Definition 5.1. Let k be a field, A an overring of k. The ring A is said to be **geometrically regular** if, for all finite field extensions k' of k, the ring $A' = A \otimes_k k'$ is regular.

Corollary 5.3. a) Every regular overring of a perfect field is geometrically regular. .

b) Every regular overring of an algebraically closed field is geometrically regular.

Remark. Let again $A' = A \otimes_k k'$. Some of the properties of A' can be deduced from those of A and of the field extension k' of k. This process of deduction is known as **ascent**. Conversely, some of the properties of A can be deduced from those of A'. This latter process of deduction is known as **descent**.

§6. COMPLETION AND NORMALIZATION

6A. **Completion.** Let A be a noetherian local ring, m its maximal ideal. It is well known(see Corollary after Proposition 5 in B.C.A., III, §3, no. 2) that $\cap m^n = (0)$. This implies that the collection $\{m^n\}$ can be taken as the basis of a filter of neighborhoods of 0 in a (unique) Hausdorff topology which is consistent with the ring structure of A (i.e. A is a Hausdorff topological ring).

The set $\hat{A}$ of (equivalence classes of) Cauchy sequences of elements of A can be given a topological ring structure which is obviously **complete** (i.e. every Cauchy sequence in $\hat{A}$ is convergent). We refer the reader to the third chapter of B.C.A. for the proof of the above statements, as well as for the

proof of the following ones, for which we give references to be found in the above mentioned third chapter.

1) The canonical homomorphism

$$j:A \to \hat{A}$$

is a monomorphism (§2, no. 12, since A is Hausdorff).

2) $\hat{A}$ is a noetherian local ring, with unique maximal ideal $\hat{m} = m\hat{A}$. (§3, no. 4, Corollary to Proposition 8, and §2, no. 12, Corollary 2)

3) $\hat{A}$ is a faithfully flat A-module (§3, no. 5, proposition 9)

4) $\hat{A} = \varprojlim(A/m^n)$ (§3, no. 6)

5) $A/m \simeq \hat{A}/\hat{m} = \hat{A}/m\hat{A}$. (Apply equation (21) in §3, no. 12, and 2) above.)

Example. Let P(X, Y) be the polynomial $Y^2 - X^2 (X + 1)$ whose variety of zeros in the affine plane is the cubic with double point represented in the figure. Let B = $\mathbb{C}[X,Y]/(P)$ and let m be the maximal ideal of B generated by (the equivalence classes of) X and Y. Let A = B_m. One easily sees that

A is an integral domain, but, as we shall see later, $\hat{A}$ is not integral, it has in fact two distinct minimal prime ideals. (See Theorem 6.5.)
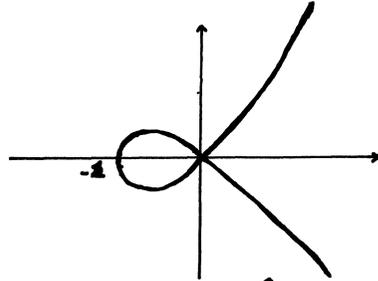
We now consider the ascent and descent properties of the local morphism A → $\hat{A}$.

Proposition 6.1. A noetherian local ring A is,

respectively, regular or C-M if, and only if, its completion $\hat{A}$ is regular or C-M. If $\hat{A}$ is, respectively, reduced or normal, then so is A.

   Proof: The morphism $A \to \hat{A}$ is flat. Hence we can apply the results of §5. Since $\hat{A}/m\hat{A} = A/m$, the first assertion of our proposition is a consequence of theorem 5.5 and corollary 5.1 respectively. The second assertion follows from theorem 5.7.

   The converse of the second statement in proposition 5.1 is false, as shown by a counter example due to Nagata. If, however, the fibers of the morphism $\text{Spec}(\hat{A}) \to \text{Spec}(A)$ (called formal fibers) are regular or geometrically regular, then a simple application of theorems 5.4 and 5.6, and propositions 4.5 and 4.6 shows that, when A is either reduced or normal, then so is $\hat{A}$.

   Having introduced complete local rings, we turn our attention to the study of some of their properties.

   Definition 6.1. Let A, B be noetherian local rings, with maximal ideals m, n respectively. Let $\varphi : A \to B$ be a local homomorphism. B is called a Cohen algebra over A if the following three properties hold:

   i) B is complete

   ii) B is A-flat

   iii) B/mB is a separable field extension of A/m.

   A trivial example of Cohen algebra is a separable field extension of a field.

   We state without proof two theorems, which will be used in the proof of the main result of this section. (For the proofs see E.G.A., Chap. 0_{IV}, 19.3.10 and 19.7.2.)

