GAUSSIAN UPPER BOUNDS FOR HEAT KERNELS OF A CLASS OF NONDIVERGENCE OPERATORS

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Abstract. Let Ω be a subset of a space of homogeneous type. Let A be the infinitesimal generator of a positive semigroup with Gaussian kernel bounds on $L^2(\Omega)$. We then show Gaussian heat kernel bounds for operators of the type bA where b is a bounded, complex valued function.

1. Introduction

Behaviour of heat kernels has long been an active topic in functional analysis and partial differential equations. In the past few years, it is known that heat kernel bounds such as Gaussian bounds or Poisson bounds imply various useful properties for operators such as L^p spectral invariance [1], [9], bounded holomorphic functional calculi on L^p spaces [11], $L^p - L^q$ maximal regularity for abstract Cauchy problems [12], [7], L^p-analyticity of the semigroup [15]. A large class of divergence form differential operators on the Euclidean space $\mathbb{R}^D$ are known to possess Gaussian heat kernel bounds, see [8], [2], [4] and their references. However, nondivergence operators with L^∞ coefficients, or even with uniformly continuous coefficients, do not possess Gaussian bounds in general, [5], [6]. Hence we can only hope Gaussian heat kernel bounds for specific classes of nondivergence form operators.

The nondivergence form operators $-b\Delta$ on the Euclidean space $\mathbb{R}^D$ were studied in [14]. It was proved that if $b : X \to \mathbb{C}$ is any bounded measurable function on $\mathbb{R}^D$ such that

$$\Re b(x) \geq \delta > 0 \text{ for a.e. } x \in X \qquad (1)$$

then the kernel $k_t(x, y)$ of the semigroup e^{-tbA} has an upper bound with polynomial decay in $|x - y|$. It was also observed that one can improve it to exponential decay by controlling the constants in the upper bound.

The proof in [14] depends on the specific Laplacian Δ through certain estimates using Sobolev embedding and contraction property of the heat semigroup $e^{-t\Delta}$ on $\mathbb{R}^D$. Note that since b is complex-valued, the semigroup $e^{-tb\Delta}$ is no longer contractive.

This result was extended in [10] where it was shown that if the heat kernel $p_t(x,y)$ of an operator A has a Gaussian bound

$$0 \leq p_t(x,y) \leq \frac{C}{t^{D/2}} e^{-c\frac{d^2(x,y)}{t}}. \tag{2}$$

then a similar upper bound holds for the heat kernels of bA. The proof in [10] relies on the Trotter product formula and estimates on the resolvents. During a seminar, the second named author was asked by E. B. Davies if the result in [10] was still true in more general setting like manifolds where the heat kernel bounds take the form

$$0 \leq p_t(x,y) \leq \frac{C}{v(x,\sqrt{t})} e^{-c\frac{d^2(x,y)}{t}}.$$

This is a natural question and this paper is to give a positive answer when the underlying space is a space of homogeneous type or a subset of a space of homogeneous type.

Throughout this paper, C, C', c and c' denote positive constants whose value may change from line to line

2. Main result

Let (X,d,μ) denote a metric space equipped with a $\sigma-$finite measure μ. We assume that X satisfies the doubling property

$$v(x,2r) \leq M v(x,r) \quad \forall x \in X, \forall r > 0, \tag{3}$$

where M is a constant and $v(x,r)$ denotes the volume of the ball with center x and radius r.

Suppose that $-A$ is the generator of a bounded analytic semigroup e^{-tA} on $L^2(X,\mu)$ which has a kernel $p_t(x,y)$ satisfying a Gaussian upper bound

$$0 \leq p_t(x,y) \leq \frac{C}{v(x,\sqrt{t})} e^{-c\frac{d^2(x,y)}{t}} \tag{4}$$

for some positive constants C, c and for all $t > 0$.

The aim of this paper is to show that if $b : X \to \mathbb{C}$ is any bounded measurable function on X which satisfies condition (1), then e^{-tbA} has a kernel $k_t(x,y)$ which satisfies a similar Gaussian upper bound, that is

$$|k_t(x,y)| \leq \frac{C'}{v(x,\sqrt{t})} e^{-c'\frac{d^2(x,y)}{t}} \tag{5}$$

for some positive constants C, c and for all $t > 0$.

