THE HEAT-FLOW METHOD IN CONTACT GEOMETRY

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Contact geometry treats such questions as the existence and classification of contact structures on manifolds of odd dimension and specified topological structure. See inequality (1) below. The geometric/analytic approach treated in this report introduces parabolic systems of partial differential equations (PDEs) in a way which complements the more algebraic methods, which until now are better known in contact geometry.

This is a report on joint work in progress with Hansjörg Geiges of the University of Leiden, Netherlands and Matthias Schwarz of the University of Leipzig, Germany. Many of the specific results reported on here appeared first in a paper [1] by Steve Altschuler, which introduced the heat-flow method to study contact structures, and in a recent preprint [2] of Altschuler and Lani Wu.

1. Introduction to Contact Geometry

Many of the participants in this conference apply analytical methods to geometrically motivated problems, or use geometric methods to strengthen their analysis. However, it cannot be assumed that everyone is familiar with all of the most modern concepts and techniques of differential geometry. For that reason, this section will be devoted to an introduction to contact geometry appropriate for analysts, among others, and may be skipped by those with a good knowledge of the area. I was until rather recently a complete novice in this area of geometry, and the reader should not expect a polished nor absolutely concise presentation. See [4], [5] and [6] for more complete references to the literature. I expect that analysts will be interested to see this novel application of parabolic operators.

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A *hyperplane distribution* ξ in an open set M of $\mathbb{R}^{2n+1}$, or in a smooth $(2n+1)$-dimensional manifold M, specifies a subspace ξ_x of dimension $2n$ in $\mathbb{R}^{2n+1}$, or rather in the tangent space to M at each point $x \in M$, which depends smoothly on the point x.

1.1. **Example: a foliation.** A familiar example of a hyperplane distribution would be the two-dimensional distribution in $\mathbb{R}^3$ spanned by the vector fields $e_1(x) = (1, 0, 2x_1)$ and $e_2(x) = (0, 1, 2x_2)$. Here we have written $x = (x_1, x_2, x_3)$. This distribution is especially easy to visualize, since $e_1(x)$ and $e_2(x)$ are a basis for tangent vectors to the family of paraboloids of revolution $x_3 - x_1^2 - x_2^2 = C$, for various real constants C. This family of surfaces is a *foliation* of $\mathbb{R}^3$, which means that every point of $\mathbb{R}^3$ lies on one of the surfaces, the surfaces and the family are smooth, and in some neighborhood of any point, the family looks like the family of coordinate planes $x_3 = \text{const}.$, up to a local diffeomorphism. In this situation, we say that the distribution is *integrable,* meaning in this case, where the first and second components of e_1 and e_2 are $(1, 0)$ and $(0, 1)$, that their third components $2x_1$ and $2x_2$ are simultaneously the partial derivatives of a scalar function, locally. (The scalar function is $x_1^2 + x_2^2 + C$, of course.) Integrability is equivalent to saying that for any two vector fields V, W in ξ, the Lie bracket $[V, W]$ also lies in ξ. Alternatively, we may describe a hyperplane distribution as the kernel of a nowhere vanishing differential 1-form α. (A 1-form is the dual of a vector field, so that for any vector field V, $\alpha(V)$ defines a scalar function and depends linearly and pointwise on V.) Given ξ, the 1-form α is determined up to a nonvanishing scalar factor by the requirement that $\alpha(e_1) = \alpha(e_2) = 0$, where e_1, e_2 form a local basis for the distribution ξ. (Computationally, α has the same components as the cross product of e_1 and e_2.) The integrability condition for the distribution ξ may be written in terms of the 1-form α as an identity between 3-forms: $\alpha \wedge d\alpha = 0$. (The exterior derivative $d\alpha$ of a 1-form α is the 2-form defined by the alternating part of the matrix of first partial derivatives; the wedge product of differential forms is the alternating part of their tensor product.)

