SINGULARITIES AND THE WAVE EQUATION ON CONIC SPACES

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Introducing polar coordinates around a point in Euclidian space reduces the Euclidian metric to the degenerate form

$$dr^2 + r^2\, d\omega^2 \tag{1}$$

where r is the distance from the point and $d\omega^2$ is the round metric on the sphere. If X is an arbitrary manifold with boundary, the class of *conic metrics* on X is modeled on this special case. Namely, a conic metric is a Riemannian metric on the interior of X such that for some choice of the defining function x of the boundary ($x \in \mathcal{C}^\infty(X)$ with $\partial X = \{x = 0\}$, $x \geq 0$, $dx \neq 0$ on ∂X), the metric takes the form

$$g = dx^2 + x^2 h \text{ on } X^\circ = X \backslash \partial X, \text{ near } \partial X.$$

Here h is a smooth symmetric 2-cotensor on X such that $h_0 = h|_{\partial X}$ is a metric on ∂X.

In fact a general conic metric can be reduced to a form even closer to (1) in terms of an appropriately chosen product decomposition of X near ∂X, that is, by choice of a smooth diffeomorphism

$$[0, \epsilon)_x \times \partial X \xrightarrow{F} O \subset X,\ O \text{ an open neighborhood of } \partial X\,. \tag{2}$$

The normal variable in $x \in [0, \epsilon)$ is then a boundary defining function, at least locally near ∂X, and the slices $F \restriction_{x=x_0}$ have given diffeomorphisms to ∂X. Now such a product decomposition can be chosen so that

$$F^* g = dx^2 + x^2 h_x, \text{ in } x < \epsilon, \tag{3}$$

where h_x is a family of metrics on ∂X.

This reduced form is closely related to the behavior of geodesics near the boundary. Up to orientation and parameterization there is a unique geodesic reaching the boundary at a given point p. In particular the normal fibration of X near ∂X given by the segments $F([0, \epsilon) \times \{p\})$, $p \in \partial X$, consists of geodesics which hit the boundary, each at the corresponding point p.

We shall discuss here the behavior of solutions to the wave equation

$$(D_t^2 - \Delta)u = 0 \text{ on } \mathbb{R} \times X^\circ \tag{4}$$

when X is endowed with a conic metric, Δ is the associated (positive) Laplacian on functions, and $D_t = -i\partial/\partial t$. For simplicity we take X to be compact. It is only really important that ∂X be compact.

Our primary concern is to describe the phenomenon of the propagation of singularities for solutions to (4). To do so it is necessary to understand the behavior of solutions in a way related to the functional analytic domain of Δ. For the moment we simply say that we are dealing with 'admissible' solutions. This condition is explained further below.

In the interior of X the propagation of singularities, described precisely in terms of the notion of wavefront set, was treated in detail by Hörmander ([4]). We paraphrase Hörmander's result here as

> "Singularities travel along null bicharacteristics, which in the case of the wave equation project to time-parameterized geodesics."

Thus, in the microlocal sense of singularities described by the wave front set, a *bicharacteristic segment*, which covers a light ray, either consists completely of singularities for a given solution or the solution has no singularity along it.

This quite adequately describes the propagation of singularities except where a light ray hits the boundary at some point and at some time. Here a 'splitting' of singularities will usually take place. This is generally called a *diffractive* effect. The contrapositive of this effect can be succinctly stated as follows:

> "If no singularity reaches the boundary at time $\bar{t}$ then no singularity leaves at time $\bar{t}$."

The point here is that the regularity along any one of the 'radial' rays leaving the boundary at a given time is related, in general, to the singularities on all the incoming rays (although there are two separate components, as described below) arriving at the boundary at that time. Thus, even if singularities arrive at the boundary at time $\bar{t}$ along just one ray, they will in general depart along all rays leaving the boundary at time $\bar{t}$.