As we mentioned above if v is polynomial in r and independent of the centre x, i.e., $v(x,r) = cr^D$ for all $x \in X$, $r > 0$, condition (4) becomes condition (2) and a similar estimate to (5) (with $v(x,\sqrt{t})$ replaced by

$t^{D/2}$) was shown in [10]. In a slightly more general setting, Gaussian lower bounds of $k_t(x,y)$ were also studied in [16].

The proof of our main result relies on the same strategy as in [10] in the sense that it is based on the estimates of powers of the resolvents and the Trotter product formula for semigroups. However, the local nature of the upper bound $(v(x,\sqrt{t}))^{-1}$ requires different approach from the case of the uniform bound $t^{-D/2}$. Indeed, an uniform bound on the kernel of an operator can be obtained through the cross norm of the operator itself from the space L^1 to the space L^∞ but this approach is clearly not enough to produce the upper bound of the type $(v(x,\sqrt{t}))^{-1}$. We overcome this problem by using the resolvent equation which implies the representation (8), together with careful estimates on their kernels; see estimates (9) and (10).

We also need to know how to pass from the upper bound (4) on heat kernels to estimates on kernels of powers of the resolvents and vice versa. The following theorem gives that equivalence.

Theorem 1. *Suppose that $-A$ is the generator of a semigroup e^{-tA} which is bounded analytic with angle ν on $L^2(X,\mu)$. The following assertions are equivalent*

(α) *e^{-tA} has a kernel $p_t(x,y)$ which satisfies the estimate (5) for all $t>0$*

(β) *For all $\lambda>0$ and large enough integer m, $(\lambda I + A)^{-m}$ has a kernel $R_{\lambda,m}(x,y)$ which satisfies*

$$|R_{\lambda,m}(x,y)| \le \frac{C}{|\lambda|^m v(x,\frac{1}{\sqrt{|\lambda|}})} e^{-c\sqrt{|\lambda|}d(x,y)} \tag{6}$$

(γ) *for all $\theta \in [0,\nu)$, $\lambda \in \Sigma(\theta+\frac{\pi}{2})$ and large enough integer m, $(\lambda I + A)^{-m}$ has a kernel $R_{\lambda,m}(x,y)$ which satisfies (6) where $\Sigma(\alpha)$ denotes the sector $\{z\in\mathbb{C}, |\arg(z)|<\alpha\}$.*

Remark.

(a) The doubling property (3) implies the strong homogeneity property

$$\forall x \in X, r>0, \lambda \ge 1, \quad v(x,\lambda r) \le M\lambda^n v(x,r) \tag{7}$$

for some $n>0$. This property will be used in our proof.

(b) The condition m large enough in the above theorem can be replaced by $m>n$.

(c) The main result of this paper is Theorem 2 but Theorem 1 is also of independent interest.

We will use Theorem 1 and the Trotter product formula to prove the following.

Theorem 2. *Assume that $-A$ generates a bounded analytic semigroup which has a kernel $p_t(x,y)$ satisfying the upper bound (4). Assume that $b \in L^\infty(X,\mu,\mathbb{C})$, satisfies (1) and the operator $-bA$ generates a bounded analytic semigroup e^{-tbA} on $L^2(X,\mu)$. Then e^{-tbA} has a kernel $k_t(x,y)$ which satisfies (5).*

Remarks.

(a) Sufficient conditions in terms of b and A which ensure that $-bA$ generates a bounded analytic semigroup are given in [10]. For example, $-bA$ always generates a bounded analytic semigroup if A is self-adjoint.

(b) In the above theorem we assumed that $p_t(x,y)$ is positive but the result is still true if we replace the assumption (4) by

$$|p_t(x,y)| \le h_t(x,y) \le \frac{C}{v(x,\sqrt{t})} e^{-c\frac{d^2(x,y)}{t}} \tag{4'}$$

where $h_t(x,y)$ is a positive heat kernel. In this case, the right hand side of the estimate (14) below will be $\|b^{-1}\|_\infty (\lambda c_0 I + H)^{-1}|f|$, where H denotes the generator of the semigroup whose kernel is $h_t(x,y)$. The rest of the proof needs no change. As an example, we obtain a Gaussian bound for the heat kernel of bA where $A = \sum_{k=1}^{D} (\frac{\partial}{\partial x_k} - ia_k)^2$ the magnetic Laplacian on $\mathbb{R}^D$. It is well known that the heat kernel of A satisfies (4′) with $h_t(x,y)$ the classical heat kernel of the Laplacian, see [13] for example.