A *contact structure* is a hyperplane distribution which is *maximally non-integrable.* In terms of Lie brackets, we may write $\omega(V, W)$ for the transversal component $\alpha([V, W])$ of the Lie bracket of two vector fields V, W in ξ. This makes ω a 2-form. The integrability condition requires that $\omega \equiv 0$; for ξ to be a contact structure, we require not merely that $\omega \neq 0$ but far more: that the $2n$-form $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$ be nowhere zero on ξ. Via the appropriate Riemannian metric, this is equivalent to saying that ω defines an almost-complex structure on the hyperplane distribution ξ. Restricted to $\xi = \ker \alpha$, ω is the same as $-d\alpha$. Thus, the contact criterion may be written entirely in terms of the 1-form α :

$$\alpha \wedge d\alpha^n \neq 0. \tag{1}$$

Note that inequality (1) depends on ξ but is independent of the choice of 1-form α, since if $\widetilde{\alpha} = f\alpha$ for some nonvanishing scalar function f, then $\widetilde{\alpha} \wedge d\widetilde{\alpha}^n \equiv f^{n+1}\alpha \wedge d\alpha^n$. Note also that since $d\alpha$ is a two-form, the left-hand side of (1) is a differential form of degree $2n+1$, so on $\mathbb{R}^{2n+1}$ or M^{2n+1}, it has only one real component. In this sense, contact structures and contact forms only have their full meaning in domains and manifolds of *odd* dimension. A 1-form α on a $(2n+1)$-manifold which satisfies inequality (1) is called a *contact form.*

Inequality (1) is unusual, in the context of geometric analysis, for two reasons: it is a strict partial differential *inequality,* and it is an underdetermined "system" consisting of one real, first-order, fully nonlinear partial differential inequality for the $2n+1$ real components $a_i(x)$ of the 1-form α. Specifically, in the 5-dimensional case $n = 2$, we may write α in local coordinates $(x_0, \dots, x_4)$ as

$$\alpha = \sum_{i=0}^{4} a_i(x)\, dx_i.$$

Then the contact inequality (1) is equivalent to the inequality

$$\sum_{\sigma} \operatorname{sgn}(\sigma)\, a_{\sigma(0)} \frac{\partial a_{\sigma(1)}}{\partial x_{\sigma(3)}} \frac{\partial a_{\sigma(2)}}{\partial x_{\sigma(4)}} \neq 0,$$

where the sum is over all permutations σ of $\{0, 1, 2, 3, 4\}$. Systems of partial differential equations of this general form are rather poorly understood at present. In the case of contact geometry, however, we shall see that there is a parabolic method available to attack inequality (1); see Section 2 below.

1.2. **Example: the standard contact structure.** A familiar example of a contact structure would be the two-plane distribution ξ in $\mathbb{R}^3$ with the subspace ξ_x at the point $x = (x_1, x_2, x_3)$ having basis vector fields $e_1(x) = (x_1, x_2, 0)$ and $e_2(x) = (-x_2, x_1, r^2)$, where we have written $r^2 = x_1^2 + x_2^2$. In order to visualize ξ, we note that e_1 is the horizontal vector field pointing away from the x_3-axis, and that e_2 is a vector orthogonal to e_1 and with slope r, as measured from the (x_1, x_2)-plane. Then the distribution ξ is not a foliation, which may be seen as follows. Suppose $(x_1(t), x_2(t))$, $0 \leq t \leq T$, describes a closed curve in the (x_1, x_2)-plane. Since ξ_x is never vertical, there is a unique way to lift this curve to a curve $x(t) = (x_1(t), x_2(t), x_3(t))$ in $\mathbb{R}^3$, so that the tangent vector $x'(t)$ is always in the distribution $\xi_{x(t)}$. If ξ were integrable, then the space curve would stay on the same surface of the foliation, and would therefore be a closed curve. However, to be specific, suppose that the plane curve $(x_1(t), x_2(t))$ describes the boundary $\partial\Omega$, in the positive sense, of $\Omega =$ one-fourth of an annulus: in polar coordinates (r, θ), Ω is given by $a < r < b$, $0 < \theta < \pi/2$. Then along each of the two straight sides $\theta \equiv 0$, $\theta \equiv \pi/2$, the tangent vector lifts to a multiple of $e_1 = (x_1, x_2, 0)$, so $x_3(t)$ remains constant. But along the quarter-circle $r \equiv b$, $0 < \theta < \pi/2$, the tangent vector lifts to a multiple of $e_2 = (-x_2, x_1, r^2)$, so $x_3(t)$ increases by the slope times the length in the plane $= \pi b^2/2$. Returning along the quarter-circle $r \equiv a$, as θ decreases from $\pi/2$ to 0, $x_3(t)$ decreases by $\pi a^2/2$. Thus, the change in $x_3(t)$ as t increases from 0 to T is $\pi(b^2 - a^2)/2$, which is exactly twice the area of the quarter-annulus Ω.

