PRINCIPAL SERIES AND WAVELETS

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ABSTRACT. Recently Antoine and Vandergheynst [1, 2] have produced continuous wavelet transforms on the n-sphere based on a principal series representation of $SO(n, 1)$. We present some of their calculations in a more general setting, from the point of view of Fourier analysis on compact groups and spherical function expansions.

1. Coherent States

We begin with Antoine and Vandergheynst's definition of a coherent state, as presented in [1, 2]. Here G is a locally compact group.

- Suppose that X is a homogeneous space of G, $X = G/H$, equipped with a G-invariant measure.
- Let $(U, L^2(Y))$ be a unitary representation of G on some Lebesgue space $L^2(Y)$.
- Assume there is a Borel cross section

$$\sigma : X \longrightarrow G, \qquad \sigma(x) H = x, \qquad \forall x \in X.$$

- Say that $\eta \in L^2(Y)$ is *admissible* $\mathrm{mod}(H, \sigma)$ when

$$\int_X |\langle U(\sigma(x)) \eta | \varphi \rangle|^2 \, dx < \infty, \quad \forall \varphi \in L^2(Y).$$

- The orbit of an admissible vector η under $\sigma(X)$,

$$\{U(\sigma(x)) \eta : x \in X\}$$

  is called a *coherent state*.

Note that there are other variations on the theme of "restricted square integrability", such as the case described in [3].

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2. Frames

Suppose now that η is an admissible vector in $L^2(Y)$. Define a linear operator

$$A_{\sigma,\eta} : L^2(Y) \longrightarrow L^2(Y)$$

by

$$\langle A_{\sigma,\eta}\varphi_1|\varphi_2\rangle = \int_X \langle \varphi_1|U(\sigma(x))\eta\rangle \langle U(\sigma(x))\eta|\varphi_2\rangle\, dx, \quad \forall \varphi_1, \varphi_2 \in L^2(Y).$$

When this has a bounded inverse, say that the coherent state is a *frame*.

When the orbit of η under $\sigma(X)$ is a frame of $L^2(Y)$ there is the *continuous wavelet transform*,

$$W_\eta : L^2(Y) \longrightarrow L^2(X)$$

defined by

$$W_\eta\varphi(x) = \langle \varphi|U(\sigma(x))\eta\rangle, \quad \forall \varphi \in L^2(Y).$$

This operator is one-to-one and its range $\mathcal{H}_\eta$ is complete with respect to the inner-product:

$$\langle W_\eta\varphi|W_\eta\psi\rangle_{\mathcal{H}_\eta} = \left\langle W_\eta\varphi|W_\eta A^{-1}_{\sigma,\eta}\psi\right\rangle_{L^2(X)}, \quad \psi, \varphi \in L^2(Y).$$

Hence there is a unitary isomorphism $W_\eta : L^2(Y) \longrightarrow \mathcal{H}_\eta$.

3. The setting

For the calculations which we will describe here, the ingredients are:

- G is a noncompact connected semisimple Lie group with finite centre and Cartan involution θ.
- K is the corresponding maximal compact subgroup.
- $G = KAN$ is an Iwasawa decomposition.
- M is the centralizer of A in K.
- $X = G/N$.
- $Y = K/M$.
- U is a certain principal series action of G on $L^2(K/M)$, to be defined below.
- Assume that (K, M) is a Gel'fand pair.

See Knapp's book for details [5, page 119].

4. Decompositions

There are *Iwasawa projections* $\mathtt{K} : G \to K$, $\mathtt{A} : G \to A$, $\mathtt{N} : G \to N$, for which

$$g = \mathtt{K}(g)\mathtt{A}(g)\mathtt{N}(g), \qquad \forall g \in G.$$

The Haar measure on G is given in terms of that of K and right Haar measure of AN, [5, page 139] with

$$dg = dk\, d_r(an).$$

The measure on K is normalized so that

$$\int_K dk = 1.$$

There is a mapping $\log : A \to \mathfrak{a}$ with

$$\exp(\log(a)) = a, \qquad \forall a \in A.$$

For each $\nu \in \mathfrak{a}^*$ let

$$a^\nu = e^{\nu(\log(a))}, \quad \forall a \in A.$$

5. Invariant Integration

There is the special functional $\rho \in \mathfrak{a}^*$ determined by the structure of the group G. For $f \in C_c(G)$ the integral formula for Haar measure on G is

$$\int_G f(x)\, dx = \int_K \int_A \int_N f(kan)\, a^{2\rho}\, dndadk.$$

See [6, Prop. 7.6.4] for details.