We can replace the space X which satisfies the doubling property (3) by Ω where Ω is any subet of X and the result of Theorem 2 is still true. More specifically, the following theorem can be proved by using the same proof as that of Theorem 2.

Theorem 3. *Assume that $-A$ generates a bounded analytic semigroup on $L^2(\Omega)$ which has kernel $p_t(x,y)$ satisfying*

$$0 \le p_t(x,y) \le \frac{C}{v^X(x,\sqrt{t})} e^{-c\frac{d^2(x,y)}{t}} \tag{4''}$$

for all $t > 0$, all $x, y \in \Omega$, where $v^X(x,\sqrt{t})$ denotes the volume of the ball with centre x, radius $\sqrt{t}$ in the space X. Assume that $b \in L^\infty(\Omega,\mu,\mathbb{C})$, satisfies (1) and the operator $-bA$ generates a bounded analytic semigroup e^{-tbA} on $L^2(\Omega)$.

Then e^{-tbA} has a kernel $k_t(x,y)$ which satisfies

$$|k_t(x,y)| \le \frac{C'}{v^X(x,\sqrt{t})} e^{-c'\frac{d^2(x,y)}{t}} \tag{5''}$$

for all $t > 0$, all $x, y \in \Omega$.

Remarks.

In Theorems 2 and 3, we only give heat kernel bounds on $k_t(x,y)$ for $t > 0$. Using Proposition 3.3 of [11], we can extend the results to obtain similar heat kernel bounds on $k_z(x,y)$ for complex z in some sector of the complex plane.

Applications

We give a few applications of our Theorems 2 and 3:

(a) Let A be the Laplace-Beltrami operator on a manifold $\mathbf{M}$ which satisfies a Sobolev inequality and the doubling property. Then A generates a positive semigroup with Gaussian heat kernel bounds (4). It follows from Theorem 2 that the semigroup e^{-tbA} has Gaussian heat kernel bounds (5).

(b) Let A be a divergence form elliptic operator with real, symmetric coefficients acting on a bounded domain Ω of $\mathbb{R}^D$ with Neumann boundary conditions. Assume that the boundary of Ω satisfies the extension property. Then A generates a positive semigroup with Gaussian heat kernel bounds [8]. More specifically,

$$0 \leq p_t(x,y) \leq C \max\{1, \frac{1}{t^{D/2}}\} e^{-c\frac{d^2(x,y)}{t}} \qquad (4''').$$

It follows from Theorem 3 that the semigroup e^{-tbA} has Gaussian heat kernel bounds

$$|k_t(x,y)| \leq C' \max\{1, \frac{1}{t^{D/2}}\} e^{-c'\frac{d^2(x,y)}{t}} \qquad (5''').$$

(c) As a consequence of heat kernel bounds, the two operators bA in applications (a) and (b) above have the following properties:

(i) L^p spectral invariance: The connected components of their resolvent sets which contain the positive real line on L^p spaces are independent of p, $1 \leq p \leq \infty$,

(ii) The $L^p - L^q$ maximal regularity for abstract Cauchy problems,

(iii) If bA has a bounded holomorphic functional calculus on L^2, then it has a bounded holomorphic functional calculus on L^p, $1 < p < \infty$.

3. The Proofs.

Proof of Theorem 1. We first show $(\alpha) \Rightarrow (\beta)$.
Let $\lambda > 0$. The Laplace transform gives

$$(\lambda I + A)^{-m} = \frac{1}{m!} \int_0^\infty t^{m-1} e^{-\lambda t} e^{-tA} dt.$$

Hence the kernel $R_{\lambda,m}(x,y)$ of $(\lambda I + A)^{-m}$ is given by

$$|R_{\lambda,m}(x,y)| \leq \frac{C}{m!}\int_0^\infty \frac{t^{m-1}e^{-\lambda t}}{v(x,\sqrt{t})} e^{\frac{-cd^2(x,y)}{t}} dt.$$

Using the fact that $\lambda t + \frac{d^2(x,y)}{t} \geq \sqrt{\lambda} d(x,y)$, we have

$$|R_{\lambda,m}(x,y)| \leq \frac{Ce^{-c'\sqrt{\lambda}d(x,y)}}{m!}\left[\int_0^{\lambda^{-1}} \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})}dt + \int_{\lambda^{-1}}^\infty \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})}dt\right].$$