In fact, for any domain Ω in the (x_1, x_2)-plane, the change in $x_3(t)$ as $(x_1(t), x_2(t))$ describes $\partial\Omega$ equals twice the area of Ω. This may be seen by computing a form α_0 so that $\xi = \ker \alpha_0$:

$$\alpha_0 = x_2\, dx_1 - x_1\, dx_2 + dx_3.$$

Since $x'(t)$ is in the distribution $\xi_{x(t)}$, we get $\alpha_0(x'(t)) = 0$, which means that $x_3'(t) = -x_2(t)x_1'(t) + x_1(t)x_2'(t)$; hence the change $x_3(T) - x_3(0)$ in the height as $(x_1(t), x_2(t))$ goes around $\partial\Omega$ equals the integral around $\partial\Omega$ of $-x_2\, dx_1 + x_1\, dx_2$, which is twice the area of Ω.

The 1-form α_0 is the *standard contact form* on $\mathbb{R}^3$, and ξ is the *standard contact structure.* More precisely, this is the rotationally symmetric version; the contact form $x_2\, dx_1 + dx_3$ is translationally invariant in two coordinate directions, and is also known as “the” standard contact form. In higher dimensions, the standard contact form on $\mathbb{R}^{2n+1}$ is

$$\alpha_0 = dx_0 + \sum_{k=1}^{n} (x_{2k}\, dx_{2k-1} - x_{2k-1}\, dx_{2k}), \tag{2}$$

which is invariant under the $(n+1)$-dimensional group generated by rotation in the
(x_{2k-1}, x_{2k})-plane, $1 \le k \le n$, plus translation along the x_0-axis.

A natural question is: when are two contact forms equivalent? The local version of this question has a surprisingly simple answer:

Theorem 1.1. (Darboux): *Let α be a contact form on a neighborhood of x in $\mathbb{R}^{2n+1}$. Then on a smaller neighborhood of x, there is a diffeomorphism into $\mathbb{R}^{2n+1}$ such that α is mapped to the standard contact form α_0.*

Darboux' Theorem may be interpreted as saying that the contact condition (1) is a very "soft" condition, as compared to the familiar partial differential *equations* traditionally treated by geometric analysts. This softness is apparent from the recent work of Gromov, Eliashberg and others on noncompact manifolds, which showed, for example, that any noncompact, odd-dimensional manifold which has a hyperplane distribution with an almost-complex structure also carries a contact structure (see [8] and references therein.)

1.3. **Global non-uniqueness: the Lutz Twist.** Since contact structures are locally unique, it might seem reasonable to think that a topologically simple space like $\mathbb{R}^3$ has only one contact structure up to a change of coordinates. However, there are subtle criteria which distinguish other contact structures on $\mathbb{R}^3$ from the standard α_0.

Recall the description in subsection 1.2 of basis vector fields e_1, e_2 for the standard contact structure on $\mathbb{R}^3$: e_2 is orthogonal to the radial vector e_1, and has slope r, which means that it makes an angle $\varphi = \arctan r$ with the (x_1, x_2)-plane. As $r \to \infty$, e_2 becomes vertical, so $\varphi \to \pi/2$. Instead, suppose that $\varphi = \varphi(r)$ increases beyond $\pi/2$ to make one or more revolutions before slowly approaching $\arctan r + 2\pi m$ ($m \in \mathbb{Z}$) as $r \to R < \infty$. Outside the cylinder $r < R$, the contact structure may be continued smoothly, to join up with the standard contact structure. This construction is known as the *Lutz twist* (see [9].)

In terms of the contact form, in cylindrical coordinates (r, θ, x_3), $\alpha_0 = dx_3 - r^2\, d\theta$ is replaced by $\alpha = h_0(r)dx_3 - h_1(r)\, d\theta$ for some functions $h_0, h_1 : [0, \infty) \to \mathbb{R}$ with $h_1'h_0 - h_1h_0' > 0$, and with $h_0(r) = 1, h_1(r) = r^2$ for all $r \geq R$. Then h_1 and h_0 are related to the angle φ by $rh_0(r)\tan\varphi(r) = h_1(r)$.