There are, however, some important exceptions to this general spreading of singularities. For instance, if X is a conic manifold with 'trivial' conic metric defined by the blowup of a point in a smooth Riemannian manifold. In this case, admissible solutions are just the lifts of solutions in the usual sense and, because of Hörmander's theorem on interior singularities, the singularities are carried outward only on the one ray continuing the incoming ray in the original manifold.

For a general conic metric there is a similar notion of the 'geometric continuation' of an incoming geodesic which hits the boundary. For a trivial conic metric obtained from a blowup, the boundary metric h_0 is the standard metric on the sphere. The geometrically related incoming and outgoing rays hit this sphere at antipodal points; these can also be thought of as the points separated by geodesics of length π on the unit sphere. In the case of a general conic metric we may mimic this by defining the relation

$$G(p) = \{q \in \partial X;\ \exists \text{ a geodesic in } \partial X \text{ for } h_0 \text{ of length } \pi \text{ with end points } p, q\}. \tag{5}$$

In general of course, $G(p)$ is not smooth. Generically it is a hypersurface with Lagrangian singularities; it is always the projection of a smooth Lagrangian relation.

A geometric refinement of the diffraction result is obtained by considering the order of singularities with respect to Sobolev spaces and an additional 'second microlocal' regularity condition. For simplicity suppose that the (admissible) solution u is singular only near ∂X and only near a single incoming ray hitting the boundary at time $\bar{t}$ and at the point p. In the past (for $t < \bar{t}$) we may suppose that the solution is locally in some Sobolev space H^s. Suppose further that the singularities of the solution are not too strongly focused on ∂X insofar as *tangential smoothing* raises the overall regularity, that is, for some $k, \ell > 0$,

$$(\Delta_{\partial X} + 1)^{-k} u \in H^{s+\ell}_{\mathrm{loc}} \text{ in } t < \bar{t} \text{ near } \partial X, \tag{6}$$

where $\Delta_{\partial X}$ is the Laplacian on ∂X with respect to the metric h_0, extended to act on a neighborhood of X using the product decomposition (2). Under these two assumptions and the additional requirement that

$$0 < \ell < \frac{n}{2}, \tag{7}$$

we obtain the following 'geometric theorem'.

> "If an admissible solution is singular only near an incoming ray arriving at ∂X at time $\bar{t}$ and (6) and (7) hold, then on outgoing rays with initial point in the complement of $G(p)$,
>
> $$u \in H^{s+\ell-\frac{1}{2}-\epsilon}_{\mathrm{loc}} \ \forall\ \epsilon > 0 \text{ in } t > \bar{t} \text{ near } \partial X." \tag{8}$$

When slightly generalized and refined, as described below, this result applies to the fundamental solution

$$\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$$

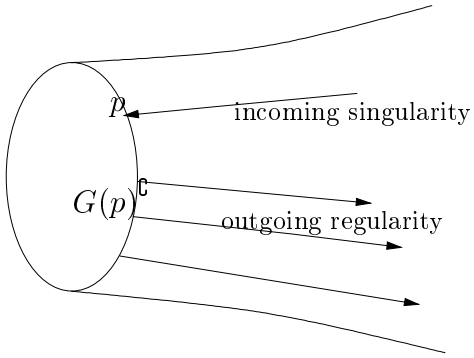

FIGURE 1. Given the tangential smoothing property, there is greater regularity on outgoing rays from points not π-related to incoming rays carrying singularities.

with pole close to ∂X and with $\ell < \frac{n-1}{2}$. The diffractive theorem merely tells us that if the pole is specified at $(\bar{x}, p)$ at $t = 0$, then singularities cannot emanate from ∂X except at time $t = \bar{x}$. On the other hand, while 'strong' singularities can emanate from all points in $G(p)$, this geometric theorem tells us that the solution is microlocally more regular on rays starting from ∂X at $t = \bar{x}$ but with initial point outside $G(p)$. We can in fact use the conormality of the fundamental solution to obtain a sharper result than (8): the analogue of (8) for the fundamental solution yields $H^{s+l-\epsilon}$ regularity, i.e. we obtain one-half derivative of improvement over the general case.

