in \partial\Omega$ and surface measure μ. The function f is assumed to be left monogenic in a neighbourhood of $\overline{\Omega}$, but $\zeta \mapsto \tilde{f}(\zeta)$ is an analytic $\mathbb{C}_{(n)}$-valued function as the closed set $\gamma(\zeta)$ varies inside Ω. Moreover, $\tilde{f}$ equals f on $\Omega \cap \mathbb{R}^n$ by the usual Cauchy integral formula of Clifford analysis, so if f is, say, the monogenic extension of a polynomial $p : \mathbb{C}^n \to \mathbb{C}$ restricted to $\mathbb{R}^n$, then $\tilde{f}(\zeta) = p(\zeta)$, as expected. In this way, for each left monogenic function f defined in a neighbourhood of $\gamma(\zeta)$, in a natural way we associate an analytic function $\tilde{f}$ defined in a neighbourhood of ζ.

It is clear that if $\zeta = \xi + i\eta$ lies in a sector in $\mathbb{C}^n$, say, $|\eta| \le |\xi| \tan\nu$, then the monogenic spectrum $\gamma(\zeta)$ lies in a corresponding sector in $\mathbb{R}^{n+1}$. More precisely, we have

Proposition 3.3. *Let* $\zeta \in \mathbb{C}^n \setminus \{0\}$ *and* $0 < \omega < \pi/2$. *Then* $\gamma(\zeta) \subset S_\omega(\mathbb{R}^{n+1})$ *if and only if*

$$|\zeta|_{\mathbb{C}}^2 \ne (-\infty, 0] \quad \text{and} \quad |\Im(\zeta)| \le \Re(|\zeta|_{\mathbb{C}}) \tan\omega. \tag{15}$$

Proof. The statement is trivially valid if $\zeta \in \mathbb{R}^n \setminus \{0\}$, so suppose that $\Im(\zeta) \ne 0$. Then the monogenic spectrum $\gamma(\zeta)$ of ζ given by (12) is a subset of $S_\omega(\mathbb{R}^{n+1})$ if and only if there exists $0 < \theta \le \omega$ such that the cone

$$H_\theta^+ = \{x_0 e_0 + \vec{x} \in \mathbb{R}^{n+1} : x_0 > 0,\ x_0 = |x| \tan\theta\ \}$$

is tangential to the boundary of $\gamma(\zeta)$. A calculation shows that H_θ^+ is tangential to the boundary of $\gamma(\zeta)$ for all $\zeta = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}^n$, satisfying

$$|\eta|^2 = \sin^2\theta(|\xi|^2 + \tan^2\theta |P_\eta \xi|^2). \tag{16}$$

Here $P_\eta : u \mapsto \langle u, \eta\rangle \eta/|\eta|^2$, $u \in \mathbb{R}^n$, is the projection operator onto $\mathrm{span}\{\eta\}$.

To relate condition (16) to the inequality (15), suppose that $m = m_0 e_0 + \vec{m}$ is the unit vector normal to H_θ such that $\vec{m}$ lies in the direction of η. Hence, $m_0 = \cot\theta |\vec{m}|$, $\tan\theta = |\vec{m}|/m_0$ and $P_\eta \xi = \langle \xi, \vec{m}\rangle \vec{m}/|\vec{m}|^2$. Then equation (15) becomes

$$\eta = \sin\theta (m_0^2 |\xi|^2 + \langle \xi, \vec{m}\rangle^2)^{1/2} \frac{\vec{m}}{|\vec{m}| m_0}.$$

But $|m_0 e_0 + \vec{m}| = 1$, so $(\cot^2\theta + 1)|\vec{m}|^2 = 1$. We have $|\vec{m}| = \sin\theta$ and

$$\eta = (m_0^2 |\xi|^2 + \langle \xi, \vec{m}\rangle^2)^{1/2} \frac{\vec{m}}{m_0}. \tag{17}$$