This new contact structure is *overtwisted,* that is, there is a topological disk $D \subset \mathbb{R}^3$ with $\alpha|_D$ nowhere zero along ∂D and $\alpha|_{\partial D} \equiv 0$. In fact, let r_0 be the first value of r with $\varphi(r_0) = \pi$. Then we may choose $D = \{(r, \theta, x_3) : x_3 = r_0^2 - r^2, 0 \leq \theta \leq 2\pi, 0 \leq r \leq r_0\}$. It may be shown that no such disk exists in $\mathbb{R}^3$ with the standard contact structure.

1.4. **Compact Manifolds.** What about *compact* manifolds? The only known obstruction to the existence of an orientable contact structure on an oriented, odd-dimensional manifold M^{2n+1} is the requirement that some hyperplane distribution on M should have an almost-complex structure; this can be written as a topological condition on M, that the even-dimensional Stiefel-Whitney classes w_{2i} (certain natural cohomology classes with $\mathbb{Z}/2\mathbb{Z}$ coefficients) are in the image of cohomology with integer coefficients. However, there are many manifolds which satisfy this condition but have not been shown to carry a contact structure. Specifically, one would like to know *whether there is a contact structure on the odd-dimensional torus* T^{2n+1}.

We shall assume for the rest of this paper that manifolds are **compact, oriented** and have **no boundary.** A readily visualized example is the interesting case of the torus T^{2n+1}, which is just the cube $[-\pi, \pi]^{2n+1} \subset \mathbb{R}^{2n+1}$ after opposite faces have been identified.

A contact form may be found on the three-torus T^3 as the first case of a classical construction. Begin on the *two*-dimensional torus T^2; introduce local coordinates (q_0, q_1) on T^2 and then extend these coordinates to the 4-dimensional phase space, or cotangent bundle, $T^*(T^2)$. Then a cotangent vector η_x at the point $x = (q_0, q_1)$ has components (p_0, p_1), meaning that $\eta_x = p_0\, dq_0 + p_1\, dq_1$. (In certain applications, (q_0, q_1) are coordinates of position and (p_0, p_1) are components of the momentum vector.) Then $\omega = dp_0 \wedge dq_0 + dp_1 \wedge dq_1$ is the natural symplectic form on phase space $T^*(T^2)$. One notes that ω is the exterior derivative $d\alpha$, where α is the canonical 1-form $p_0\, dq_0 + p_1\, dq_1$ on phase space. When α is restricted to the unit-sphere bundle $M^3 := \{(x,p) : x \in T^2, p \in T^*_x(T^2), |p|^2 = 1\}$, it satisfies the contact condition (1). Here $|p|^2 = p_0^2 + p_1^2$. The verification of inequality (1) reduces to showing that $p_0\, \partial|p|^2/\partial p_0 + p_1\, \partial|p|^2/\partial p_1 \neq 0$ on M. Meanwhile, on T^2, there is a global basis of tangent vector fields, which implies that the unit sphere bundle M^3 of T^2 is $T^2 \times S^1 = T^3$. In coordinates (q_0, q_1, θ) for $T^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, we have $\alpha = \cos\theta\, dq_0 + \sin\theta\, dq_1$. This is the most natural construction for a contact structure on T^3.

The construction above generalizes to higher dimensions. Let N be an oriented $(n+1)$-dimensional manifold, equipped with a Riemannian metric, and introduce local coordinates $(q_0, \ldots, q_n, p_0, \ldots, p_n)$ for the cotangent bundle T^*N of N, where $(q_0, \ldots, q_n)$ are local coordinates on N and a cotangent vector is represented as $\sum_{i=0}^n p_i\, dq_i$. Let α be the canonical 1-form $\sum_{i=0}^n p_i\, dq_i$. When α is restricted to the unit-sphere bundle M^{2n+1}, defined as $\left\{(q,p) : q \in N, p \in T^*_q N, |p|^2 = 1\right\}$, it satisfies the contact condition (1). That is, the unit sphere of the cotangent bundle of any manifold carries a natural contact structure. This is how contact structures arise naturally, on suitable energy surfaces in Hamiltonian systems.

