WHAT'S NEW FOR THE BELTRAMI EQUATION ?

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ABSTRACT. The existence theorem for quasiconformal mappings has found a central rôle in a diverse variety of areas such as holomorphic dynamics, Teichmüller theory, low dimensional topology and geometry, and the planar theory of PDEs. Anticipating the needs of future researchers we give an account of the "state of the art" as it pertains to this theorem, that is to the existence and uniqueness theory of the planar Beltrami equation, and various properties of the solutions to this equation.

This paper surveys the recent work of the authors' paper [14] and parts of our monograph [15]. Readers interested in more details, and in particular rather greater discussion of related work by others, should consult those works. In what follows we use fairly standard notation, in particular $\mathbf{B}$ denotes a disk, usually the unit disk, in the complex plane $\mathbb{C}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.

The Beltrami equation has a long history. Gauss first studied the equation, with smooth coefficients, in the 1820's while investigating the problem of existence of isothermal coordinates on a given surface. The complex Beltrami equation was intensively studied by Morrey in the late 1930's, and he established the existence of homeomorphic solutions for measurable μ [18]. It took another 20 years before Bers recognised that homeomorphic solutions are quasiconformal mappings. Since then these ideas have found diverse applications in a variety of areas such as holomorphic dynamics, Teichmüller theory, low dimensional topology and geometry, and the planar theory of PDEs.

1. PDEs

There is a strong interaction between linear and non-linear elliptic systems in the plane and quasiconformal mappings. The most general first order linear (over $\mathbb{R}$) elliptic system takes the form

$$\overline{\partial}\, f = \mu_1\, \partial f + \mu_2\, \overline{\partial f}$$

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where μ_1 and μ_2 are complex valued measurable functions such that

$$|\mu_1(z)| + |\mu_2(z)| \leq \frac{K-1}{K+1} < 1 \quad a.e.\ \Omega$$

The complex Beltrami equation is simply that equation which is linear over the complex numbers.

$$\overline{\partial} f(z) = \mu(z)\, \partial f(z) \tag{1}$$

Classically one assumes ellipticity bounds:

$$\|\mu\|_\infty = \frac{K-1}{K+1} < 1 \tag{2}$$

When $\mu = 0$ we have the Cauchy–Riemann system.

These sets of equations are particular cases of the genuine non-linear first ord er system

$$\overline{\partial} f = H(z, \partial f)^M \tag{3}$$

where $H : \Omega \times \mathbb{C} \to \mathbb{C}$ is Lipschitz in the second variable,

$$|H(z,\zeta) - H(z,\xi)| \leq \frac{K-1}{K+1}\, |\zeta - \xi|, \qquad H(z,0) \equiv 0$$

A feature of (3) is that the difference of two solutions need not solve the same equation but it is K–quasiregular (the term used to describe non-injective quasiconformal functions). Thus quasiconformal maps are a central tool used to establish *a priori* estimates needed for the existence and uniqueness.

2. Classical Regularity Theory

Typically one seeks solutions to the Beltrami equation in the Sobolev space $W^{1,2}_{loc}(\Omega)$. However, the solutions to this equation have the following striking regularity result finally established in complete form by K. Astala (the Area Distortion Theorem) [2].

Theorem 2.1. *Let μ be a measurable function defined in Ω with $\|\mu\|_\infty = k < 1$. Let f be any solution to the Beltrami equation with $f \in W^{1,q}_{loc}(\Omega)$, $q > 1 + k$. Then $f \in W^{1,p}_{loc}(\Omega)$ for all $p < 1 + \frac{1}{k}$.*

Moreover, there may be solutions in $W^{1,1+k}_{loc}(\Omega)$ not in any higher Sobolev space, and there may be solutions in $W^{1,2}_{loc}(\Omega)$ not in $W^{1+1/k}_{loc}(\Omega)$.

Notice the indices p and q in the above result form a Hölder conjugate pair.

The Neumann iteration procedure based on invertibility of the Beltrami operator $\mathbf{I} - \mu\mathcal{S}$, ($\mathcal{S}$ the complex Hilbert transform) yields a representation formula (first found by Bojarski [5]) and existence.

Theorem 2.2. *Let μ be a measurable function in $\Omega \subset \mathbb{C}$ and suppose $\|\mu\|_\infty < 1$. Then there is a homeomorphic solution $g : \Omega \to \mathbb{C}$ to the Beltrami equation.*

Moreover every $W^{1,2}_{loc}(\Omega, \mathbb{C})$ solution f is of the form

$$f(z) = \Psi(g(z))$$

where $\Psi : g(\Omega) \to \mathbb{C}$ is a holomorphic function.

This last fact is known as "factorisation".

3. A Fundamental Example

This example reflects what is possible in the degenerate elliptic setting, and shows why it is necessary to use the Orlicz-Sobolev spaces in order to discuss the fine properties of solutions.