Theorem 6.1. Let B be a Cohen algebra, over a noetherian local ring A, and let C be a complete noetherian local ring which is an A-algebra under a local homomorphism $\varphi : A \to C$. Let J be a closed ideal in C. Then for any A-homomorphism $\Psi : B \to C/J$ there exists a local A-homomorphism $\theta : B \to C$ such that the following diagram commutes

$$B \overset{\theta}{\to} C$$
$$\Psi \searrow \swarrow$$
$$C/J$$

Theorem 6.2. Let A be a noetherian local ring, $\mathcal{m}$ its maximal ideal, $k = A/\mathcal{m}$, K a separable extension of k. Then there exists a unique (up to A-isomorphisms) Cohen algebra B over A, such that $B/\mathcal{m}B \simeq K$.

We denote by Z_p the localization of the ring Z at the principal prime ideal pZ. The rings $\widehat{Z_p}$ are called the complete prime local rings. One trivially sees that every local ring is a Z_p-algebra for an appropriate prime $p \geq 0$, and contains it, and that every complete local ring is a $\widehat{Z_p}$-algebra, again for an appropriate prime p.

With this in mind we give the following

Definition 6.2. A ring B is called a Cohen ring if it is a Cohen algebra over a complete prime local ring.

As an easy application of theorem 6.2 we obtain

Proposition 6.2. For each separable extension L of the field $\prod_p = Z_p/pZ_p$ ($\prod_0 = Q$) there exists a unique (up to isomorphisms) Cohen ring B, over $\widehat{Z_p}$, having L as residual field.

This clearly describes all Cohen rings.

We can now state and prove the main theorem of this section, namely:

Theorem 6.3. (Cohen Structure Theorem for Complete Local Rings). Let A be a complete, noetherian local ring. Then:

1) There exists either a Cohen ring W or a field k such that $A \simeq W[[T_1,\ldots,T_n]]/\mathfrak{a}$, for an appropriate ideal $\mathfrak{a} \subset W[[T_1,\ldots,T_n]]$. If A contains a field k one can take W = k.

2) If in addition A is an integral domain, then there exists a subring $B \subset A$ such that the following properties hold:

a) B is isomorphic either to $k[[T_1,\ldots,T_n]]$, where $k = A/\mathfrak{m}$, or to $W[[T_1,\ldots,T_n]]$, where W is a Cohen ring.

b) A and B have the same residue field.

c) A is a finitely generated B-module.

3) If in addition A is regular, then A is isomorphic either to $k[[T_1,\ldots,T_n]]$, $k = A/\mathfrak{m}$, or to $W[[T_1,\ldots,T_n]]$, W a Cohen ring.

Remark. The above theorem classifies all complete noetherian regular local rings, as we asserted in the section on regular local rings.

Proof: If A contains a field, say k", let P denote its prime field. We have the diagram $P \to A \to k = A/\mathfrak{m}$ whence k is a Cohen algebra over P, since P is perfect. Therefore, by theorem 6.1, we obtain the commutative diagram

$$
\begin{array}{ccc}
k & \to & A/\mathfrak{m} \\
& \searrow \quad \nearrow & \\
u & A &
\end{array}
$$

and u is necessarily injective, i.e. A contains a copy k' of k.

If A contains no field, then A is a $\widehat{Z}_p$-algebra, for some appropriate prime p > 0, (otherwise A contains $Z_{(0)}$ = Q), and char(k) = p. By theorem 6.2 there exists a Cohen ring W over $\widehat{Z}_p$ such that its residue field is isomorphic to k. Since A is a $\widehat{Z}_p$-algebra and $\mathcal{m}$ is closed in A we can apply theorem 6.1 in this case also, and obtain the commutative diagram

$$\begin{array}{ccc} W & \to & A/\mathcal{m} \\ & \searrow & \nearrow \\ u & A & \end{array}$$

where u is a local homomorphism.

Let now $x_1,\ldots,x_n$ be a set of elements of $\mathcal{m}$. Define a map $v:W[[T_1,\ldots,T_n]] \to A (v:k[[T_1,\ldots,T_n]] \to A)$ according to the following rules

    1) $v|W = u$ $(v|k = u)$

    2) $v(T_i) = x_i$, $i = 1,\ldots,n$

The completeness of A and the fact that the x_i's are trivially topologically nilpotent, guarantee the existence and uniqueness of the homomorphism v. (See B.C.A., III, §4, no. 5)

Having disposed of the above preliminaries, we proceed with the proof of the three statements of the theorem.

    1) Take $\{x_i\}$ $i = 1,\ldots,n$ to be a set of generators of $\mathcal{m}$. Let $\mathcal{n}$ be the maximal ideal of $W[[T_1,\ldots T_n]]$ (of $k[[T_1,\ldots,T_n]]$). Consider the homomorphism

$$gr(v):gr_{\mathcal{n}}(W[[T_1,\ldots,T_n]]) \to gr_{\mathcal{m}}(A)$$

$$(gr(v):gr_{\mathcal{n}}(k[[T_1,\ldots,T_n]]) \to gr_{\mathcal{m}}(A))$$

Since $W \to A/\mathcal{m}$ ($k \to A/\mathcal{m}$) is surjective, the choice of $x_1, \ldots, x_n$ shows that $gr(u)$ is surjective. Then by Corollary 2 of B.C.A., III, §2, no. 8, we have that v is surjective, and 1) is proved.