In the special case in which the metric g takes precisely the 'product' form

$$g = dx^2 + x^2 h(y, dy), \tag{9}$$

near the boundary, Cheeger and Taylor [1, 2] have given an explicit analysis of the fundamental solution constructed by separation of variables. (See also the discussion by Kalka-Menikoff [6].) They Sobolev regularity they obtain is the same, and is therefore optimal in general. They also show that the outgoing solution is conormal and compute the precise order. A version of the results of Cheeger-Taylor has been established in the analytic category by Rouleux [14]. Lebeau [7, 8] has also obtained a diffractive theorem in the setting of manifolds with corners in the analytic category.

Detailed proofs of the results in this paper will appear in [11].

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1. Friedrichs extension

To describe the admissibility condition, near the boundary, for solutions to (4) we first describe the domain of the Laplacian for a conic metric. We take the Friedrichs extension of Δ. By definition, Δ is associated to the Dirichlet form

$$F(u,v)=\int_X \langle du,dv\rangle_g\,dg\,,\ u,v\in\mathcal{C}^\infty_c(X^\circ) \tag{10}$$

Hence, dg is the metric volume form; in this case

$$dg=\varphi x^{n-1}\,dx\,dh_0 \text{ near } \partial X,\quad n=\dim X\,,\ \varphi\in\mathcal{C}^\infty,\varphi>0\,.$$

The inner product in (10) is that induced, by duality, by the metric on T^*X°. Following Friedrichs we define,

$$D(\Delta^{1/2})=\operatorname{clos}\left\{\mathcal{C}^\infty_{\mathrm{c}}(X^\circ)\text{ w.r.t. }F(u,u)+\|u\|^2_{L^2_g}\right\},$$

whenever X is a compact conic manifold with boundary of dimension $n\geq 2$.

This is a Hilbert space with dense injection $D(\Delta^{1/2})\hookrightarrow L^2_g(X)$ so there is a dual injection $L^2_g(x)\hookrightarrow(D(\Delta^{1/2}))'$. The natural operator $\Delta:D(\Delta^{1/2})\to(D(\Delta^{1/2}))'$ is determined by

$$(\Delta u,\varphi)_{L^2_g}=F(u,\varphi)\ \forall\ u,\varphi\in D(\Delta^{1/2}).$$

Then the Friedrichs extension of Δ is the unbounded operator with domain

$$D(\Delta)=\left\{u\in D(\Delta^{1/2}),\ \Delta u\in L^2_g(x)\right\}\,.$$

It is a self-adjoint, non-negative operator and in this case has discrete spectrum of finite multiplicity. This allows its complex powers to be defined by reference to an eigenbasis. The real powers are isomorphisms off the null space, which consists precisely of the constants. Each of the powers is therefore a Fredholm map

$$\Delta^s:D(\Delta^s)\to L^2_g(X)\ \forall\ s\in\mathbb{R}.$$

with null space the constants and range the orthocomplement of the constants. The domains form a scale of Hilbert spaces, and

$$D(\Delta^s)\hookrightarrow D(\Delta^t)\text{ is dense }\forall\ s\geq t$$

with $D(\Delta^0)=L^2_g(X)$.

For our purposes it is also important to note when the domains consist of extendible distributions, i.e. those dual to $\dot{\mathcal{C}}^\infty(X)$. This is the case only for $s > -\frac{n}{4}$ and more precisely

$$\dot{\mathcal{C}}^\infty(X) \hookrightarrow D(\Delta^s) \hookrightarrow \mathcal{C}^{-\infty}(X) \tag{11}$$

are dense inclusions for $-\frac{n}{4} < s < \frac{n}{4}$. The limits of this range correspond to the occurrence of formal solutions of $\Delta u = 0$.