Theorem 3.1. *Let $\mathcal{A} : [1, \infty) \to [1, \infty)$ be a smooth increasing function with $\mathcal{A}(1) = 1$ and such that*

$$\int_1^\infty \frac{\mathcal{A}(s)}{s^2} ds < \infty. \tag{4}$$

Then there is a Beltrami coefficient μ compactly supported in the unit disk, $|\mu(z)| < 1$, with the following properties:

1. *The ellipticity bound $K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$ satisfies*

$$\int_{\mathbf{B}} e^{\mathcal{A}(K(z))} dz < \infty \tag{5}$$

2. *Every $W^{1,1}_{loc}(\mathbf{B})$–solution to the Beltrami equation*

$$f_{\overline{z}} = \mu f_z \qquad a.e. \quad \mathbf{B} \tag{6}$$

   continuous at the origin is constant.

3. *There is a bounded solution $w = f(z)$ to the Beltrami equation in the space weak-$W^{1,2}(\mathbf{B}) \subset \bigcap_{1 \le q < 2} W^{1,q}(\mathbf{B})$ which homeomorphically maps the punctured disk onto the annulus $1 < |w| < R$.*

A few remarks.

First, $W^{1,1}_{loc}(\mathbf{B})$ is really the weakest space in which one can begin to discuss what it means to be a solution.

Secondly, the integrability condition (4) implies that $\mathcal{A}$ is sublinear. As examples the function

$$\mathcal{A}(\tau) = \frac{\tau}{(\log \tau)^{1+\epsilon}} \tag{7}$$

satisfies (4) for all $\epsilon > 0$, but not for $\epsilon = 0$. More generally, if we put

$$\log_1 \tau = \log \tau, \qquad \log_{n+1} \tau = \log(\log_n \tau)$$

we have the *iterated logarithm* functions. Then for each $n > 0$ the function

$$\frac{\tau}{\log_1(1+\tau)\ \log_2(e+\tau)\ \log_3(e^e+\tau)\cdots(\log_n(e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}+\tau))^{1+\epsilon}}$$

again satisfies (4) for all $\epsilon > 0$, but not for $\epsilon = 0$.

Finally, (3) is no accident. We can prove that if $e^{\mathcal{A}(K)} \in L^p_{loc}(\mathbf{B})$ for some $p \geq 0$ and certain types of $\mathcal{A}$ (and in particular the $\log\log\ldots$ examples), with

$$\int_{\mathbf{B}} e^{\mathcal{A}(K(z))} dz = \infty$$

then there is a homeomorphic (and hence continuous) solution in $W^{1,p}(\mathbf{B})$ for all $p < 2$. In fact the solution lies in an Orlicz–Sobolev class just below $W^{1,2}_{loc}$

4. Mappings of Finite Distortion

We next give a general definition of the mappings which mostly occur. Roughly, solutions to a Beltrami equation with ellipticity bounds which are pointwise finite will be mappings of finite distortion as soon as they are ACL–*absolutely continuous on lines.*

Definition A mapping $f : \Omega \to \mathbb{C}$ is said to have *finite distortion* if:

1. $f \in W^{1,1}_{loc}(\Omega)$,
2. The Jacobian determinant, $J(z, f) = \det Df(z)$, of f is locally integrable and does not change sign in Ω
3. There is a measurable function $\mathcal{K} = \mathcal{K}(z) \geq 1$, finite almost everywhere, such that f satisfies the *distortion inequality*

$$|Df(z)|^2 \leq \mathcal{K}(z)\ |J(z,f)| \qquad a.e.\ \ \Omega \tag{8}$$

Notice that the hypotheses are not sufficient to guarantee that $f \in W^{1,2}_{loc}(\Omega)$ unless the distortion function $\mathcal{K}$ is bounded. Nor do they imply that the Jacobian does not vanish on a set of positive measure.

The motivational philosophy behind the condition that the distortion function is exponentially integrable is now clear. We wish to exploit the $BMO - \mathcal{H}^1$ duality (and even more refined versions of this) to achieve uniform estimates on approximating sequences of solutions.

5. Maximum Principle and Continuity

One of the first tasks is to establish a maximum type principle and modulus of continuity estimates.

A continuous function $u : \Omega \to \mathbb{R}$ defined in a domain Ω is *monotone* if

$$\operatorname{osc}_{\mathbf{B}} u \leq \operatorname{osc}_{\partial \mathbf{B}} u$$

for every ball $\mathbf{B} \subset \Omega$. This definition in fact goes back to Lebesgue in 1907 where he first showed the relevance of the notion of monotonicity to elliptic PDEs in the plane. In order to handle very weak solutions of differential inequalities, such as the distortion inequality, we need to extend this concept, dropping the assumption of continuity, and to the setting of Orlicz–Sobolev spaces.