As mentioned in [13, p67], the set of all $\zeta = \xi + i\eta$ with $\eta \neq 0$ satisfying (17) is equal to the set of all $\zeta = \xi + i\eta$ with $\eta \neq 0$ satisfying

$$|\zeta|^2_{\mathbb{C}} \neq (-\infty, 0] \quad \text{and} \quad \eta = \Re(|\zeta|_{\mathbb{C}})\frac{\vec{m}}{m_0}.$$

Because $|\vec{m}|/m_0 = \tan\theta \leq \tan\omega$, we obtain the desired equivalence by letting $\vec{m}$ vary over all directions in $\mathbb{R}^n$. □

For each $0 < \omega < \pi/2$, let $S_\omega(\mathbb{C}^n)$ denote the set of all $\zeta \in \mathbb{C}^n$ satisfying condition (8) and let $S^\circ_\omega(\mathbb{C}^n)$ be its interior.

Corollary 3.4. *Let $f : S^\circ_\omega(\mathbb{R}^{n+1}) \to \mathbb{C}_{(n)}$ be a left monogenic function such that the restriction $\tilde{f}$ of f to $\mathbb{R}^n \setminus \{0\}$ takes values in $\mathbb{C}$. Then $\tilde{f}$ is the restriction to $\mathbb{R}^n \setminus \{0\}$ of an analytic function defined on $S^\circ_\omega(\mathbb{C}^n)$*

The sectors $S_\omega(\mathbb{C}^n) \subset \mathbb{C}^n$ and $S_\omega(\mathbb{R}^{n+1}) \subset \mathbb{R}^{n+1}$ are dual to each other in the sense that the mapping

$$(\omega, \zeta) \mapsto G_\omega(\zeta), \quad \omega \in \mathbb{R}^{n+1} \setminus S_\omega(\mathbb{R}^{n+1}),\ \zeta \in S^\circ_\omega(\mathbb{C}^n)$$

is two-sided monogenic in ω and analytic in ζ.

The sector $S_\omega(\mathbb{C}^n)$ arose in [10] as the set of $\zeta \in \mathbb{C}^n$ for which the exponential functions

$$e_+(x, \zeta) = e^{i\langle \vec{x}, \zeta\rangle} e^{-x_0|\zeta|_{\mathbb{C}}} \chi_+(\zeta), \quad x = x_0 e_0 + \vec{x},$$

have decay at infinity for all $x \in \mathbb{R}^{n+1}$ with $\langle x, m\rangle > 0$ and all unit vectors $m = m_0 e_0 + \vec{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 \geq \cot\omega|\vec{m}|$.

4. Joint spectral theory of sectorial multiplication operators

By means of the higher-dimensional analogue (4) of the Riesz-Dunford functional calculus, we can form functions $f(A)$ of a noncommuting system A of operators uniformly of type ω for left monogenic functions f defined on a sector $S_\nu(\mathbb{R}^{n+1})$, $\omega < \nu < \pi/2$, provided that f has decay at zero and infinity.

The observations of the preceding section mean that we can do something similar for the commutative system of multiplication operators in the sector $S_\omega(\mathbb{C}^n)$, although these are not uniformly of type ω. The problem mentioned in the Introduction of connecting monogenic functions defined on a sector in $\mathbb{R}^{n+1}$ with analytic functions defined on a sector in $\mathbb{C}^n$ can be reformulated simply in terms of studying the monogenic functional calculus for multiplication operators.

More precisely, let $0 < \omega < \pi/2$ and set $X = L^2(S_\omega(\mathbb{C}^n), \mathbb{C}_{(n)})$. Integration is with respect to Lebesgue measure on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. Let

M_j be the operator of multiplication by the j'th coordinate function defined on $S_\omega(\mathbb{C}^n)$, that is, the domain $\mathcal{D}(M_j)$ of M_j is the set of all functions $\psi \in X$ such that the function $M_j\psi$ defined by

$$(M_j\psi)(\zeta) = \zeta_j\psi(\zeta), \quad \zeta \in S_\omega(\mathbb{C}^n), \tag{18}$$

belongs to X, and the unbounded operator M_j is given by formula (18) for each $\psi \in \mathcal{D}(M_j)$. Set $M = (M_1, \dots, M_n)$. Then M is a commuting n-tuple of normal operators. Unlike the n-tuple D_Σ of operators mentioned in the Introduction, the existence of a joint functional calculus for M is not an issue.