When one applies the same construction to $N = T^3$, $n = 2$, one finds a contact 1-form α_1 on the 5-dimensional unit sphere bundle of T^*N. But the unit sphere bundle M^5 is now $T^3 \times S^2$, not T^5. However, T^5 can still be given a contact structure, as was first shown by Lutz [9]. Another way to find a contact structure on T^5 is to apply the following result of Gromov (see [8] and [5], p. 456):

Theorem 1.2. : *If M_2 is a branched covering of M_1, branched along a codimension-2 submanifold Σ of M_1, and if M_1 has a contact form α_1 whose restriction to Σ also makes Σ into a contact manifold, then M_2 has a contact form α close to the pullback of α_1.*

In our case, T^2 may be written as a branched double cover $F : T^2 \to S^2$ of the sphere, branched simply over four points of S^2, which we may assume are the four equidistant points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ along the equator $\{p_2 = 0\}$ of $S^2 \subset \mathbb{R}^3$.

We construct a branched covering $\widetilde{F} : M_2 \to M_1$ from $M_2 = T^5 = T^3 \times T^2$ to $M_1 = T^3 \times S^2$, by twisting F, as follows. Let $q = (q_0, q_1, q_2)$ be coordinates for $T^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, and let $p = (p_0, p_1, p_2)$ be coordinates for S^2, where $p_0^2 + p_1^2 + p_2^2 = 1$. For each $q_2 \in \mathbb{R}/2\pi\mathbb{Z}$, let $\Phi(q_2) : S^2 \to S^2$ be the rotation in the (p_0, p_1)-plane by angle q_2, leaving p_2 fixed. Then $\widetilde{F} : M_2 \to M_1$ is defined by $\widetilde{F}(q_0, q_1, q_2, z) := (q_0, q_1, q_2, \Phi(q_2)(F(z)))$. $\widetilde{F} : M_2 \to M_1$ is a branched covering, with branch locus

$$\Sigma = \{(q, p) \in T^3 \times S^2 : p_0 = \cos(q_2 + k\pi/2),$$
$$p_1 = \sin(q_2 + k\pi/2),\ p_2 = 0,\ k \in \mathbb{Z}\}.$$

Σ has four connected components Σ_k, $k = 0, 1, 2, 3$, each of which projects diffeomorphically onto the the T^3 factor of M_1.

Write α_1 for the canonical contact 1-form $\sum_{i=0}^{2} p_i\, dq_i$ on M_1, viewed as the unit cotangent bundle of T^3. On each component Σ_k of Σ, we have $\alpha_1 \mid_{\Sigma_k} = \cos(q_2 + k\pi/2)\, dq_0 + \sin(q_2 + k\pi/2)\, dq_1$. We compute $(\alpha_1 \wedge d\alpha_1) \mid_{\Sigma_k} = -dq_0 \wedge dq_1 \wedge dq_2$, $k = 0, 1, 2, 3$, which shows that Σ is a (disconnected) contact 3-manifold with contact form $\alpha_1 \mid_\Sigma$.

We may now apply Theorem 1.2 to find a contact form α on $M_2 = T^5$ which is close to the pullback of α_1. Thus, the 5-torus T^5 has a contact structure.

The existence of a contact structure on the 7-torus, and on numerous higher-dimensional manifolds, was unclear until now.

2. The Heat Flow

Recall that we are assuming that manifolds are compact, connected, oriented and have no boundary. In addition, we will assume that a Riemannian metric has been chosen.

A property of parabolic PDEs familiar to analysts is the *strong maximum principle:* if the solution $f(t, x)$ satisfies at the initial time $f(0, x) \geq 0$ but $f(0, x) \not\equiv 0$, then at time $t > 0$, $f(t, x)$ will be positive everywhere. That is, heat flows instantaneously to warm a connected domain. This property makes parabolic methods ideal for the study of strict inequalities such as the contact inequality (1). The idea is to use a hands-on construction to make $f(0,x) > 0$ for x in an appropriate, possibly quite small, open set, while $f(0,\cdot) \geq 0$ everywhere, and then to replace $f(0,x)$ with the strictly positive solution $f(t,x)$ at some small positive time t. Since $f(t,x)$ is close to $f(0,x)$ in certain strong norms, other relevant conditions will be maintained.