Definition. A real valued function $u \in W^{1,P}(\Omega)$ is said to be *weakly monotone* if for every ball $\mathbf{B} \subset \Omega$ and all constants $m \leq M$ such that

$$|M-u| - |u-m| + 2u - m - M \in W_0^{1,P}(\mathbf{B}) \tag{9}$$

we have

$$m \leq u(x) \leq M \tag{10}$$

for almost every $x \in \mathbf{B}$.

For continuous functions (9) holds if and only if $m \leq u(x) \leq M$ on $\partial\mathbf{B}$. Then (10) says we want the same condition in $\mathbf{B}$, that is the maximum and minimum principles.

Here, and in what follows we assume, unless otherwise stated, that the Orlicz function P satisfies

$$\int_1^\infty P(t)\,\frac{dt}{t^3} = \infty \tag{11}$$

and that the function $t \mapsto t^{\frac{5}{8}}$ is convex. For example Orlicz functions of the form

$$P(\tau) = \frac{\tau^2}{\log_1(1+\tau)\ \log_2(e+\tau)\ \log_3(e^e+\tau)\cdots(\log_n e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}+\tau)}$$

are of this form.

The Orlicz–Sobolev space $W^{1,P}_{loc}(\Omega)$ consists of functions which, together with their first derivatives, lie in the space $L^P(\Omega)$. Thus in the example given above we are looking in Zygmund type spaces just below $W^{1,2}_{loc}(\Omega)$.

Lemma 5.1. *Let Ω be a bounded domain and suppose that $u \in W^{1,P}(\Omega) \cap C(\overline{\Omega})$ is weakly monotone. Then*

$$\min_{\partial\Omega} \, u \leq u(x) \leq \max_{\partial\Omega} \, u \tag{12}$$

for every $x \in \Omega$.

The paper [17] by Manfredi should be mentioned as the beginning of the systematic study of weakly monotone functions.

We now recall a fundamental monotonicity result in the Orlicz–Sobolev classes.

Theorem 5.1. *The coordinate functions of mappings with finite distortion in $W^{1,P}(\Omega)$ are weakly monotone.*

There is a particularly elegant geometric approach to the continuity estimates of monotone functions. The idea goes back to Gehring in his study of the Liouville theorem in space where he developed the Oscillation Lemma.

We need the *P–modulus* of continuity $\Xi_P(\tau)$ defined for $0 \leq \tau < 1$ as follows. For $\tau > 0$ the value t of Ξ_P at τ is uniquely determined by the equation

$$\int_1^{1/\tau} P(st) \, \frac{ds}{s^3} = P(1). \tag{13}$$

Certainly Ξ_P is a non-decreasing function with

$$\lim_{\tau \to 0} \Xi_P(\tau) = 0. \tag{14}$$

Given the transcendental nature of the equation one must solve, it is impossible in all but the most elementary situations, to calculate Ξ. Here are a few explicit formulas for $\Xi(\tau)$ which exhibit the correct asymptotics for τ near 0.

$$P(t) = t^2, \quad \Xi(\tau) = |\log \tau|^{-\frac{1}{2}}$$

More generally for all $\alpha > 0$ we have,

$$P(t) = t^2 \log^{\alpha-1}(e+t), \ \alpha > 0, \quad \Xi(\tau) \approx |\log \tau|^{-\frac{\alpha}{2}}$$

$$P(t) = \frac{t^2}{\log(e+t)}, \quad \Xi(\tau) \approx [\log|\log \tau|]^{-\frac{1}{2}},$$

and finally

$$P(t) = \frac{t^2}{\log(e+t)\log\log(3+t)}, \quad \Xi(\tau) \approx [\log\log|\log \tau|]^{-\frac{1}{2}},$$

We now have the fundamental modulus of continuity estimate.

Theorem 5.2. *Let $u \in W^{1,P}(\mathbf{B})$ be weakly monotone in $\mathbf{B} = \mathbf{B}(z_0, 2R)$. Then for all Lebesgue points $a, b \in \mathbf{B}(z_0, R)$ we have*

$$|u(a) - u(b)| \leq 16\pi R \|\nabla u\|_{\mathbf{B},P}\, \Xi\left(\frac{|a-b|}{2R}\right). \tag{15}$$

In particular, u has a continuous representative for which (15) holds for all a and b in the disk $\mathbf{B}(z_0, R)$.

In the statement above we have used

$$\|\nabla u\|_{\mathbf{B},P} = \inf\left\{\frac{1}{\lambda} : \frac{1}{|\mathbf{B}|}\int_{\mathbf{B}} P(\lambda|\nabla u|) \leq P(1)\right\} \tag{16}$$

to denote the P-average of ∇u over the ball $\mathbf{B}$.