2) We consider two cases:

<u>Case 1</u>. A contains a copy k' of k = $A/\mathcal{m}$ (see preliminary remarks). Let dim A = n and let $y_1, \ldots, y_n \in \mathcal{m}$ be a system of parameters of A. Define B = $k[[T_1, \ldots, T_n]]$ and consider the homomorphism $v: B \to A$ as constructed in the proof of 1).

<u>Case 2</u>. A does not contain a field. Since A is an integral, local domain, for an appropriate prime integer p > 0, A contains a copy of Z_p. (See remark preceding definition 6.2) Identify Z_p with its image in A and note that $p \in \mathcal{m}$ and that p is not a zero divisor of A, hence by proposition 3.3, p can be imbedded in a system $\{p, y_1, \ldots, y_{n-1}\}$ of parameters of A. From the commutative diagram (see preliminary remarks to proof)

$$W \searrow \quad \to \quad A/\mathcal{m} \nearrow$$
$$u \searrow \quad \nearrow$$
$$A$$

and the fact that $u(1_W) = 1_A$ we see that $p' \neq 0$ where p' denotes the element $p \cdot 1$ of W. Since u is local, $p' \in \mathcal{n}'$, the maximal ideal of W. Let B = $W[[T_1, \ldots, T_n]]$, and define the homomorphism $v: B \to A$ as in the proof of 1). Note that $v(T_i) = y_i$, i=1,...,n-1, and $v(p') = p$.

In either case 1) or case 2) we have obtained a homomorphism $v: B \to A$, where B = $W[[T_1, \ldots, T_r]]$, r = n, n - 1, W a field or a Cohen algebra over A respectively. We assert:

    i)   B and A have isomorphic residue field k

   ii)   A is a finitely generated B-module

  iii)   v is injective.

Clearly the above three assertions imply 2) of the theorem. We proceed to prove them.

    i)   We leave as an exercise to the reader the proof that, for any local ring C, the two local rings C, $C[[T_1,\ldots,T_n]]$ have isomorphic residue fields.

   ii)   By the construction of v we clearly have, letting n be the maximal ideal of B,

$$n\,A \subset m \subset A$$

Furthermore, since nA is generated by a system of parameters of A, nA is an ideal of definition of A (Definition 2.5 and Proposition 2.1), and therefore $m \supset n\,A \supset m^h$ for some integer $h > 0$. We have $A/n\,A = (A/m^h)/(n\,A/m^h)$.

   Now, A/m is (trivially) a finitely generated B-module, and since m^q/m^{q+1} is a finitely generated A/m-module, m^q/m^{q+1} is a finitely generated B-module for all $q > 0$. From the exact sequences

$$0 \to m/m^2 \to A/m^2 \to A/m \to 0$$
$$0 \to m^2/m^3 \to A/m^3 \to A/m^2 \to 0$$
$$0 \to m^{h-1}/m^h \to A/m^h \to A/m^{h-1} \to 0$$

we obtain (proceeding by induction), that A/m^h is a finitely generated B-module, and therefore $A/n\,A$, as a quotient module of A/m^h, is also a finitely generated B-module.

Let $\{\overline{a}_j\}_{j=1,\ldots t}$ be a set of generators of $A/\eta\,A$ over B,
and let $a_j \in A$ such that $\overline{a}_j = a_j + \eta\,A$, $j = 1,\ldots,t$. Let F be
the submodule of A generated over B by $a_1,\ldots,a_t$. Then
$\eta\,A + F = A$. Since B is complete we can apply (ii) of
Corollary 3 of B.C.A.'s, III, §2, no. 9, and obtain that A is a
finitely generated B-module, and   assertion ii) is proved.

To prove (iii) we observe first of all that, since A is an
integral domain, ker(v) is a prime ideal of B. Furthermore,
since W is an integral domain, B is an integral domain. Finally,
both in case 1 and case 2 we have dim(B) = dim(A). This is seen
by observing that, in case 1, $T_1,\ldots,T_n$ is a system of
parameters of B, while in case 2, p', $T_1,\ldots,T_{n-1}$ is a system
of parameters of B. Therefore ker(v) = 0, otherwise
dim(B) > dim(A). Assertion 2) of the theorem is proved.

3) Let $y_1,\ldots,y_n$ be a regular system of parameters of A,
with $y_1 = p$ if A contains no field. (See case 2 in the proof of
2)). We obtain a homomorphism

$$v:W[[T_1,\ldots,T_r]] \to A$$

where r = n or n − 1 and W is either the field k or a Cohen ring
over $\widehat{Z}_p$, according as A does or does not contain a field. By
the proof of 1) v is surjective, and by the proof of 2) v is
injective, whence 3) follows. The theorem is proved.

$$\longleftarrow \qquad\qquad\qquad \longrightarrow$$

6B. <u>Normalization.</u> In this part of the present section all
rings shall be assumed to be integral domains unless otherwise
specified. If A is one such ring and L is a field containing A

(and containing a fortiori the field of fractions K of A), we denote by A'_L the integral closure of A in L, i.e. the subring of L consisting of all those elements of L satisfying an equation of integral dependence over A. A'_L is called the normalization of A in L.