We can use KA to parametrize G/N and the G-invariant integral on G/N is given by

$$\int_{G/N} F(y)dy = \int_K \int_A F(kaN)a^{2\rho}\, dadk$$

for $F \in C_c(G/N)$. Hence, we take the Borel section $\sigma : G/N \to G$ to be

$$\sigma(kaN) = ka, \qquad \forall a \in A, k \in K.$$

6. Induced Representations

Consider the space of continuous covariant functions:

$$\mathbf{I}(G) = \left\{ f \ : \ \begin{array}{c} f : G \to \mathbb{C} \text{ continuous} \\ f(gman) = a^{-\rho} f(g), \\ \forall g \in G, m \in M, a \in A, n \in N \end{array} \right\}.$$

Left translation by elements of G preserves the property of covariance:

$$\left(U(g)f\right)(x) = f\left(g^{-1}x\right), \qquad \forall g, x \in G, f \in \mathbf{I}(G).$$

$$U(g) : \mathbf{I}(G) \longrightarrow \mathbf{I}(G), \quad \forall g \in G.$$

For a covariant function $f \in \mathbf{I}(G)$,

$$f(x) = f(\mathtt{K}(x)\mathtt{A}(x)\mathtt{N}(x)) = \mathtt{A}(x)^{-\rho} f(\mathtt{K}(x)), \quad \forall x \in G.$$

Equip $\mathbf{I}(G)$ with the inner product

$$\langle f_1 | f_2 \rangle = \int_K f_1(k)\overline{f_2(k)}\, dk$$

and norm

$$\|f\| = \left(\int_K |f(k)|^2 \, dk \right)^{1/2}.$$

The completion of $\mathbf{I}(G)$ is

$$\mathcal{H}_U \cong L^2(K/M).$$

The action of G on $\mathcal{H}_U$ is an example of a *principal series representation*, see section 8.3 of Wallach's book [6]. For our purposes, the essential fact is that $U|_K$ is the regular representation of K on a subspace of $L^2(K)$. If $f \in L^2(K/M)$, extend it to be an element of $\mathcal{H}_U$ by assigning

$$f(kan) = a^{-\rho} f(k).$$

Notice that if $f \in L^2(K/M)$,

$$U(g)f(k) = \mathtt{A}(g^{-1}k)^{-\rho} f(\mathtt{K}(g^{-1})k), \quad k \in K, g \in G.$$

For each $g \in G$ the action of $U(g)$ extends to a continuous linear operator on $\mathcal{H}_U$. It is a unitary representation:

$$\langle U(g)f_1 | U(g)f_2 \rangle = \int_K (U(g)f_1)(k)\overline{(U(g)f_2)(k)}\, dk$$

$$= \int_K \mathtt{A}(g^{-1}k)^{-2\rho} f_1(\mathtt{K}(g^{-1}k))\overline{f_2(\mathtt{K}(g^{-1}k))}\, dk. = \langle f_1 | f_2 \rangle$$

Lemma 1. *The representation $(U, \mathcal{H}_U)$ is unitary. When restricted to K, it is the action of K by left translation on $L^2(K/M)$.*

7. Fourier analysis on the compact group K

We review some basic facts about analysis on compact groups. Let $\widehat{K}$ be the dual object of K, consisting of a maximal set of inequivalent irreducible unitary representations (γ, V_γ) of K.

For each integrable function f on K there is the Fourier series:

$$f(x) = \sum_{\gamma \in \widehat{K}} d_\gamma f * \chi_\gamma(x).$$

Convolution with a character is

$$f * \chi_\gamma(x) = \int_K f(y)\, \mathrm{tr}(\gamma(y^{-1})\gamma(x))\, dy = \mathrm{tr}(\widehat{f}(\gamma)\gamma(x))$$

where the Fourier coefficient is

$$\widehat{f}(\gamma) = \int_K f(x)\gamma(x^{-1})\, dx = \int_K f(x)\gamma(x)^*\, dx.$$

The Fourier coefficients are linear transformations

$$\widehat{f}(\gamma) \in \mathrm{Hom}_{\mathbb{C}}(V_\gamma, V_\gamma).$$

Fourier coefficients of convolutions are products of Fourier coefficients:

$$\begin{aligned}(f * g)^\wedge(\gamma) &= \int_K \int_K f(x) g(x^{-1}y)\gamma(y^{-1})\, dx dy \\ &= \int_K \int_K f(x) g(x^{-1}y)\gamma(y^{-1}xx^{-1})\, dx dy \\ &= \widehat{g}(\gamma)\widehat{f}(\gamma).\end{aligned}$$