Theorem 5.3. *Every mapping with finite distortion in the Orlicz–Sobolev class $W^{1,P}_{loc}(\Omega)$, is continuous.*

6. Liouville Type Theorem

Here is a first taste of the power of Theorem 5.1.

Theorem 6.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a mapping of finite distortion whose differential belongs to $L^P(\mathbb{C})$. Then f is constant.*

The proof consists in showing that $R\, \|\nabla u\|_{\mathbf{B},P} \to 0$ as $R \to \infty$ in (15) using the Dominated Convergence Theorem.

7. Solutions

A *principal solution* is a homeomorphism $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with

1. a discrete set E (the singular set) such that $h \in W^{1,1}_{loc}(\mathbb{C} \setminus E)$,
2. the Beltrami equation $h_{\overline{z}}(z) = \mu(z) h_z(z)$ holds for a.e. $z \in \mathbb{C}$, and
3. we have the normalisation $h(z) = z + o(1)$ at ∞

It will become clear that the key to understanding the Beltrami equation and its local solutions is in the existence and uniqueness properties of the principal solutions.

A function f, not necessarily a homeomorphism, is a *very weak solution* if it satisfies:

- there is a discrete set $E_f \subset \Omega$ (the singular set) such that $h \in W^{1,1}_{loc}(\Omega \setminus E_f)$.
- the Beltrami equation

  $$f_{\overline{z}}(z) = \mu(z) f_z(z)$$

  holds for almost every $z \in \Omega$

The point is that one expects a solution f to be the composition of the principal solution h with a meromorphic function φ defined on $h(\Omega)$,

$$f = \varphi \circ h,$$

(the Stoilow factorisation theorem). Thus away from the poles, the solution f is locally as good as the principal solution.

7.1. **Uniqueness of Principal Solutions.** Here is the most general uniqueness result that we are aware of.

Theorem 7.1. *Every elliptic equation*

$$h_{\overline{z}} = H(z, h_z)$$

admits at most one principal solution in the Sobolev-Orlicz class $z + W^{1,P}(\mathbb{C})$

We use the term $z + W^{1,P}(\mathbb{C})$ to denote the mappings h with $|h_{\overline{z}}| + |h_z - 1| \in L^P(\mathbb{C})$. As far as the ellipticity is concerned, we assume that there is a measurable compactly supported function $k : \mathbb{C} \to \mathbf{B}$ such that for almost every $z \in \mathbb{C}$ and all $\zeta, \xi \in \mathbb{C}$

$$|H(z,\zeta) - H(z,\xi)| \le k(z)|\zeta - \xi|$$

Proof. Let h be a solution to the equation. Thus $h_{\overline{z}}(z) = 0$ for z sufficiently large. The point is that given two principal solutions h^1 and h^2, the mapping $f = h^1 - h^2$ has finite distortion and its differential $Df = Dh^1 - Dh^2$ belongs to $L^P(\mathbb{C})$. To see this note

$$\begin{aligned} |(h^1 - h^2)_{\overline{z}}| &= |h^1_{\overline{z}} - h^2_{\overline{z}}| \\ &= |H(z, h^1_z) - H(z, h^2_z)| \\ &\le k(z)|(h^1 - h^2)_z| \end{aligned}$$

whence $J(z, h^1 - h^2) \ge 0$. It follows that f is constant from the Liouville theorem. The normalisation at ∞ implies that this constant is 0.

8. Stoilow Factorisation

We now state that if a Beltrami equation admits a homeomorphic solution, then all other solutions in the same class are obtained from this solution via composition with a holomorphic mapping.

Theorem 8.1. *Suppose we are given a homeomorphic solution* $h \in W^{1,P}_{loc}(\Omega)$ *to the Beltrami equation*

$$h_{\overline{z}} = \mu(z) h_z \qquad a.e. \quad \Omega \tag{17}$$

Then every solution $f \in W^{1,P}_{loc}(\Omega)$ takes the form

$$f(z) = \phi(h(z)), \qquad z \in \Omega$$

where $\phi : h(\Omega) \to \mathbb{C}$ is holomorphic.

9. Failure of Factorisation

Now an example which shows that for fairly nice solutions one cannot expect a factorisation theorem even in the case of bounded distortion.

Theorem 9.1. *Let $K > 1$ and $q_0 < \frac{2K}{K+1}$. Then there is a Beltrami coefficient μ supported in the unit disk with the following properties.*

- $\|\mu\|_\infty = \frac{K-1}{K+1}$,
- *The Beltrami equation $h_{\overline{z}} = \mu h_z$ admits a Hölder continuous solution $h \in z + W^{1,q_0}(\mathbb{C})$ which fails to be in $W^{1,2}_{loc}(\mathbb{C})$,*
- *The solution h is not quasiregular, and therefore not the principal solution, nor obtained from the principal solution by factorisation.*

10. Distortion in the Exponential Class

Theorem 10.1. *There exists a number $p_0 > 1$ such that every Beltrami equation*

$$h_{\overline{z}}(z) = \mu(z)\; h_z(z) \qquad a.e. \quad \mathbb{C}$$

with Beltrami coefficient μ such that

$$|\mu(z)| \leq \frac{K(z) - 1}{K(z) + 1}\, \chi_{\mathbf{B}}(z)$$

and

$$e^K \in L^p(\mathbf{B}) \tag{18}$$

with $p \geq p_0$, admits a unique principal solution $h \in z + W^{1,2}(\mathbb{C})$.