Altschuler in [1] considers any orientable compact 3-manifold M: he combines the Lutz twist with the strong maximum principle to construct a contact form on M. (The result was proved using entirely different methods in [10]; see also [4].) Altschuler's technique is to start with a foliation, or equivalently with a 1-form α_1 satisfying $\alpha_1 \wedge d\alpha_1 \equiv 0$, and then to use the Lutz twist to construct a 1-form α_2 satisfying $\alpha_2 \wedge d\alpha_2 > 0$ on a certain open set U, with $\alpha_2 = \alpha_1$ near ∂U. The resulting 1-form α_2 on all of M satisfies $\alpha_2 \wedge d\alpha_2 \geq 0$; such 1-forms have been called *confoliations* by Eliashberg and Thurston [4]. Altschuler then defines a degenerate parabolic system of equations for a 1-form $\alpha(t,x)$, $0 < t < \varepsilon$, $x \in M$, and uses α_2 as the initial condition at time $t = 0$. The system of PDEs is chosen so that the scalar quantity $f(t,x) := *(\alpha \wedge d\alpha)$, which is initially nonnegative everywhere and strictly positive on U, becomes everywhere positive for small $t > 0$. One difficulty is that the system of PDEs is *degenerate* parabolic, so that "heat" will flow reliably only in certain directions. Altschuler defines the system of equations so that heat flows in directions tangent to $\ker \alpha_2$, which is the original foliation $\ker \alpha_1$ on the more troublesome set $M \backslash U$, and ensures that the Lutz twist was carried out so that the open set U meets each leaf of $\ker \alpha_1$.

A nonlinear version of the system of equations Altschuler uses on a 3-manifold is

$$\frac{\partial \alpha}{\partial t} = *\left(\alpha \wedge df\right), \text{where } f(t,x) = *(\alpha \wedge d\alpha). \tag{3}$$

Here, for a p-form β on an oriented Riemannian $(2n+1)$-manifold, $*\beta$ is a $(2n+1-p)$-form, the *Hodge star* of β, which depends linearly on β and is defined at each point so that for any oriented orthonormal coframe $\theta_0, \ldots, \theta_{2n}$ of 1-forms, $*(\theta_p \wedge \ldots \wedge \theta_{2n}) = \theta_0 \wedge \ldots \wedge \theta_{p-1}$. The system (3) appears quite complicated, but it may be dealt with successfully by the following trick. The real-valued function $f(t,x)$ satisfies a single degenerate parabolic PDE:

$$\frac{\partial f}{\partial t} = *\left(\alpha \wedge d *\left(\alpha \wedge df\right)\right) + \langle \alpha \wedge df, d\alpha \rangle. \tag{4}$$

Thus, the system (3) uncouples weakly, in the sense that α appears in the PDE (4) only as a coefficient. Once $f(t,x)$ is determined, the equation (3) for $\alpha(t,x)$ becomes a parameterized system of ODEs. Of course, the unknown 1-form α also appears in the coefficients of (4), so this version of Altschuler's method succeeds by requiring t to remain small, implying that $\alpha(t,\cdot)$ is close to the initial 1-form α_2.

More generally, on a $(2n+1)$-manifold, choose a $(2n-1)$-form Ω, and consider the system of equations

$$\frac{\partial\alpha}{\partial t} = *\left(\Omega\wedge df\right), \text{where } f(t,x) := *(\Omega\wedge d\alpha). \tag{5}$$

The PDE satisfied by f is now

$$\frac{\partial f}{\partial t} = *\left(\Omega\wedge d*\left(\Omega\wedge df\right)\right) + *\left(\frac{\partial\Omega}{\partial t}\wedge d\alpha\right). \tag{6}$$

Again, the system (5) uncouples weakly. Further, we have

Proposition 2.1. : *Equation (6) is a weakly parabolic PDE, a degenerate heat equation, where the right-hand side defines a second-order partial differential operator, which is strongly elliptic when restricted to the distribution $H \subset TM$ given by*

$$H = (\ker(*\Omega))^{\perp}.$$

Recall that for a 2-form β on M, $\ker\beta_x := \{\, v \in T_xM \mid \beta(v,\cdot) = 0 \text{ on } T_xM\,\}$. The coefficients of the principal part of the PDE (6) at (t,x) are A^TA, where the skew-symmetric matrix A represents $*\Omega_x$ in coordinates which are orthonormal at x; the subspace H_x is spanned by the columns of A^TA.