For $t \in [\lambda^{-1}, \infty)$, we obviously have $v(x,\sqrt{t}) \geq v(x,\sqrt{\lambda^{-1}})$. Hence the second term in the square bracket satifies

$$\begin{aligned}\int_{\lambda^{-1}}^\infty \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})}dt &\leq \frac{1}{v(x,\sqrt{\lambda^{-1}})}\int_{\lambda^{-1}}^\infty t^{m-1}e^{-c"\lambda t}dt \\ &= \frac{1}{v(x,\sqrt{\lambda^{-1}})}\int_1^\infty \lambda^{-m}s^{m-1}e^{-c"s}ds \\ &= \frac{C}{\lambda^m v(x,\sqrt{\lambda^{-1}})}.\end{aligned}$$

For the first term of the square bracket, we have

$$\int_0^{\lambda^{-1}} \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})}dt = \frac{1}{\lambda^m}\int_0^1 \frac{s^{m-1}e^{-c"s}}{v(x,\sqrt{\frac{s}{\lambda}})}ds.$$

We now apply the strong homogeneity property (7) to deduce that

$$v(x,\sqrt{\lambda^{-1}}) \leq Ms^{\frac{-n}{2}}v(x,\sqrt{\frac{s}{\lambda}}) \quad \forall s \in (0,1].$$

This implies that

$$\int_0^{\lambda^{-1}} \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})}dt \leq \frac{C}{\lambda^m v(x,\sqrt{\lambda^{-1}})}\int_0^1 s^{m-\frac{n}{2}-1}e^{-c"s}ds$$

and the last integral is finite for $m > \frac{n}{2}$. This shows the desired implication.

We now show $(\beta) \Rightarrow (\gamma)$. Firstly, we show an upper bound for $R_{m,\lambda}(x,y)$ for $\lambda \in \Sigma(\theta + \frac{\pi}{2})$.
By the resolvent equation (iterated $m-$times) we have

$$(\lambda I + A)^{-2m} = (\lambda I + A)^{-m}(I + (|\lambda| - \lambda)(\lambda I + A)^{-1})^{2m}(\lambda I + A)^{-m}$$

from which it follows that

$$R_{\lambda,2m}(x,y) = \int_X R_{|\lambda|,m}(x,z)(LR_{|\lambda|,m}(.,y))(z)d\mu(z) \tag{8}$$

where $L = (I + (|\lambda| - \lambda)(\lambda I + A)^{-1})^{2m}$. In order to apply L to $z \to R_{|\lambda|,m}(z, y)$ we need to know that the latter is in $L^2(X, \mu)$. Let us assume this for the moment.
By Cauchy-Schwarz inequality and (8) we have

$$|R_{\lambda,2m}(x, y)| \le \|R_{|\lambda|,m}(x, .)\|_2 \ \|LR_{|\lambda|,m}(., y)\|_2.$$

By the analyticity assumption on e^{-tA} we have

$$\sup_{\lambda \in \Sigma(\theta+\frac{\pi}{2})} \|\lambda(\lambda I + A)^{-1}\|_{\mathcal{L}(L^2)} < \infty.$$

In particular,

$$\|LR_{|\lambda|,m}(., y)\|_2 \le M\|R_{|\lambda|,m}(., y)\|_2$$

with some constant M independent of λ. We then obtain

$$|R_{\lambda,2m}(x, y)| \le \|R_{|\lambda|,m}(x, .)\|_2 \ \|R_{|\lambda|,m}(., y)\|_2 \tag{9}$$

We now estimate $\|R_{|\lambda|,m}(x, .)\|_2$. Using heat kernel bound (4), we have

$$\begin{aligned}
&\int_X | \ R_{|\lambda|,m}(x, z)|^2 d\mu(z) \\
&\quad \le \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^2} \int_X e^{-c\sqrt{|\lambda|}d(x,z)} d\mu(z) \\
&\quad = \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^2} \sum_{k=0}^{\infty} \int_{\{\frac{k}{\sqrt{|\lambda|}} \le d(x,z) \le \frac{k+1}{\sqrt{|\lambda|}}\}} e^{-c\sqrt{|\lambda|}d(x,z)} d\mu(z) \\
&\quad \le \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^2} \sum_{k=0}^{\infty} v(x, \frac{k+1}{\sqrt{|\lambda|}}) e^{-ck} \\
&\quad \le \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})} \sum_{k=0}^{\infty} (k+1)^n e^{-ck}
\end{aligned}$$

where we used (7) to obtain the last inequality. We now obtain from this and (9) that

$$|R_{\lambda,2m}(x, y)| \le \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^{1/2} v(y, \frac{1}{\sqrt{|\lambda|}})^{1/2}}. \tag{10}$$