2. Wave group

The Cauchy problem for the wave equation

$$\begin{gathered}(D_t^2 - \Delta)u = 0, \text{ on } \mathbb{R} \times X^\circ \\ u|_{t=0} = u_0, \ D_t u|_{t=0} = u_1\end{gathered} \tag{12}$$

has a unique solution

$$\begin{gathered}u \in C^0(\mathbb{R}; D(\Delta^{1/2})) \cap C^1(\mathbb{R}; L^2_g(x)) \\ \forall\ (u_0, u_1) \in \mathcal{E} = \mathcal{E}_1 = D(\Delta^{1/2}) \oplus L^2_g(X)\,.\end{gathered}$$

These 'finite energy solutions' are the main object of study here. More generally, with the equation interpreted in $\mathcal{C}^{-\infty}(\mathbb{R}; D(\Delta^{\frac{s}{2}-1}))$ the Cauchy problem has a unique solution

$$\begin{gathered}u \in C^0(\mathbb{R}; D(\Delta^{\frac{s}{2}})) \cap C^1(\mathbb{R}; D(\Delta^{\frac{s}{2}-\frac{1}{2}})) \\ \forall\ (u_0, u_1) \in \mathcal{E}_s = D(\Delta^{\frac{s}{2}}) \oplus D(\Delta^{\frac{s}{2}-\frac{1}{2}})\,.\end{gathered} \tag{13}$$

The regularity hypothesis on the solution can be weakened to

$$u \in L^2_{\mathrm{loc}}(\mathbb{R}; D(\Delta^{\frac{s}{2}})) \cap H^1_{\mathrm{loc}}(\mathbb{R}; D(\Delta^{\frac{s}{2}-\frac{1}{2}}))$$

without changing the unique solvability.

Notice that these calculations are consistent under decrease of s. Furthermore, partial hypoellipticity in t shows that the solution to (12) satisfies

$$u \in H^{-k}_{\mathrm{loc}}(\mathbb{R}; D(\Delta^{\frac{s}{2}+\frac{k}{2}})) \quad \forall\ k \in \mathbb{R}. \tag{14}$$

An *admissible solution* to the wave equation is one that satisfies

$$u \in H^p_{\mathrm{loc}}(\mathbb{R}; D(\Delta^{\frac{q}{2}})) \text{ for some } q, p \in \mathbb{R} \tag{15}$$

with (4) holding in $H^{-p-2}_{\mathrm{loc}}(\mathbb{R}; D(\Delta^{\frac{q}{2}-1}))$. Such a solution automatically satisfies (14) for some s.

These statements can be reinterpreted in terms of the wave group

$$U(t) : \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \begin{pmatrix} u(t) \\ D_t u(t) \end{pmatrix},\ U(t) : \mathcal{E}_s \to \mathcal{E}_s\ \forall\ s. \tag{16}$$

3. Hörmander's theorem

Let M be a manifold without boundary. The wave front set of a distribution $u \in \mathcal{C}^{-\infty}(M)$ is a closed subset of the cosphere bundle

$$\mathrm{WF}(u) \subset S^*M.$$

It may be defined by decay properties of the localized Fourier transform, or the FBI (Fourier–Bros–Iagonitzer) transform, or by testing with pseudodifferential operators. The projection $\pi(\mathrm{WF}(u)) \subset M$ is exactly the $\mathcal{C}^\infty$ singular support, the complement of the largest open subset of M to which u restricts to be $\mathcal{C}^\infty$.

A refined notion of wavefront set is the Sobolev-based wavefront set, denoted WF_s; this is a closed subset of S^*M, where now the projection is the complement of the largest open subset of M to which u restricts to be H^s.

If u satisfies a linear differential equation, $Pu = 0$, then

$$\mathrm{WF}(u) \subset \Sigma(P) \subset S^*M$$

when $\Sigma(P)$ is the characteristic variety of P, the set on which its (homogeneous) principal symbol, p, vanishes.

If p is real then the symplectic structure on T^*M, or the contact structure on S^*M, defines a 'bicharacteristic' direction field V_P on S^*M, tangent to $\Sigma(P)$. The integral curves of V_P are called bicharacteristics; those lying in $\Sigma(P)$ are called null bicharacteristics.