In particular A is called <u>normal</u> (<u>integrally</u> closed) if $A'_K = A$. From now on we shall write A' for A'_K.

Examples of normal and not-normal rings abound in Algebraic Geometry. The following two rings are easily seen to be not normal (in both cases the element T is integral over the given ring, but outside it):

Let $R_1 = \mathbb{C}[T^2, T^3]$, $R_2 = \mathbb{C}[T^2-1, T(T^2-1)]$, $m_1 = T^2 R_1$, $m_2 = (T^2-1)R_2$. Then $A_1 = (R_1)m_1$; $A_2 = (R_2)m_2$.

In both cases we have $K = \mathbb{C}(T)$, and

$$A'_1 = A_1[T]$$

$$A'_2 = A_2[T]$$

In the case of A'_2 we see that it has two maximal ideals, namely $(T-1)A'_2$ and $(T+1)A'_2$ (see figure). In this case the number of maximal ideals in A'_2 equals the number of minimal prime ideals in the completion $\widehat{A_2}$ of A_2. (See example on page 101). This is in fact a situation that repeats itself in many cases as we shall later see.

With reference to the above two examples, if L is a finite extension of K one easily sees that in these cases A'_{i_L} is a finitely generated A_i-module. In fact it is well known (E.
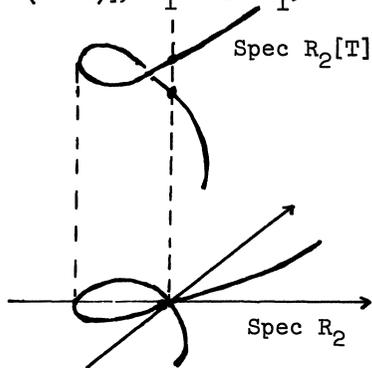
Noether) that if A is noetherian and char(K) = 0, then for any finite extension L of K the ring A'_L is a finitely generated A-module.

If char(K) $\neq$ 0 however, the situation is completely different. Nagata has given examples where, respectively, A is a discrete valuation ring, a noetherian local ring of dimension 2, a noetherian local ring of dimension 3, [L:K] < ∞ and, respectively A'_L is not a finite A-module, A'_L is not noetherian, A' is not noetherian.

We are therefore led to the following

Definition 6.3. An integral domain A, with field of fractions K, is said to be Japanese if, for every finite extension L of K, [L:K] < ∞, A'_L is a finitely generated A-module. A is said to be universally Japanese if every finitely generated algebra over A (in particular A itself) is Japanese.

Proposition 6.3. Let A be a noetherian integral domain, K its field of fractions. If, for every finite, purely inseparable field extension K' of K, $A'_{K'}$ is a finitely generated A-module, then A is Japanese.

Proof: The proof is based on the following two statements:

a) For every finite, field extension L of K there exists a finite field extension $\overline{L}$ of L such that every polynomial f(X) $\in$ K[X] with a root in $\overline{L}$ factors completely in $\overline{L}$.

b) If $\overline{L}$ is the field constructed in a) above there exists a field K', K $\subset$ K' $\subseteq$ $\overline{L}$ such that K' is purely inseparable over K and L is separable algebraic over K'.

See Theorem 14 of Zariski-Samuel "Commutative Algebra",

Volume I, Chapter II. Now, by assumption $A'_{K'}$ is a finitely

generated A-module, and by proposition 18 of B.C.A., V, §1,

no. 6, the integral closure of $A'_{K'}$ in $\overline{L}$ is a finitely

generated $A'_{K'}$-module.

Clearly such integral closure is $A'_{\overline{L}}$, and we have there-

fore proved that $A'_{\overline{L}}$ is a finitely generated A-module. Since

$A'_L \subset A'_{\overline{L}}$, and A is noetherian, A'_L is a finitely generated

A-module, and the proposition is proved.

When char(K) = 0 every normal ring is (trivially!) Japanese.

The following theorem, the main one in this section, gives

us a large class of Japanese rings.

Theorem 6.4. (Nagata) Every noetherian complete, local,

integral domain is Japanese.

The proof uses two lemmas, the second due to Tate, which we

proceed to state and prove.

Lemma 6.1. Let A be a ring and x an element of A which is

not a zero divisor. If $\mathfrak{p} = x \cdot A$ is a prime ideal of A, then the

inverse image of the ideal $x^n A_{\mathfrak{p}}$ under the canonical

homomorphism $\varphi : A \to A_{\mathfrak{p}}$ is the ideal $x^n A$.

Proof: Clearly $\varphi(x^n A) \subset x^n A_{\mathfrak{p}}$, whence $x^n A \subset \varphi^{-1}(x^n A_{\mathfrak{p}})$.