Now, Proposition 3.3 of [11] shows that the strong homogeneity property (7), together with the bound (10) for $\lambda > 0$, imply that

$$|R_{\lambda,2m}(x, y)| \le \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^{1/2} v(y, \frac{1}{\sqrt{|\lambda|}})^{1/2}} e^{-c"\sqrt{|\lambda|}} d(x, y)$$

for $\lambda \in \Sigma(\theta + \frac{\pi}{2})$. This inequality and (7) imply (6) for $\lambda \in \Sigma(\theta + \frac{\pi}{2})$.

We now show $(\gamma) \Rightarrow (\alpha)$. The proof is standard but we give the details for the sake of completeness.
Using the inverse Laplace transform we have

$$p_t(x,y) = \frac{m-1}{2\pi i t^{m-1}} \int_{\Gamma_R} e^{\lambda t} R_{\lambda,m}(x,y) d\lambda$$

where $\Gamma_R = \{re^{-i\alpha}, r \geq R\} \cup \{Re^{i\alpha}, |\phi| \leq \alpha\} \cup \{re^{i\alpha}, r \geq R\} := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and where $\alpha \in (\frac{\pi}{2}, \nu + \frac{\pi}{2})$ is a given constant and $R = \max(\frac{1}{t}, \frac{d^2(x,y)}{t^2})$.
Using the assertion 3, we can write that for some constants C, c, c' independent of R

$$|p_t(x,y)| \leq \frac{C}{t^{m-1}} \int_{\Gamma_R} \frac{e^{\Re\lambda t}}{|\lambda|^m v(x, \sqrt{|\lambda|^{-1}})} e^{-c\sqrt{|\lambda|}d(x,y)} d|\lambda| \tag{11}$$

The doubling property (7) implies that for $|\lambda| \geq R \geq t^{-1}$,

$$v(x, \sqrt{t}) \leq M(|\lambda|t)^{\frac{n}{2}} v(x, \sqrt{|\lambda|^{-1}}). \tag{12}$$

Hence,

$$\begin{aligned} &\tfrac{1}{t^{m-1}} \int_{\Gamma_1 \cup \Gamma_3} \tfrac{e^{\Re\lambda t}}{|\lambda|^m v(x, \sqrt{|\lambda|^{-1}})} \; e^{-c\sqrt{|\lambda|}d(x,y)} d|\lambda| \\ &\qquad \leq \frac{C}{v(x,\sqrt{t})} \int_R^\infty (\lambda t)^{-m+\frac{n}{2}} e^{-c'\lambda t} e^{-c\sqrt{\lambda}d(x,y)} d\lambda \\ &\qquad \leq \frac{C}{v(x,\sqrt{t})} e^{-c\sqrt{R}d(x,y)} e^{-\frac{c'}{2}tR} \int_1^\infty s^{-m+\frac{n}{2}} e^{-\frac{c'}{2}s} ds \\ &\qquad \leq \frac{C}{v(x,\sqrt{t})} e^{-c\sqrt{R}d(x,y)} e^{-\frac{c'}{2}tR} \dots \end{aligned}$$

Using the fact that $R = \max(\frac{1}{t}, \frac{d^2(x,y)}{t^2})$, the last term is dominated by

$$\frac{C}{v(x,\sqrt{t})} e^{-c\frac{d^2(x,y)}{t}}.$$

Again by (12) we can bound the third term (i.e., $\int_{\Gamma_2}$) in the right hand side of (11) by

$$\frac{C}{v(x,\sqrt{t})} \int_{|\lambda|=R} (Rt)^{-m+\frac{n}{2}} e^{c'Rt} e^{-c\sqrt{R}t} t d|\lambda|.$$

This term is clearly dominated by

$$\frac{C}{v(x,\sqrt{t})} (Rt)^{-m+\frac{n}{2}+1} e^{-c\sqrt{R}t}$$

which gives the desired bound on $p_t(x,y)$. $\diamondsuit$

Proof of Theorem 2. Suppose that $b \in L^\infty(X,\mu,\mathbb{C})$ satisfies (1) and that $-bA$ generates a bounded analytic semigroup on $L^2(X,\mu)$. For any $\lambda > 0$ we write

$$(\lambda I + bA)^{-1} = (\lambda b^{-1} + A)^{-1} b^{-1} \tag{13}$$

Our aim is to prove the following pointwise inequalities which is valid a.e. for all $f \in L^2(X,\mu)$

$$|(\lambda I + bA)^{-1} f| \leq \|b^{-1}\|_\infty (\lambda c_0 I + A)^{-1} |f| \tag{14}$$

where $c_0 = \frac{\delta}{\|b\|_\infty^2}$. The proof of (14) was given in [10] but we repeat it here to keep this paper self sufficient.