Theorem 1 (Hörmander)**.** *Let P be a (pseudo)-differential operator with real principal symbol. If $Pu = 0$ then $WF(u) \subset \Sigma(P)$ is a union of maximally extended null bicharacteristics. The same result also holds with* WF *replaced by* WF_s *for any s.*

In our case, $M = \mathbb{R} \times X^\circ$ so $T^*M = T^*\mathbb{R} \times T^*X^\circ$. The principal symbol of the d'Alembertian is $\tau^2 - |\cdot|_g^2$, where τ is the dual variable to t and $|\cdot|_g$ is the (dual) metric on T^*X°. Then

$$\Sigma(P) = \Sigma_+(P) \cup \Sigma_-(P) \subset S^*M$$

when $\Sigma_I(P) \cong \mathbb{R} \times S^*X^\circ$ are the disjoint parts of $\Sigma(P)$ in $\tau > 0$ and $\tau < 0$. In this representation of $\Sigma(P)$ the null bicharacteristics are geodesics on X°, lifted canonically to S^*X°, with t as affine parameter. Thus, for the wave equation over X°, Hörmander's theorem does indeed reduce to the informal propagation statement described above.

Combined with standard results relating the singularities of the solution to singularities of the initial data, Hörmander's theorem applied to the wave equation on a conic manifold yields complete information on

the behavior of singularities except along bicharacteristics lying above geodesics which hit the boundary.

4. Diffractive theorem

On parametrized geodesic segments with an end point on the boundary, the defining function x is either strictly increasing or strictly decreasing near the boundary. For each sign of τ and for each $\bar{t} \in \mathbb{R}$ the bicharacteristics covering such geodesics which hit the boundary at $t = \bar{t}$ and along which t is increasing (resp. decreasing) as x decreases, form a smooth submanifold of $T^*(\{t < \bar{t}\} \times X)$ (resp. $T^*(\{t > \bar{t}\} \times X)$). We denote these 'radial' surfaces (near ∂X) by

$$R_{\pm,I}(\bar{t}) \text{ and } R_{\pm,O}(\bar{t}) \subset \Sigma(P)$$

where $\pm$ is the sign of τ and I, O refers to whether these are 'incoming' or 'outgoing' and hence, equivalently, whether they lie in $t < \bar{t}$ or $t > \bar{t}$.

Theorem 2 (Diffractive theorem)**.** *If u is an admissible solution to* (4) *then for any $\bar{t} \in \mathbb{R}$, $s \in \mathbb{R}$, $\sigma = \pm$,*

$$R_{\sigma,I}(\bar{t}) \cap \mathrm{WF}_s(u) = \emptyset \Rightarrow R_{\sigma,O}(\bar{t}) \cap \mathrm{WF}_s(u) = \emptyset.$$

Here, $\mathrm{WF}_s(u)$ is the wave front set computed relative to the Sobolev space H^s, locally in the interior.

This is a precise form of the diffractive result described informally above. Notice that the singularities for different signs of τ are completely decoupled. This does not, however, represent any refinement in terms of propagation along the underlying geometric rays, since all geodesics are covered by bicharacteristics with τ fixed and of either sign.

The proof of this result is discussed briefly below in §7.

5. Geometric theorem

Consider a geodesic on X which hits the boundary at a point $p \in \partial X$. An open set of perturbations of the geodesic, meaning geodesics starting near some interior point on the geodesic and with initial tangent close to the tangent to the geodesic, will miss the boundary. A limit of such curves as the perturbation vanishes consists of three segments. The first is the incoming geodesic segment. The second is a geodesic segment in the boundary, of length π. The third is the outgoing geodesic from the end point of the boundary segment, which is therefore a point in $G(p)$ as defined in (5) (see Figure 2). Thus it is reasonable to suppose that, amongst the outgoing bicharacteristics leaving the boundary at time $\bar{t}$, those with initial points in $G(p)$ will be more