Define left translation on K by

$$_x f(y) = f(x^{-1}y), \qquad \forall x, y \in K,$$

and the composition with inversion

$$f^\vee(x) = f(x^{-1}), \qquad \forall x \in K.$$

Fourier coefficients of left translates satisfy

$$\left(_x f\right)^\wedge(\gamma) = \int_K f(x^{-1}y)\gamma(y^{-1}xx^{-1})\, dy = \widehat{f}(\gamma)\gamma(x^{-1})$$

Fourier coefficients of adjoints satisfy

$$(\overline{g}^\vee)^\wedge(\gamma) = \widehat{g}(\gamma)^*.$$

The $L^2(K)$ inner product can be viewed as a convolution:

$$\int_K f(x)\overline{g(x)}\, dx = \int_K f(x)\overline{g}^\vee(x^{-1})\, dx = f * \overline{g}^\vee(1).$$

For $f, g \in L^2(K)$, the Fourier series of their convolution is absolutely convergent, see [4],

$$f * g(x) = \sum_{\gamma \in \widehat{K}} d_\gamma f * g * \chi_\gamma(x)$$

f and g in $L^2(K)$:

$$f * g(x) = \sum_{\gamma \in \widehat{K}} d_\gamma \operatorname{tr}\left(\widehat{g}(\gamma)\widehat{f}(\gamma)\gamma(x)\right),$$

$$\int_K f(x)\overline{g(x)}\,dx = \sum_{\gamma \in \widehat{K}} d_\gamma \operatorname{tr}(\widehat{f}(\gamma)\widehat{g}(\gamma)^*),$$

$$\|f\|_2^2 = \sum_{\gamma \in \widehat{K}} d_\gamma \left\|\widehat{f}(\gamma)\right\|_{\phi_2}^2.$$

In particular, for each $\gamma \in \widehat{K}$,

$$\|\widehat{f}(\gamma)\|_{\phi_2}^2 = d_\gamma \|f * \chi_\gamma\|_2^2.$$

See Appendix D of Hewitt and Ross [4] for details about the norms

$$\|\cdot\|_{\phi_p}, \quad 1 \le p \le \infty.$$

If $h \in L^1(K)$ then

$$f \mapsto f * h, \quad L^2(K) \longrightarrow L^2(K),$$

is a bounded linear operator which commutes with left translation. Similarly,

$$f \mapsto h * f, \quad L^2(K) \longrightarrow L^2(K),$$

is a bounded linear operator which commutes with right translation. The norm of both of these operators is

$$\sup_{\gamma \in \widehat{K}} \left\|\widehat{h}(\gamma)\right\|_{\phi_\infty}.$$

8. Homogeneous Spaces

Now we return to dealing with functions on K/M, which we identify with right-M-invariant functions on K.

For each $\gamma \in \widehat{K}$, let

$$V_\gamma^M = \{v \in V_\gamma \;:\; \gamma(m)v = v, \quad \forall m \in M\}$$

and $P_\gamma : V_\gamma \longrightarrow V_\gamma^M$, the orthogonal projection on to this subspace.

Let μ be the normalized Haar measure on M. Its Fourier coefficients are

$$\widehat{\mu}(\gamma) = P_\gamma, \quad \forall \gamma \in \widehat{K}.$$

If $f \in L^1(K/M)$ then

$$f = f * \mu, \quad \Longrightarrow \quad \widehat{f}(\gamma) = P_\gamma \widehat{f}(\gamma), \quad \forall \gamma \in \widehat{K}.$$

We are restricting our attention to the case where (K, M) is a *Gel'fand pair*, which means that

$$\dim \left(V_\gamma^M\right) \leq 1, \quad \forall \gamma \in \widehat{K}.$$

Lemma 2. *If (K, M) is a Gel'fand pair and $f \in L^1(K/M)$, then for all $\gamma \in \widehat{K}$,*

$$rank(\widehat{f}(\gamma)) \leq 1 \qquad and \qquad (V_\gamma^M)^\perp \subseteq ker(\widehat{f}(\gamma)^*).$$

Lemma 3. *If (K, M) is a Gel'fand pair and $f \in L^1(K/M)$, then for all $\gamma \in \widehat{K}$,*

$$\widehat{f}(\gamma)\widehat{f}(\gamma)^* = \|\widehat{f}(\gamma)\|_{\phi_2}^2 P_\gamma.$$