To prove $x^n A \supset \varphi^{-1}(x^n A_{\mathfrak{p}})$ we proceed by induction. If n = 1 and

$y \in A$ is such that $\frac{y}{1} = \frac{xa}{f}$, $f \notin \mathfrak{p}$, then, for some $g \notin \mathfrak{p}$

$gfy = gxa$. Since $\mathfrak{p}$ is prime $gf \notin \mathfrak{p}$, whence $y \in \mathfrak{p} = x A$ and we

are done in this case. For the general case, let $b \in A$ such

that $b/1 = x^n a/s$, $s \notin \mathfrak{p}$. Then for some $s' \notin \mathfrak{p}$, $s'sb = s'x^n a$,

whence $b \in \mathfrak{p}$ and therefore $b = x b'$. Therefore

$x(s'sb' - s'x^{n-1} a) = 0$ and since x is not a zero divisor in A,

$s'sb' - s'x^{n-1} a = 0$, whence $b'/1 \in x^{n-1} A_p$. By induction we have $b' \in x^{n-1} A$ and the lemma is proved.

Lemma 6.2. (Tate) Let A be a noetherian integral domain, $x \neq 0$ an element of A. Assume that the following conditions hold:

 i) A is integrally closed

 ii) The ideal $p = x \cdot A$ is a prime ideal of A and A is complete and Hausdorff for the p-adic topology

 iii) A/xA is Japanese.

Then A is Japanese.

Proof: Let K be the field of fractions of A, K' a finite extension of K. By Proposition 6.3 it suffices to show that $A'_{K'}$ is a finitely generated A-module when K' is a purely inseparable extension of K, say $(K')^q \subset K$, $q = p^e$, $0 < p = char(K)$. (As we remarked after Proposition 6.3, if $char(K) = 0$ A is trivially a Japanese ring.) Let K(y) be a purely inseparable extension of K such that $y^q = x$. Then, if $K'' = K' \cdot K(y)$, we have $(K'')^q \subset K$. Furthermore, if $A'_{K''}$ is a finitely generated A-module, so is $A'_{K'}$. Hence we can assume that there exists $y \in K'$ such that $y^q = x$. Denote $A'_{K'}$ by A'. Since A is integrally closed we have $A' \cap K = A$, whence
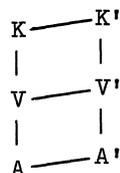
$$A' = \{x' \in K' \,|\, x'^q \in A\}$$

Let now $V = A_p$; $m = p A_p = x A_p$. Since the maximal ideal of V is generated by <u>one</u> regular element (A is an integral domain) V is a discrete valuation ring. In fact, by part d) of theorem 4.1, V is a regular, one dimensional local ring, hence by

proposition 9 of B.C.A., Chapter VI, §3, V is a discrete valuation ring. Let V' be the integral closure of V in K'. Since V is integrally closed (Corollary 4.1), V' ∩ K = V, and therefore

$$V' = \{x' \in K' | x'^q \in V\}.$$

By Corollary 2 of B.C.A., Chapter VI, §8, no 6, V' is a valuation ring, and by Corollary 3, Chapter VI, §8, no 1, V' is a discrete valuation ring. Letting $\mathfrak{m}'$ denote the maximal ideal of V', by Proposition 5, Chapter VI, §8, no 5, V'/$\mathfrak{m}'$ is an extension of finite degree of V/$\mathfrak{m}$, and

$$\mathfrak{m}' = \{x' \in K' | x'^q \in \mathfrak{m}\}.$$

$$
\begin{array}{ccc}
K & \longrightarrow & K' \\
| & & | \\
V & \longrightarrow & V' \\
| & & | \\
A & \longrightarrow & A'
\end{array}
$$

We prove the following three statements:

a) $\mathfrak{m}'^n \cap A' = y^n A'$

b) The x A'-adic topology on A' is Hausdorff

c) A'/xA' is a finitely generated A-module.

To prove a) we observe first that, since $y^q = x \in \mathfrak{m}$, $y \in \mathfrak{m}'$, and that clearly $y \in A'$. Hence $y^n A' \subset \mathfrak{m}'^n \cap A'$. Conversely, let $x' \in \mathfrak{m}'^n \cap A'$, and let $x' = y^n z'$, $z' \in K'$. We need to show $z' \in A'$. Now, since $x' \in \mathfrak{m}'^n$, we can write

$$x' = \Sigma\, t_1' \ldots t_n' \qquad t_j' \in \mathfrak{m}'$$

whence $(x')^q = \Sigma(t_1')^q \ldots (t_n')^q$ and by the above characterization of $\mathfrak{m}'$, $(t_j')^q \in \mathfrak{m}$, whence $(x')^q \in \mathfrak{m}^n$. Furthermore, by the characterization of A', $(x')^q \in A$, whence $(x')^q \in \mathfrak{m}^n \cap A$.

By lemma 6.1 $m^n \cap A = x^n A$, and we therefore obtain

$$y^{nq}(z')^q = (x')^q \in x^n A$$

whence $x^n(z')^q \in x^n A$, and, from the fact that A is an integral domain, $(z')^q \in A$. Therefore $z' \in A'$ and statement a) is proved.