There are examples to show that in order for there to be a principal solution in the natural Sobolev space $z + W^{1,2}(\mathbb{C})$ it is necessary that the exponent p at (18) is large, at least $p \geq 1$.

As a matter of fact, somewhat more is true in Theorem 10.1. The higher the exponent of integrability of e^{pK} the better the regularity of the solution. That is even beyond L^2, such as $L^2 \log L$ with any $\alpha \geq 0$, see [16].

The situation is different if the integrability exponent of e^K is smaller than the critical exponent p_0. Here the principal solution need not be in $z + W^{1,2}(\mathbb{C})$, but we still obtain a satisfactory class of solutions.

Theorem 10.2. *Suppose the distortion function $K = K(z)$ for the Beltrami equation is such that $e^K \in L^p(\mathbf{B})$ for some positive p. Then the equation admits a unique principal solution h*

$$h \in z + W^{1,Q}(\mathbb{C}), \qquad Q(t) = t^2 \log^{-1}(e+t) \tag{19}$$

Moreover, every $W^{1,Q}_{loc}(\Omega)$ solution is factorisable.

11. Distortion in the Subexponential Class

We assume here that the Beltrami coefficient is supported in the unit disk $\mathbf{B}$.

Theorem 11.1. *There is a number $p_* \geq 1$ such that every Beltrami equation whose distortion function has*

$$\exp\left(\frac{K(z)}{1+\log K(z)}\right) \in L^p(\mathbf{B})$$

for $p > p_*$, admits a unique principal solution $h \in z + W^{1,Q}(\mathbb{C})$ with Orlicz function $Q(t) = t^2 \log^{-1}(e+t)$. Moreover we have

- Modulus of Continuity*;*

$$|h(a) - h(b)|^2 \leq \frac{C_K}{\log\log(1 + \frac{1}{|a-b|})} \tag{20}$$

  for all $a, b \in 2\mathbf{B}$.
- Inverse*; The inverse map $g = h^{-1}(w)$ has finite distortion $\mathcal{K} = \mathcal{K}(w)$ and*

$$\log \mathcal{K} \in L^1(\mathbb{C})$$

- Factorization*; each solution $g \in W^{1,Q}_{loc}(\Omega)$ to the equation*

$$g_{\overline{z}} = \mu(z) g_z, \qquad a.e. \quad \Omega$$

  admits a Stoilow factorisation

$$g(z) = \Phi \circ h(z) \tag{21}$$

  where Φ is holomorphic in $h^{-1}(\Omega)$. In particular, all non-constant solutions in $W^{1,Q}_{loc}(\Omega)$ are open and discrete.

12. Existence Theory.

Various reductions show the important case to be when the Beltrami coefficient μ is compactly supported in the unit disk $\mathbf{B}$. Then any solution is analytic outside the unit disk.

12.1. **Results from Harmonic Analysis.** The existence proof presented here exploits a number of substantial results in harmonic analysis. The arguments clearly illustrate the important rôle that the higher integrability properties of the Jacobians have to play. The critical exponent p_0 in Theorem 10.1 depends only on the constants in three inequalities which we now state.

The first is a direct consequence of [8].

Theorem 12.1. (Coifman, Lions, Meyer, Semmes) *The Jacobian determinant* $J(x,\phi)$ *of a mapping* $\phi \in W^{1,2}(\mathbb{C})$ *belongs to the Hardy space* $H^1(\mathbb{C})$ *and we have the estimate*

$$\|J(x,\phi)\|_{H^1(\mathbb{C})} \le C_1 \int_{\mathbb{C}} |D\phi|^2 \tag{22}$$

Next we have from [9]

Theorem 12.2. (Coifman, Rochberg) *Let* μ *be a Borel measure in* $\mathbb{C}$ *such that its Hardy–Littlewood maximal function* $M(x,\mu)$ *is finite at a single point (and therefore at every point). Then* $\log M(x,\mu) \in BMO(\mathbb{C})$ *and its norm is bounded by an absolute constant,*

$$\|\log M(x,\mu)\|_{BMO} \le C_2. \tag{23}$$

Finally we shall need the constant C_3 which appears in the H^1-BMO duality theorem of Fefferman, [12].