In the nonlinear version of Altschuler's heat flow on a 3-manifold, as we have seen, one chooses $\Omega = \alpha$, the evolving 1-form itself. We would like to apply Proposition 2.1 in this case. At a given small time $t > 0$, the 1-form $\alpha(t,\cdot)$ never vanishes, so we may complete to a local orthonormal basis $(\theta_1,\theta_2,\theta_3)$ with α a nonvanishing scalar multiple of θ_1. We compute $\ker(*\alpha) = \ker(*\theta_1) = \ker(\theta_2\wedge\theta_3) = \mathbb{R}e_1$, and therefore $H = (\ker(*\alpha))^{\perp} = \mathbb{R}e_2 + \mathbb{R}e_3 = \ker\alpha$. In particular, for small positive t, the distribution H is close to $\ker\alpha_2$. Thus, if α_2 is a contact form on an open set U which is a neighborhood of some point on each leaf of the original foliation $\ker\alpha_1$, then heat will flow out of U to warm each point of M.

In general, one may show that

Lemma 2.2. : *If Ω is locally decomposable as a product of 1-forms, then*

$$H := (\ker(*\Omega))^{\perp} = \ker\Omega.$$

3. The Higher Lutz Twist

For higher dimensions $2n + 1$, in the recent paper of Altschuler and Wu [2], the degenerate parabolic system (5) is studied with the choice $\Omega = \alpha \wedge (d\alpha)^{n-1}$. They show the existence of a smooth solution for all positive time, via a parabolic regularization. (More precisely, in [2] Altschuler and Wu consider a partly linearized degenerate parabolic system which is easier to analyze than equation (5), and slightly more complicated, but has consequences equivalent to those of equation (5).) The PDE (6) now becomes

$$\frac{\partial f}{\partial t} = n * (\Omega \wedge d * (\Omega \wedge df)) + \langle \Omega \wedge df, (d\alpha)^n \rangle. \tag{7}$$

Thus, the system of equations uncouples in the same sense as in the 3-dimensional case $n = 1$ (compare equations (3) and (4).)

Another part of their paper carries out a higher analogue of the Lutz twist for the five-dimensional product case $M^5 = N^3 \times F^2$, using a contact structure on the 3-dimensional manifold N and its parallelizability. They are thereby able to prove that every product 5-manifold of this form carries a contact structure. Incidentally, this gives another construction of a contact form on the 5-torus T^5.

Let us proceed in an analogous, but in applications rather different, fashion. Consider a $(2n+1)$-manifold $M^{2n+1} = N^{2n-1} \times F^2$ which is the product of a contact $(2n-1)$-manifold (N, α_N) and an oriented surface F. We shall write α_1 for the 1-form on $M = N \times F$ pulled back from α_N. For simplicity, assume that (N, α_N) has a closed *Reeb orbit* γ. This means that $\gamma'(s) \neq 0$ and that $d\alpha(\gamma'(s), v) = 0$ for all parameter values s along the curve γ and for all vectors $v \in T_{\gamma(s)}M$. Then, according to an extension of Darboux' Theorem 1.1, in some neighborhood W of γ in N, there are multipolar coordinates $(z, r_1, \theta_1, \dots, r_{n-1}, \theta_{n-1})$, $r_1^2 + \dots + r_{n-1}^2 < R^2, \theta_k \in \mathbb{R} \mod 2\pi$, so that α_N is the standard contact form (2), which in these coordinates means that

$$\alpha_N = dz + \sum_{k=1}^{n-1} r_k^2 \, d\theta_k.$$

In a small ball $B \subset F^2$, let polar coordinates (r_n, θ_n) be chosen, $0 \leq r_n < R$, $\theta_n \in \mathbb{R} \mod 2\pi$. For some choice of real-valued functions $h_k(r_1, \dots, r_n)$, $0 \leq k \leq n$, define

$$\alpha_2 = h_0(r_1, \dots, r_n) \, dz + \sum_{k=1}^{n} h_k(r_1, \dots, r_n) \, d\theta_k \, . \tag{8}$$