We now prove b). Since $xA' = y^q A'$, the xA'-adic topology on A' and the yA'-adic topology on A' clearly coincide. Furthermore, by a) the yA'-adic topology on A' is induced by the m'-adic topology on V', which is Hausdorff since V' is a local ring. Therefore the xA'-adic topology of A' is Hausdorff.

Next, we prove c). We have $y^q = x$, and therefore $A'/xA' = A'/y^q A'$. The exact sequences

$$0 \to y^k A'/y^{k+1} A' \to A'/y^{k+1} A' \to A'/y^k A' \to 0 \qquad 0 < k \le q-1$$

show that it suffices to show that A'/yA' and $y^k A'/y^{k+1} A'$, $k = 1, \ldots, q-1$ are finitely generated A-modules. The diagram

$$
\begin{array}{ccccccc}
0 \to & yA' & \to & A' & \to & A'/yA' & \to 0 \\
 & \varphi \downarrow & & \varphi \downarrow & & \bar{\varphi} \downarrow & \\
0 \to & y^{k+1} A' & \to & y^k A' & \to & y^k A'/y^{k+1} A' & \to 0 \qquad k = 1, \ldots, q-1
\end{array}
$$

where $\varphi(\xi) = y^k \xi$ and $\bar{\varphi}$ is the induced homomorphism, shows that $\bar{\varphi}$ is an isomorphism, since φ is. Hence it suffices to show that A'/yA' is a finitely generated A-module. Now, by a) $yA' = m' \cap A'$, whence $A'/yA' \simeq A'/m' \cap A'$ and $A'/m' \cap A'$ is a submodule of V'/m'. Also, since A' is integral over A, $A'/m' \cap A'$ is integral over A/p. Since V'/m' is a finite extension of V/m, and A/p is Japanese by assumption, the integral closure of A/p in V'/m' is a finitely generated

A/p -module, since clearly V/m is the field of fractions of
A/p . Therefore $A'/m' \cap A'$ is contained in a finitely generated
A/p -module, and is hence itself finitely generated A/p -module
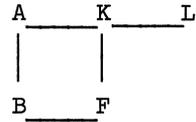(A is noetherian). Therefore $A'/m' \cap A'$ is a finitely
generated A-module, and c) is proved.

Let now $\hat{A'}$ denote the completion of A' in the xA'-adic
topology which, by b) is Hausdorff. Therefore we have that $\hat{A'}$
contains an isomorphic copy of A', and we identify the two, i.e
we have $A' \subset \hat{A'}$. By statement 6) at the beginning of section
6A we have $\hat{A'}/x\hat{A'} \simeq A'/xA'$, and in the proof of c) we actually
showed that A'/xA' is a finitely generated A/xA-module. Since A
is complete and Hausdorff in the xA-adic topology, we can apply
part ii) of Proposition 14 of B.C.A., Chapter III, §2, no 11,
and obtain that $\hat{A'}$ is a finitely generated A-module. Therefore
$A' \subset \hat{A'}$ is also finitely generated over A, and the lemma is
proved.

We now proceed with the proof of theorem 6.4, namely that
every complete noetherian local domain is Japanese.

By Cohen's Structure theorem, since A is an integral domain,
A contains a ring B which is regular and such that A is a
finitely generated B-module. Therefore A is integral over B,
and hence it suffices to prove that B is Japanese, since for
every finite extension L of the field of fractions K of A we
have $A'_L = B'_L$, and L is a finite extension of the field of
fractions F of B. Therefore it suffices to prove the theorem
with the additional assumption that A is regular.

We proceed by induction on n = dim(A). If n = 0, since A

is integral, it follows that A is a field,
trivially a Japanese ring. Assume $n > 0$
and let $x \in A$ be an A-regular element,
$x \notin \mathit{m}^2$. Then A/xA is again regular
(Corollary 4.2) and complete and $\dim(A/xA) = n-1$. Since A/xA is
regular, xA is a prime ideal. By the induction assumption A/xA
is Japanese. Furthermore, A being complete and Hausdorff in the
m-adic topology, it is so a fortiori in the xA-adic topology.
Since A is regular, it is integrally closed, and by lemma 6.2
A is Japanese. The theorem is proved.

Corollary 6.1. Let A be a complete, local, noetherian
integral domain, K the field of fractions of A, K' a finite field
extension of K. The integral closure A' of A in K' is a local
ring.

Proof. By theorem 6.4 A' is a finitely generated A-module.
Therefore A' is complete in the mA'-adic topology (B.C.A.
Chapter III, §2, no 12, Corollary 1), and semi-local (B.C.A.
Chapter IV, §2, no 5, Corollary 3), and mA' is an ideal of
definition of A'. Therefore the mA'-adic topology on A' is
equivalent to the r-adic topology, where r denotes the
radical of A'. By proposition 18 of B.C.A., Chapter III, §2,
no 13, (applied to A') we have $A' = \prod_{i=1}^{q} A'_i$, where each A'_i is a
local ring, $i = 1, \ldots, q$. Since A' is an integral domain, $q = 1$
and A' is a local ring,   q.e.d.