Theorem 12.3. (Fefferman) *For* $K \in BMO(\mathbb{C})$ *and* $J \in H^1(\mathbb{C})$ *we have*

$$\left| \int K(x)\ J(x)\ dx \right| \le C_3 \|K\|_{BMO} \|J\|_{H^1} \tag{24}$$

Having these prerequisites we can reveal that the exponent in Theorem 10.1 is

$$p_0 = 8C_1C_2C_3. \tag{25}$$

13. Sketch of Proof for Theorem 10.1

We again refer the reader to [14] for more details, but the basic ideas can be found here.

Let K be the distortion function. Set

$$A_p = \int_{\mathbb{C}} \left[e^{pK(z)} - e^p \right]\ dz < \infty$$

We approximate μ by smooth functions μ_ν via mollification. We have $K_\nu \leq K$ and the uniform bound

$$\int_{\mathbb{C}} \left[e^{pK_\nu(z)} - e^p\right] \, dz \leq \int_{\mathbb{C}} \left[e^{pK(z)} - e^p\right] \, dz = A_p \tag{26}$$

Put $\gamma = \frac{p}{2}$ to see $\left[e^{\gamma K_\nu(z)} - e^\gamma\right] \in L^2(\mathbb{C})$.

Next, the maximal function of $e^{\gamma K_\nu(z)}$ is finite everywhere,

$$M(z, e^{\gamma K_\nu}) = e^\gamma + M(z, e^{\gamma K_\nu} - e^\gamma)) \tag{27}$$

and this last term is a constant plus a function in $L^2(\mathbb{C})$.

Now consider the BMO functions

$$\mathcal{K}_\nu(z) = \frac{1}{\gamma} \log M(z, e^{\gamma K_\nu}) \tag{28}$$

By Theorem 12.2, the BMO norm of this function does not depend on ν,

$$\|\mathcal{K}_\nu\|_{BMO} \leq \frac{2C_2}{p} \tag{29}$$

Moreover this function pointwise majorises the distortion function. We need a uniform L^2 bound for $\mathcal{K}_\nu$. Clearly

$$\begin{aligned}
&\frac{1}{|2\mathbf{B}|} \int_{2\mathbf{B}} \mathcal{K}_\nu(z)^2 \, dz \\
&= \frac{1}{4\gamma^2 |2\mathbf{B}|} \int_{2\mathbf{B}} \log^2 \, [M(z, e^{\gamma K_\nu})]^2 \\
&\leq \frac{1}{4\gamma^2} \log^2 \, \frac{1}{|2\mathbf{B}|} \int_{2\mathbf{B}} [M(z, e^{\gamma K_\nu} - e^\gamma) + e^\gamma]^2 \\
&\leq \frac{1}{4\gamma^2} \log^2 \, \left[2e^{2\gamma} + \frac{2C}{|2\mathbf{B}|} \int_{\mathbb{C}} (e^{\gamma K_\nu} - e^\gamma)^2\right]
\end{aligned}$$

Here we have used the L^2 inequality for the maximal operator. This gives

$$\int_{2\mathbf{B}} |\mathcal{K}_\nu(z)|^2 \, dz \leq C_4 \log^2(1 + A_p) \tag{30}$$

where C_4 is an absolute constant.

Let us now return to the mollified Beltrami equation,

$$f^\nu_{\bar z} = \mu_\nu(z) f^\nu_z \tag{31}$$

We look for a C^1–solution of (31) in the form

$$f^\nu_z(z) = e^{\sigma(z)}, \qquad f^\nu_{\bar z}(z) = \mu_\nu(z) e^{\sigma(z)} \tag{32}$$

where $\sigma \in W^{1,p_\nu}(\mathbb{C},\mathbb{C})$, for some $p_\nu > 2$, is compactly supported. The necessary and sufficient condition for σ is that

$$(e^\sigma)_{\overline{z}} = (\mu_\nu e^\sigma)_z$$

or equivalently

$$\sigma_{\overline{z}} = \mu_\nu \sigma_z + (\mu_\nu)_z \tag{33}$$

This equation is uniquely solved using the Beurling–Ahlfors transform, that is the singular integral operator defined by

$$(Sg)(z) = -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(\xi)|d\xi \wedge d\overline{\xi}}{(z-\xi)^2}$$