We first observe that the positivity of $p_t(x,y)$ implies that the resolvent $(\lambda c_0 I + A)^{-1}$ is a positivity preserving operator for $\lambda > 0$. This is also the case for the operators $(sI + \lambda \Re(\frac{1}{b}) + A)^{-1}$ for all $s, \lambda > 0$ as a consequence of the Trotter product formula (see [17])

$$e^{-t(\lambda\Re(\frac{1}{b})+A)} f = \lim_{n\to\infty} (e^{-\frac{t}{n}\lambda\Re(\frac{1}{b})} e^{-\frac{t}{n}A})^n f, \quad \forall f \in L^2(X)$$

(in [17], the formula is given for contraction semigroups but it applies in this situation since both semigroups $e^{-t\Re(\frac{1}{b})}$ and e^{-tA} are contractions for the equivalent norm $\|f\|_* := \sup_{t\geq 0} \|e^{-tA}|f|\|_2$).

Note that by this formula we have the pointwise estimate

$$e^{-t(\lambda\Re(\frac{1}{b})+A)} |f| \leq e^{-tA}|f|, \quad \forall t > 0, \text{ and } f \in L^2(X)$$

from which it follows that

$$(sI + \lambda\Re(\frac{1}{b}) + A)^{-1} = \int_0^\infty e^{-st} e^{-t(\lambda\Re(\frac{1}{b})+A)} dt$$

exists for all $s, \lambda > 0$.

Using again the Trotter product formula for $e^{-t(\frac{\lambda}{b}+A)}$, we deduce that

$$|e^{-t(\frac{\lambda}{b}+A)} f| \leq e^{-t(\lambda\Re(\frac{1}{b})+A)} |f|$$

which implies

$$|(sI + \frac{\lambda}{b} + A)^{-1} b^{-1} f| \leq \|b^{-1}\|_\infty (sI + \lambda\Re(\frac{1}{b}) + A)^{-1} |f|.$$

Since $\Re(\frac{1}{b}) \geq \frac{\delta}{\|b\|_\infty^2} := c_0$, it follows from the Trotter product formula that

$$|(sI + \frac{\lambda}{b} + A)^{-1} b^{-1} f| \leq \|b^{-1}\|_\infty (sI + \lambda c_0 I + A)^{-1} |f|.$$

Since $(\frac{\lambda}{b} + A)^{-1}$ and $(\lambda c_0 I + A)^{-1}$ exist we conclude from this inequality that

$$|(\frac{\lambda}{b} + A)^{-1} b^{-1} f| \leq \|b^{-1}\|_\infty (\lambda c_0 I + A)^{-1} |f|$$

which is the desired inequality [14].

We now choose an integer $m > n$ where n is the constant in the homogeneity property (7). Iterate (13) $m-$times, we obtain

$$|(\lambda I + bA)^{-m} f| \leq \|b^{-1}\|_\infty^m (\lambda c_0 I + A)^{-m} |f| \ . \tag{15}$$

By assumption, $p_t(x,y)$ satisfies (4) hence by Theorem 1, $(\lambda c_0 I + A)^{-m}$ has a kernel $R_{\lambda,m}(x,y)$ which satisfies (6). In particular, $v(.,\sqrt{\lambda^{-1}}) \times (\lambda c_0 I + A)^{-m}$ is bounded from L^1 into L^∞. Estimate (15) implies that $v(.,\sqrt{\lambda^{-1}})(\lambda I + bA)^{-m}$ is bounded from L^1 into L^∞, so it is given by a kernel. This implies that $(\lambda I + bA)^{-m}$ is given by a kernel. Again by (15), this kernel satisfies (6). By Theorem 1 we conclude that the kernel $k_t(x,y)$ of e^{-tbA} satisfies (5). $\diamondsuit$

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