Lemma 4. *If (K, M) is a Gel'fand pair and $f \in L^1(K/M)$, then for all $\gamma \in \widehat{K}$,*

$$\|\widehat{f}(\gamma)\|_{\phi_p} = \|\widehat{f}(\gamma)\|_{\phi_2}, \quad 1 \leq p \leq \infty.$$

Lemma 5. *If (K, M) is a Gel'fand pair and $h \in L^1(K/M)$, then the norm of the operator*

$$f \mapsto f * h, \quad L^2(K) \longrightarrow L^2(K/M),$$

is

$$\sup \left\{ \|\widehat{h}(\gamma)\|_{\phi_2} \ : \ \gamma \in \widehat{K} \right\} = \sup \left\{ \sqrt{d_\gamma}\, \|h * \chi_\gamma\|_2 \ : \ \gamma \in \widehat{K} \right\}.$$

In this lemma, if $\dim \left(V_\gamma^M\right) = 0$ then $\widehat{h}(\gamma) = 0$ and so we need only take the supremum over those γ for which $\dim \left(V_\gamma^M\right) = 1$.

9. Admissible Vectors

In [2] the unitary representation $(U, \mathcal{H}_U)$ of G is said to be *square-integrable modulo N* if there is a non-zero vector η for which

$$\int_K \int_A |\langle U(ka)\eta|\xi\rangle|^2 \, a^{2\rho} \, dadk < \infty$$

for all $\xi \in \mathcal{H}_U$. Such an η is called *admissible*.

Notice that this can be rearranged to say

$$\int_K \int_A \left|\left\langle U(a)\eta | U(k^{-1})\xi\right\rangle\right|^2 a^{2\rho} \, dadk < \infty$$

for all $\xi \in \mathcal{H}_U$. Recall that $U|_K$ is left translation.

We then find that

$$\begin{aligned}\int_K |\langle U(ka)\eta \mid \xi\rangle|^2 \, dk &= \int_K \left| \int_K (U(a)\eta)(x)\overline{\xi(kx)}\, dx\right|^2 dk \\ &= \int_K \left|(U(a)\eta) * \bar{\xi}^{\vee}(k)\right|^2 \, dk \\ &= \left\|(U(a)\eta) * \bar{\xi}^{\vee}\right\|_2^2\end{aligned}$$

Using the Plancherel formula for this,

$$\begin{aligned}\left\|(U(a)\eta) * \bar{\xi}^{\vee}\right\|_2^2 &= \sum_{\gamma} d_\gamma \operatorname{tr}\Big((U(a)\eta)^\wedge(\gamma)^* \widehat{\xi}(\gamma)\widehat{\xi}(\gamma)^*(U(a)\eta)^\wedge(\gamma)\Big) \\ &= \sum_{\gamma} d_\gamma \|(U(a)\eta)^\wedge(\gamma)\|_{\phi_2}^2 \|\widehat{\xi}(\gamma)\|_{\phi_2}^2\end{aligned}$$

We arrive at the general version of Antoine and Vandergheynst's criterion for admissibility.

Theorem 1. *If $\eta \in \mathcal{H}_U = L^2(K/M)$ has the property that*

$$\sup_{\gamma\in\widehat{K}} \int_A \|(U(a)\eta)^\wedge(\gamma)\|_{\phi_2}^2 \, a^{2\rho}\, da < \infty$$

then η is admissible.

Since the functions here are right-M-invariant, the only non-zero parts of the Fourier series correspond to those γ for which $P_\gamma \neq 0$.

Theorem 2. *If $\eta \in \mathcal{H}_U = L^2(K/M)$ is admissible and there are constants $0 < c_1 \leq c_2$ for which*

$$c_1 \leq \int_A \|(U(a)\eta)^\wedge(\gamma)\|_{\phi_2}^2 \, a^{2\rho}\, da \leq c_2$$

for all $\gamma \in \widehat{K}$ with $P_\gamma \neq 0$, then the corresponding coherent state is a frame.

We can reword this to see that the criterion for η to give rise to a frame for $L^2(K/M)$ is that there are constants $0 < c_1 \leq c_2$ for which

$$c_1 \leq d_\gamma \int_A \|(U(a)\eta) * \chi_\gamma\|_2^2 \; a^{2\rho}\, da \leq c_2,$$

for all $\gamma \in \widehat{K}$ with $P_\gamma \neq 0$.

10. Spherical Functions

Let $\widehat{K}_M$ denote the set of those $\gamma \in \widehat{K}$ with $P_\gamma \neq 0$. For each $\gamma \in \widehat{K}_M$ define the spherical function

$$\varphi_\gamma = \chi_\gamma * \mu = \mu * \chi_\gamma.$$

If $f \in L^1(K/M)$ its Fourier series is

$$\sum_{\gamma \in \widehat{K}_M} d_\gamma f * \varphi_\gamma.$$

When $K/M = S^n$, this is the usual spherical harmonic expansion.