Note that $\sigma_z = S\sigma_{\overline{z}}$ and so equation (33) reduces to

$$(\mathbf{I} - \mu_\nu S)\sigma_{\overline{z}} = (\mu_\nu)_z \tag{34}$$

As $\|S\|_2 = 1$ and as $\|\mu_\nu\|_\infty < 1$, there is $p_\nu > 2$ such that

$$\|\mu_\nu\|_\infty \|S\|_{p_\nu} < 1$$

In this case the operator $\mathbf{I} - \mu_\nu S$ has a continuous inverse. Thus

$$\sigma_{\overline{z}} = (\mathbf{I} - \mu_\nu S)^{-1}(\mu_\nu)_z \in L^{p_\nu}(\mathbb{C}) \tag{35}$$

and also

$$\sigma_z = S\sigma_{\overline{z}} \in L^{p_\nu}(\mathbb{C}) \tag{36}$$

Note that $\sigma_{\overline{z}}$ vanishes outside the support of μ_ν which is contained in $\mathbf{B}(0,2)$. Also $\sigma_z = S\sigma_{\overline{z}} = O(z^{-2})$ as $z \to \infty$. Thus $\sigma(z) \approx \frac{C}{z}$ asymptotically, for a suitable constant C. In fact

$$\sigma(z) = (T\sigma_{\overline{z}})(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\sigma_{\overline{z}}(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} \tag{37}$$

where T is the complex Riesz Potential. Hence σ is Hölder continuous with exponent $1 - \frac{2}{p_\nu}$, by the Sobolev Imbedding Theorem.

Now the solution f^ν of equation (32) is unique up to a constant as $f^\nu_{\overline{z}} = 0$ outside $2\mathbf{B}$ and as $f^\nu_z - 1 \in L^{p_\nu}(\mathbb{C})$. That is, f^ν is a principal solution to the Beltrami equation (31). It is important to realise here that the Jacobian of f^ν is strictly positive,

$$J(z, f^\nu) = |f^\nu_{\overline{z}}|^2 - |f^\nu_z|^2 = (1 - |\mu_\nu|^2)e^{2\sigma} > 0 \tag{38}$$

The Implicit Function Theorem tells us that f^ν is locally one-to-one. Another observation to make is that $\lim_{z\to\infty} f^\nu(z) = \infty$. It is an elementary topological exercise to show that $f^\nu : \mathbb{C} \to \mathbb{C}$ is a global homeomorphism of $\mathbb{C}$. It's inverse is C^1–smooth of course.

We now digress for a second to outline the existence proof in the classical setting where $K(z) \leq K < \infty$. As the sequence K_ν is uniformly bounded we find there is an exponent $p = p(K) > 2$ such that

$$\|f^\nu_{\bar z}\|_p + \|f^\nu_z - 1\|_p \leq C_K \tag{39}$$

where C_K is a constant independent of ν. Hence the Sobolev Imbedding Theorem yields the uniform bound

$$|f^\nu(a) - f^\nu(b)| \leq C_K |a-b|^{1-\frac{2}{p}} + |a-b| \tag{40}$$

The same inequality holds for the inverse map and hence

$$|f^\nu(a) - f^\nu(b)| \geq \frac{|a-b|^{\frac{p}{p-2}}}{C_K + |a-b|^{\frac{2}{p-2}}} \tag{41}$$

We may assume that $f^\nu(0) = 0$. As the p–norms of $f^\nu_{\bar z}$ and $f^\nu_z - 1$ are uniformly bounded, we may assume that each converges weakly in $L^p(\mathbb{C})$ after possibly passing to a subsequence. From the uniform continuity estimates and Ascoli's Theorem, we may further assume $f^\nu \to f$ locally uniformly in $\mathbb{C}$. Obviously f satisfies the same modulus of continuity estimates and is therefore a homeomorphism. Moreover, it follows that the weak limits of $f^\nu_{\bar z}$ and $f^\nu_z - 1$ must in fact be equal to $f_{\bar z}$ and $f_z - 1$ respectively. Hence f is a homeomorphism in the Sobolev class $z + W^{1,p}(\mathbb{C})$, that is $f_{\bar z}$ and $f_z - 1$ in $L^p(\mathbb{C})$. Finally observe that $\mu_\nu \to \mu$ pointwise almost everywhere, and hence in $L^q(\mathbb{C})$, where q is the Hölder conjugate of p. The weak convergence of the derivatives shows that f is a solution to the Beltrami equation.

Back to the more general setting. If we followed the above argument we find the L^{p_ν} bounds are useless as we cannot keep them uniform. We therefore seek an alternative route via a Sobolev-Orlicz class where uniform bounds might be available. We note the elementary inequality

$$(|u| + |v|)^2 \leq 2K(|u|^2 - |v|^2) + 4K^2|v - w|^2 \tag{42}$$

whenever u, v, w are complex numbers such that $|w| \leq \frac{K-1}{K+1}|u|$ and $K \geq 1$. We apply this inequality pointwise with

$$u = \phi^\nu_z, \quad v = \phi^\nu_{\bar z}, \quad w = \mu_\nu \phi^\nu_z$$

and $K = \mathcal{K}_\nu(z)$ as defined at (28), where

$$\phi^\nu(z) = f^\nu(z) - z \in W^{1,2}(\mathbb{C}), \quad K = \mathcal{K}_\nu(z) \tag{43}$$

and use equations (31), (29) we can write

$$(|\phi^\nu_z| + |\phi^\nu_{\bar z}|)^2 \leq 2\mathcal{K}_\nu(|\phi^\nu_z|^2 - |\phi^\nu_{\bar z}|^2) + 4(\mathcal{K}_\nu)^2|\mu_\nu|^2$$

and hence

$$|D\phi^\nu(z)|^2 \leq 2\mathcal{K}_\nu J(z,\phi^\nu) + 4|\mu_\nu \mathcal{K}_\nu|^2 \tag{44}$$