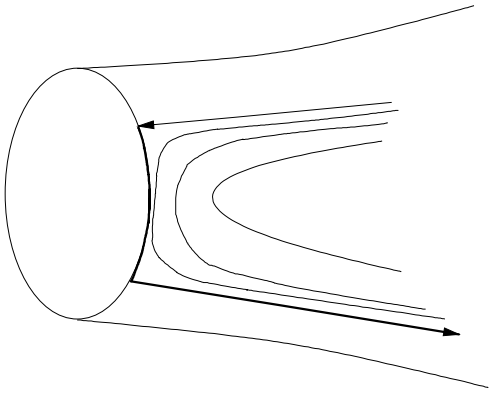

FIGURE 2. A sequence of geodesics nearly missing the boundary, and the three segments which they approach.

closely related to an incoming bicharacteristic with end point p arriving at time $\bar{t}$. We call these the geometrically-related bicharacteristics (or geodesics).

For instance, if there are incoming singularities on a single ray the singularities on the 'non-geometrically-related' outgoing bicharacteristics might be expected to be weaker than the incoming singularity. However, this is *not* in general the case. To obtain such a geometric refinement of the diffraction result we need to impose an extra 'nonfocusing' assumption.

Theorem 3 (Geometric theorem). *Let u be an admissible solution to* (4) *and let $\sigma = \pm$. Suppose that $R_{\sigma,I}(\bar{t}) \cap \mathrm{WF}_s\, u = \emptyset$ near ∂X. Suppose additionally that for some k and $0 < \ell < \frac{n}{2}$*

$$\mathrm{WF}_{s+\ell}(1 + \Delta_{\partial X})^{-k}\, u \cap R_{\sigma,I}(\bar{t}) = \emptyset. \tag{17}$$

For any $0 < r < \ell - 1/2$, if no incoming bicharacteristic hitting the boundary at time $\bar{t}$ at a point in $G(p)$ with $\operatorname{sgn}\tau = \sigma$ is in $\mathrm{WF}_{s+r}\, u$, then the outgoing bicharacteristic with initial point $p \in \partial X$ and $\operatorname{sgn}\tau = \sigma$ is not in $\mathrm{WF}_{s+r-\epsilon}\, u$ for any $\epsilon > 0$.

If in addition to (17) *we have*

$$\mathrm{WF}_{s+\ell}(xD_x + (t - \bar{t})D_t)(1 + \Delta_{\partial X})^{-k}\, u \cap R_{\sigma,I}(\bar{t}) = \emptyset. \tag{18}$$

then the same conclusion follows for all $0 < r < l$.

Thus the additional assumption (17) allows regularity on outgoing rays to be deduced from regularity in the incoming geometrically-related rays up to the corresponding level above 'background' regularity.

As already noted, this result may be applied to the fundamental solution with initial point near the boundary. If the initial pole of the fundamental solution is sufficiently close to the boundary then there is a unique short geodesic segment from it to the boundary, arriving at a point p. If $\bar{t}$ is the length of the segment then, provided $\bar{t}$ is small enough, (17) and (18) hold with $s < -\frac{n}{2} + 1$ for any $\ell < \frac{n-1}{2}$. It follows that on $R_{\pm,O}(\bar{t})$, the outgoing set, the fundamental solution is in $H^{1/2-\epsilon}$, for all $\epsilon > 0$, microlocally near the non-geometrically related rays, those with end point not in $G(p)$, whereas the general regularity is $H^{-\frac{n}{2}+1-\epsilon}$ for all $\epsilon > 0$. This is a gain of 'nearly' $\frac{n-1}{2}$ derivatives over the background regularity.

In this way we extend part of the result of Cheeger and Taylor [1, 2] in the product case (9) to the general conic case. Inspection of the fundamental solution constructed in [1] reveals the 'nearly' $\frac{n-1}{2}$ difference in smoothness between geometric and non-geometric rays to be sharp.