Indeed, the joint spectral measure $P : \mathcal{B}(S_\omega(\mathbb{C}^n)) \to \mathcal{L}(X)$ given by

$$P(B)\psi = \chi_B.\psi, \quad \psi \in X,\ B \in \mathcal{B}(S_\omega(\mathbb{C}^n)),$$

has support $S_\omega(\mathbb{C}^n)$. For any bounded Borel measurable function $f : S_\omega(\mathbb{C}^n) \to \mathbb{C}$, the bounded linear operator

$$f(M) = \int_{S_\omega(\mathbb{C}^n)} f\, dP$$

is given by the functional calculus for commuting normal operators, and explicitly, by

$$f(M) : \psi \mapsto f.\psi, \quad \psi \in X.$$

Thus we have a functional calculus for M for a class of functions far richer than uniformly bounded analytic functions f defined in a sector $S^\circ_\nu(\mathbb{C}^n)$ with $\omega < \nu < \pi/2$.

Nevertheless, it is not so obvious that bounded linear operators $f(M)$ can also be formed naturally for functions f that are monogenic in a sector $S^\circ_\nu(\mathbb{R}^{n+1})$ with $\omega < \nu < \pi/2$.

The Cauchy kernel $G_x(M)$ is defined for all $x \in N_\nu$ and all ν such that $\omega < \nu < \pi/2$ simply by setting

$$(G_x(M)\psi)(\zeta) = \psi(\zeta)G_x(\zeta), \quad \zeta \in S_\omega(\mathbb{C}^n), \tag{19}$$

for all $\psi \in X$ and $x \in N_\nu$, so that $G_x(M)$ is a left module homomorphism of X. The proof of the next lemma is straightforward and is omitted.

Lemma 4.1. *Let $\omega < \nu < \pi/2$. Then $G_x(M)$ is a bounded linear operator on X for all $x \in N_\nu \setminus \{0\}$. Furthermore, the operator valued function $x \mapsto G_x(M)$, $x \in N_\nu \setminus \{0\}$, is continuous for the strong operator topology, right monogenic in the interior of N_ν and $\|G_x(M)\| = O(|x|^{-n})$ as $|x| \to \infty$ and $|x| \to 0$ in N_ν*

Let $\omega < \nu < \pi/2$. Suppose that f is a left monogenic function defined on $S^\circ_\nu(\mathbb{R}^{n+1})$ such that for every $0 < \theta < \nu$, the bound (7) holds.

Now if $\omega < \theta < \nu$ a nd H_θ is the two-sheeted cone (8) in $\mathbb{R}^{n+1}$, then the bounded linear operator $f(M) : X \to X$ is defined by the formula

$$f(M) = \int_{H_\theta} G_x(M)\boldsymbol{n}(x)f(x)\,d\mu(x) \tag{20}$$

by the same argument by which (9) is defined. Then $f(M)$ is a left module homomorphism of X. The operator $f(M)$ is identified in the next statement.

Proposition 4.2. *Suppose that f is a left monogenic function defined on $S^\circ_\nu(\mathbb{R}^{n+1})$ satisfying the bound (7) for every $0 < \theta < \nu$ and $f(M)$ is defined by formula (20). Let $\tilde{f}$ be the $C_{(n)}$-valued analytic function defined from f by formula (14). Then $\tilde{f}$ is uniformly bounded on the sector $S_\omega(\mathbb{C}^n)$,*

$$(f(M)\psi)(\zeta) = \psi(\zeta)\tilde{f}(\zeta), \quad \zeta \in S_\omega(\mathbb{C}^n), \tag{21}$$

for all $\psi \in X$ and $\|f(M)\| = \|\tilde{f}\|_{H^\infty(S_\omega(\mathbb{C}^n))}$.

The problem remains of obtaining better bounds for the operator norm of $f(M)$ in terms of bounds of the left monogenic function defined on $S^\circ_\nu(\mathbb{R}^{n+1})$ and feeding these bounds back into formula (9) for a system A uniformly of type ω.

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