Next we integrate this and use Theorems 12.1 and 12.3 to obtain

$$\begin{aligned}
\int_{\mathbb{C}} |D\phi^\nu|^2 &\leq 2C_3\|\mathcal{K}_\nu\|_{BMO}\|J(z,\phi^\nu)\|_{H^1} + 4\int_{2\mathbf{B}} |\mathcal{K}_\nu|^2 \\
&\leq \frac{4C_1C_2C_3}{p}\int_{\mathbb{C}} |D\phi^\nu|^2 + 4C_4\log^2(1+A_p)
\end{aligned}$$

where in the latter step we have used the uniform bounds at (29) and (30).

It is clear at this point why we have chosen $p_0 = 8C_1C_2C_3$ at (25). The term $\int_{\mathbb{C}} |D\phi^\nu|^2$ in the right hand side can be absorbed in the left hand side. After doing this we obtain the uniform bounds in L^2

$$\int_{\mathbb{C}} |D\phi^\nu|^2 \leq 8C_4\log^2(1+A_p) \tag{45}$$

which read as

$$\|D\phi^\nu\|_{L^2(\mathbb{C})} \leq C_5 \log\left(\int_{\mathbf{B}} e^{pK}\right) \tag{46}$$

and in turn leaves us with the local estimate for the mapping $f^\nu(z) = \phi^\nu(z) + z$, namely

$$\|Df^\nu\|_{L^2(\mathbf{B}_R)} \leq C_5\left[R + \log\left(\int_{\mathbf{B}} e^{pK}\right)\right] \tag{47}$$

where $\mathbf{B}_R = \mathbf{B}(0,R)$. As f^ν is monotone (being a C^1 homeomorphism) we can apply the modulus of continuity estimate of Theorem 5.2,

$$|f^\nu(a) - f^\nu(b)| \leq C_6 \frac{R + \log\left(\int_{\mathbf{B}} e^{pK}\right)}{\log^{\frac{1}{2}}\left(e + \frac{R}{|a-b|}\right)} \tag{48}$$

for all $a, b \in \mathbf{B}_R$.

Now consider the inverse map to f^ν. Let us denote it by $h^\nu = (f^\nu)^{-1} : \mathbb{C} \to \mathbb{C}$. As both f^ν and h^ν are smooth diffeomorphisms we find

$$\begin{aligned}\int_{\mathbf{B}_R} |Dh^\nu(w)|^2\, dw &= \int_{\mathbf{B}_R} K(w,h^\nu)J(w,h^\nu)\, dw \\ &= \int_{h^\nu(\mathbf{B}_R)} K(z,f^\nu)\, dz \\ &\leq \int_{\mathbb{C}} (K(z)-1)\, dz + |h^\nu(\mathbf{B}_R)| \\ &\leq C_pR^2 + \int_{\mathbf{B}} (K(z)-1)\, dz\end{aligned}$$

In this last inequality we have put in the uniform bound $|h^\nu(\mathbf{B}_R)| \leq C_pR^2$. One interesting way to see this estimate (though perhaps not the easiest) is via the Koebe distortion theorem. Anyway, we have

$$\int_{\mathbf{B}(0,R)} |Dh^\nu|^2 \leq C\,(R^2 + \int_{\mathbf{B}} K) \tag{49}$$

and consequently we have the continuity estimate at (5.2) for h^ν,

$$|h^\nu(x) - h^\nu(y)|^2 \leq \frac{CR^2 \int_{\mathbf{B}} K}{\log\left(e + \frac{R}{|x-y|}\right)} \tag{50}$$

For f^ν this reads as

$$|f^\nu(a) - f^\nu(b)| \geq R\exp\left(\frac{-CR^2\int_{\mathbf{B}} K}{|a-b|^2}\right) \tag{51}$$

whenever $a, b \in \mathbf{B}(0,R)$ and $R \geq 1$. The uniform $W^{1,2}$ bounds, and the continuity estimates from above and below now enable us to pass to the limit. We find $f^\nu \to f$ and $h^\nu \to h = f^{-1}$ locally uniformly in $\mathbb{C}$ and Df^ν and Dh^ν converging weakly in $L^2_{loc}(\mathbb{C})$. As in the classical setting this implies that f is a homeomorphic solution to the Beltrami equation. Moreover $f_{\overline{z}}, f_z - 1 \in L^2(\mathbb{C})$ and the same is true of the inverse function

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