6. Spherical conormal waves

Around a given point q in a compact Riemann manifold there are 'spherical' conormal waves which are singular only on the spherical surfaces $r = \pm t$, for small t of both signs. These just correspond to conormal data at $t = 0$ at the (fictive) cone point q. An important example is the fundamental solution, in which case the result follows from Hadamard's construction. In the more general case of a conic manifold with boundary there are similar contracting, and then expanding, conormal waves.

Theorem 4. *If u is an admissible solution near ∂X and $t = 0$ which is conormal to $t = -x$ for $t < 0$ then it is conormal to $t = x$, near the boundary, for small $t > 0$.*

These conormal solutions to the wave equation in the general conic case are at the opposite extreme to those considered in the Geometric Theorem above. Namely, they are already smooth in the tangential variables, so no tangential smoothing in the sense of (6) is possible. Further analysis of the structure of these waves shows that the principal symbols undergo a transition at $x = 0$, the boundary, given by the scattering matrix for the model cone with the same boundary metric. Since this scattering matrix should have full support in general, this provides counterexamples to any extension of the geometric theorem in which the tangential smoothing condition is dropped.

7. Methods

The basic method we use is microlocal, but non-constructive. It is a direct extension of one of the proofs by Hörmander of the interior propagation theorem. This 'positive' commutator method is itself a microlocalization of the energy method for hyperbolic equations. In it a 'test' pseudodifferential operator, A, is applied to the equation and the essential positivity of the symbol of the commutator $\frac{1}{i}[P, A]$ gives a local regularity estimate on the solution.

To extend this method to cover behavior of solutions near the boundary we replace the ordinary notions of wavefront set, pseudodifferential operators and microlocalization with versions appropriately adapted to the geometry. When considering the Laplacian itself on the manifold with boundary with conic metric, the appropriate notion is that of a weighted b-pseudodifferential operator (see [13]). This for instance allows the precise description of the domains of the powers of Δ which is used at various points in the argument.

However, for the wave operators for the conic Laplacian the appropriate notion corresponds to the 'edge' calculus of pseudodifferential operators discussed originally by Mazzeo [9], arising from a filtration of the boundary (see also Schulze [15]). In this case, the manifold with boundary is $X \times \mathbb{R}$ and the fibers of the boundary are the surfaces $t = \text{const}$. Thus t is the base variable of the fibration.

To the edge calculus of pseudodifferential operators, given by microlocalization from the differential operators generated by xD_x, D_y (where the y's are tangential variables) and xD_t, we associate a notion of wavefront set. We can prove the propagation theorem analogous to that of Hörmander for this 'edge' wavefront set. However, in this new sense, $D_t^2 - \Delta$ is *not* globally of principal type but rather has two radial surfaces. These correspond to the end points of bicharacteristics arriving at, and leaving from, the boundary. At these surfaces there are restrictions on the propagation results, very closely related to those for scattering Laplacians in [10]. The construction of the test operator A, which away from the radial surfaces is essentially given by flowout along the geodesic spray on ∂X, becomes more delicate at the radial surfaces. Positivity relies on the precise form of the singularity of the Hamilton vector field there.

These propagation estimates form the basis of both the diffractive and geometric theorems. In the former we combine the estimates with a variant of the one-dimensional FBI transform, scaled with respect to the normal variable x. This reduces the diffractive result to an iterative

application of a uniqueness theorem for the Laplacian on the model, non-compact cone.

To obtain the geometric theorem, showing that the outgoing singularities on non-geometrically related rays are weaker than the incoming ones, we use a division theorem. The additional hypothesis of microlocal tangential smoothing is shown to imply that the solution actually lies in a weighted Sobolev space with a higher x weight (hence more 'divisible' by x) than is given, *a priori*, by energy conservation. This allows the microlocal propagation results indicated above to be pushed further at the outgoing radial surface and so yields the extra regularity.

8. Applications and extension

The propagation of singularities results of the type discussed above should allow estimates of the spectral counting function as shown originally by Ivrii ([5], see also [12] and [3]).

We expect these methods to extend to more complicated geometries, including manifolds with corners and iterated conic spaces.

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