If A is a noetherian, local, integral domain, it need not

be a Japanese ring. However, A is Japanese if two certain

conditions hold for the completion $\hat{A}$ of A. Namely

Proposition 6.4. Let A be a noetherian local, integral

domain, $\hat{A}$ the completion of A in the m-adic topology, K the

field of fractions of A, K' a finite field extension of K, A'

the integral closure of A in K'. Let R be the total ring of

fractions of $\hat{A}$. If

       i) $\hat{A}$ is reduced

      ii) $R \otimes_K K'$ is reduced

then A' is a finitely generated A-module.

Proof: Let $P_1, \ldots, P_t$ be the minimal prime ideals of $\hat{A}$,

and let L_i be the field of fractions of $B_i = \hat{A}/P_i$, i=1,...,t.

Since $\hat{A}$ is reduced we have $\bigcap_{i=1}^{t} P_i = (0)$ and a sequence of

inclusions

$$\hat{A} \to \prod B_i \to \prod L_i$$

with

$$R = \prod L_i.$$

Now let $A_1 = \hat{A}$, $A'_1 = A' \otimes_A A_1$, $K'_1 = K' \otimes_A A_1$, $K_1 = K \otimes_A A_1$.

We therefore have $K'_1 = K' \otimes_K K_1$. Since A_1 is a faithfully flat

A-module, it suffices to prove that A'_1 is a finitely generated

A_1-module. Furthermore, again by the flatness of A_1 over A we

have $A'_1 \subset K'_1$. Finally, letting S denote the multiplicatively

closed subset of A consisting of the non zero divisors of A, we

clearly have $K = S^{-1}A$, and $K_1 = S^{-1}A \otimes_A A_1 = S^{-1}A_1$, since S

consists of non zero divisors of A_1 also. Clearly $S^{-1}A_1 \subset R$,

whence $K_1 \subset R$. We therefore have the inclusion diagram
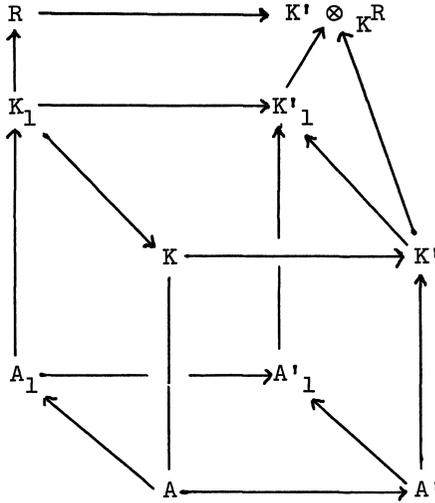
where $K'_1 \subset K' \otimes_K R$ is seen from $K_1 \subset R$ and the flatness of K' over K.

By proposition 5 of B.C.A., Chapter V, §1, no 2, A'_1 is integral over A_1, and is therefore contained in the integral closure C of A_1 in $K' \otimes_K R$.

If $a \in A$ is not a zero divisor, then a is not a zero divisor in A_1. From this we see that the L_i's are vector spaces over K. Since $R = \prod L_i$, we have $K' \otimes_K R = \prod K' \otimes_K L_i$ and, since $K' \otimes_K R$ is reduced, so are the $K' \otimes_K L_i$, $i = 1, \ldots, t$. Furthermore, since $[K':K] < \infty$, $K' \otimes_K L_i$ is finitely generated

over L_i, $i = 1,\ldots,t$. Since $K' \otimes {}_K L_i$ has no nilpotent elements

and, again, $[K':K] < \infty$, $K' \otimes {}_K L_i$ is a product $\prod_j M_{ij}$, where the

M_{ij} are <u>fields</u>, which are actually finite field extensions of L_i.

Therefore the integral closure B'_i of B_i in $K' \otimes {}_K L_i$ is, by

theorem 6.4, a finitely generated B_i-module, $i = 1,\ldots,t$, and

hence a finitely generated A_1-module. Since A'_1 is integral

over A_1 we have $A'_1 \subset \prod_{i=1}^{t} B'_i$, and therefore A'_1, being

contained in a finitely generated A_1-module, is itself a finitely

generated A_1-module, and the proposition is proved.

$\quad$ <u>Theorem 6.5</u>.$\quad$ Let A be a reduced noetherian local ring with

geometrically regular formal fibers.$\quad$ Then:

$\quad$ 1)$\quad$ $\widehat{A}$ is reduced

$\quad$ 2)$\quad$ The integral closure A' of A in its total ring of

$\qquad$ fractions is a finitely generated A-module

$\quad$ 3)$\quad$ The completion $\widehat{A'}$ of A' is isomorphic to the integral

$\qquad$ closure of $\widehat{A}$ in its total ring of fractions

$\quad$ 4)$\quad$ There exists a 1-1 correspondence between the maximal

$\qquad$ ideals of A' and the minimal prime ideals of $\widehat{A}$ given by

$$\widehat{A'}_{\mathcal{m}} \simeq \widehat{A}/\mathcal{q}$$

$\qquad$ where $\mathcal{m}$ is a maximal ideal in A' and $\mathcal{q}$ the correspond-

$\qquad$ ing minimal prime ideal in $\widehat{A}$.