To use the criterion for a frame, we need estimates on

$$d_\gamma \int_A \|(U(a)\eta) * \varphi_\gamma\|_2^2 \; a^{2\rho} \, da,$$

uniformly in $\gamma \in \widehat{K}_M$.

11. Zonal Functions

A special case occurs when η is bi-M-invariant, since it is then expanded in a series

$$\eta = \sum_{\gamma \in \widehat{K}_M} d_\gamma c_\gamma \varphi_\gamma \quad \text{with} \quad c_\gamma = \langle \eta | \varphi_\gamma \rangle .$$

But $U(a)\eta$ is also bi-M-invariant and its expansion is

$$U(a)\eta = \sum_{\gamma \in \widehat{K}_M} d_\gamma c_\gamma(a) \varphi_\gamma$$

with

$$c_\gamma(a) = \langle U(a)\eta | \varphi_\gamma \rangle = \left\langle \eta | U(a^{-1}) \varphi_\gamma \right\rangle .$$

Since the spherical functions φ_γ are matrix entries of irreducible representations,

$$\varphi_\gamma * \varphi_{\gamma'} = \begin{cases} \varphi_\gamma / d_\gamma & \text{if } \gamma = \gamma' \\ 0 & \text{if } \gamma \neq \gamma', \end{cases}$$

and $\|\varphi_\gamma\|_2^2 = 1/d_\gamma$. Hence, Theorem 2 says that a bi-M-invariant function η produces a frame for $L^2(K/M)$ when there are positive constants $c_1 \leq c_2$ for which

$$0 < c_1 \leq \int_A |c_\gamma(a)|^2 \; a^{2\rho} \, da \leq c_2$$

for all $\gamma \in \widehat{K}_M$.

12. Antoine and Vandergheynst

The results in [2] are concerned with the case where:

- $G = SO_e(1,3)$, $K \cong SO(3)$, $M \cong SO(2)$, and $K/M \cong S^2$.
- $A \cong (0,\infty)$ with multiplication, $X \cong SO(3) \times A$, $\rho = 1$.
- $\widehat{K}_M = \{0,1,2,3,\dots\}$, $d_n = 2n+1$, and the spherical functions φ_n are normalized ultraspherical polynomials.

Suppose we use spherical coordinates (θ,ϕ) to parametrize S^2. Proposition 3.4 of [2] states that if $\eta \in L^2(S^2)$ is admissible and

$$\int_0^{2\pi} \eta(\theta,\phi)\, d\phi \neq 0$$

then η gives rise to a frame. This is achieved using the spherical harmonic expansion of $U(a)\eta$ and the asymptotics of the zonal spherical functions, to get the inequality in Theorem 2 above.

In [2] there is presented a sufficient condition on a function $\eta \in L^2(S^2)$ so that it satisfies the hypotheses of Theorem 1. These are similar to the moment conditions in the Euclidean space setting, see Proposition 7 in [3]. Proposition 3.6 [2] states that if $\eta \in L^2(S^2)$ satisfies

$$\int_0^{\pi}\int_0^{2\pi} \frac{\eta(\theta,\phi)}{1+\cos(\theta)}\, \sin(\theta)\, d\theta d\phi = 0$$

then it is admissible.

References

1. J.-P. Antoine and P. Vandergheynst, *Wavelets on the n-sphere and related manifolds*, J. Math. Phys. **39** (1998), no. 8, 3987–4008.
2. ———, *Wavelets on the 2-sphere: a group-theoretical approach*, Appl. Comput. Harmon. Anal. **7** (1999), no. 3, 262–291.
3. J. E. Gilbert, R. A. Kunze, and C. Meaney, *On derived intertwining norms for the Lorentz group*, Representation theory and harmonic analysis (Cincinnati, OH, 1994), Amer. Math. Soc., Providence, RI, 1995, pp. 57–73.
4. Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups*, Springer-Verlag, New York, 1970, Die Grundlehren der mathematischen Wissenschaften, Band 152.
5. Anthony W. Knapp, *Representation theory of semisimple groups*, Princeton University Press, Princeton, N.J., 1986, An overview based on examples.
6. Nolan R. Wallach, *Harmonic analysis on homogeneous spaces*, Marcel Dekker Inc., New York, 1973, Pure and Applied Mathematics, No. 19.

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