Then α_2 will satisfy the contact inequality (1) in $U := W \times B$ provided

$$\frac{1}{r_1 \cdot \ldots \cdot r_n} \begin{vmatrix} h_0 & h_1 & \ldots & h_n \\ \partial_1 h_0 & \partial_1 h_1 & \ldots & \partial_1 h_n \\ \vdots & \vdots & & \vdots \\ \partial_n h_0 & \partial_n h_1 & \ldots & \partial_n h_n \end{vmatrix} > 0, \quad \text{for } r_i \geq 0\,. \tag{9}$$

Here the operator ∂_k denotes $\partial/\partial r_k$. Inequality (9) is equivalent to the orientation preserving local diffeomorphism property for the central projection of $(h_0, \ldots, h_n) \in \mathbb{R}^{n+1}$ to the sphere S^n. Observe that inequality (9) continues to hold when α_2 is multiplied by a positive scalar function.

Recall that we wish to carry out this higher Lutz twist on the open set U, but we need to construct α_2 on all of M. Therefore it will be necessary for the coefficients $h_k(r_1, \ldots, r_n)$ to satisfy boundary conditions on ∂U, so that the extension of α_2 to all of M by defining $\alpha_2 = \alpha_1$ on $M \backslash U$ will be smooth. However, only the oriented contact structure is important to us, which means that α_2 only needs to be defined modulo a (nonconstant) positive multiple. Specifically, there needs to hold on the boundary $h_k(r_1, \ldots, r_n) = r_k^2\, h_0(r_1, \ldots, r_n)$, $1 \leq k \leq n-1$, and $h_n(r_1, \ldots, r_n) = 0$, as well as inequality (9) in the interior of U. This requires us to find a mapping from the sector $V := \{(r_1, \ldots, r_n) \in (0, \infty)^n : r_1^2 + \ldots + r_n^2 < R^2\}$ to the sphere S^n which is a diffeomorphism of V with an open subset of S^n, having the following boundary values on ∂V. On the curved part of the boundary $\{r_1^2 + \ldots + r_n^2 = R^2\}$, we require $h_k = r_k^2\, h_0$, $1 \leq k \leq n-1$, and $h_n = 0$. For $1 \leq k \leq n$, on the face $\{r_k = 0\}$, we require $h_k = 0$, $1 \leq k \leq n$. In the five-dimensional case $n = 2$, this may be done using a conformal mapping from the quarter-disk V to the hemisphere of S^2 with a slit from an interior point to the equator removed. The boundary of the quarter-disk covers the slit twice and the equator once. For the general case $n \geq 2$, another more hands-on construction of the map from V into the hemisphere of S^n may be carried out.

Closed Reeb orbits may be rare for a given contact manifold (N, α_N), but the above procedure may be modified appropriately.

A covering argument may then be used to arrange disjoint open sets of M of the above form so that their projections from $M = N \times F$ to N cover all of N. For small time $t > 0$, the solution $\alpha(t, \cdot)$ will be close to the initial value α_2. On the complement of the union of the sets U where the higher Lutz twist has been carried out, we have $\alpha_2 = \alpha_1$. Write $\Omega_1 = \alpha_1 \wedge (d\alpha_1)^{n-1}$. Since α_1 is the pullback of the contact form α_N, we see that Ω_1 is the pullback of a volume form on N, and thus is decomposable as a product of 1-forms. It follows from

Lemma 2.2 that on the complement of the sets U, the distribution H_1 of elliptic directions for the system (5), with Ω replaced by Ω_1, equals $\ker \Omega_1$, which is the tangent plane TF to $\{y\} \times F^2$ in $TM = TN \times TF$. Therefore, for small time $t > 0$, the distribution H is close to the foliation TF. It follows by Proposition 2.1, for small time $t > 0$, that heat flows out of the union of open sets U along directions arbitrarily close to TF to warm all of $M = N \times F$. Numerous points omitted here, in part rather technical, will be treated in [7] to prove

Theorem 3.1. : *If N^{2n-1} is a compact contact manifold and F^2 is a compact oriented surface, then $M^{2n+1} = N \times F$ has a contact 1-form, which is C^2-close to a 1-form α_2 obtained from the contact form of N by means of the higher Lutz twist.*

By induction on $n = 2, 3, \dots$, with $N = T^{2n-1}$ and $F = T^2$, we deduce

Corollary 3.2. : *Any odd-dimensional torus T^{2n+1} carries a contact structure.*

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