$\quad$ <u>Proof</u>:$\quad$ 1)$\quad$ This is a direct result of theorem 5.7.$\quad$ Note

that here we only need the formal fibers to be regular.

$\quad$ 2)$\quad$ Let $\mathcal{p}_i$, $i = 1,\ldots,t$ be the minimal prime ideals of A,

and let $B_i = A/p_i$, $i = 1, \ldots, t$.

We assert that $\widehat{B_i}$ is reduced, $i = 1, \ldots, t$. In fact, let $\widehat{B_i}/\sigma\widehat{B_i}$, $\sigma \in \text{Spec}(B_i)$ be a formal fiber of B_i. Then, letting p denote the unique prime ideal of A corresponding to σ we have $\widehat{B_i}/\sigma\widehat{B_i} \cong \widehat{A}/p\widehat{A}$. I.e. that the formal fibers of B_i are isomorphic to formal fibers of A. As B_i is reduced, by proposition 5.7, $\widehat{B_i}$ is reduced, $i = 1, \ldots, t$. If L_i denotes the field of fractions of B_i, $i = 1, \ldots, t$, then $A \subset \prod_i B_i \subset \prod_i L_i$. Apply proposition 6.4, with $K = K' = L_i$.

It follows that the integral closure B'_i of B_i in L_i is a finitely generated B_i-module, hence a finitely generated A-module. Now clearly $\prod_i L_i$ is the total ring of fractions of A, whence $A' = \prod B'_i$, and 2) is proved.

3) We let $X = \text{Spec}(\widehat{A})$, $Y = \text{Spec}(A)$, $Z = \text{Spec}(A')$. Then we have canonical morphisms $\varphi : X \to Y$, $\Psi : Z \to Y$.

By 2) A' is a finitely generated A-module, and by part (ii) of Theorem 3 of B.C.A., Chapter III, §3, no 4, $\widehat{A'} \cong \widehat{A} \otimes_A A'$. Therefore, if we let $W = \text{Spec } \widehat{A'}$ we have the commutative diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\quad p \quad} & W \\
\varphi \downarrow & & \downarrow q \\
Y & \xleftarrow{\quad \Psi \quad} & Z
\end{array}
$$
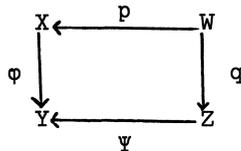
Let $w \in W$, $z = q(w)$, $y = \Psi(z)$, $x = p(w)$. Then $\varphi(x) = y$ and

$$0_x \otimes_{0_y} k(z) = (0_x \otimes_{0_y} k(y)) \otimes_{k(y)} k(z)$$

is the local ring of w in $q^{-1}(z)$.

Since A' is a finitely generated A-module, it follows

$[k(z):k(y)] < \infty$, and since $0_x \otimes_{0_y} k(y)$ is geometrically

regular, so is $0_x \otimes_{0_y} k(z)$. Therefore the formal fibers of A'

are geometrically regular. Let $m_1, \ldots, m_t$ denote the maximal

ideals of A' (A' is semi-local). By the corollary to proposition

19 of B.C.A., Chapter III, §2, no 13, $\widehat{A'} = \prod_j \widehat{A'_{m_j}}$. There-

fore, since A' has geometrically regular formal fibers, so do

the A'_{m_j}, $j = 1, \ldots, t$.

Since A' is normal and has (geometrically) regular formal

fibers, it follows from theorem 5.7 that $\widehat{A'}$ is normal. Since $\widehat{A}$

if faithfully flat, over A, the inclusions $A \subset A' \subset R$ imply, by

tensoring with $\widehat{A}$, the inclusion relations $\widehat{A} \subset \widehat{A'} \subset R \otimes_A \widehat{A}$. Now

$R \otimes_A \widehat{A}$ is clearly contained in the total ring of fractions R"

of $\widehat{A}$. Therefore $\widehat{A'}$ is a normal ring containing $\widehat{A}$ and contained

in the total ring of fractions of $\widehat{A}$. It follows that $\widehat{A'}$ is the

normalization of $\widehat{A}$ in R", and 3) is proved.

4) With the same notations as in the proof of 3), we have

$\widehat{A'} = \prod_{j=1}^{t} \widehat{A'_{m_j}}$. Let $q_1, \ldots, q_s$ denote the minimal prime ideals

of $\widehat{A}$. Since $\widehat{A}$ is reduced, we have the inclusions

$\widehat{A} \subset \prod_{i=1}^{s} \widehat{A}/q_i \subset R"$. It follows that the integral closure of $\widehat{A}$

in R" is given by $\prod_{i=1}^{s} B_i$, where B_i denotes the integral closure

of $\widehat{A}/\mathcal{O\!\!q}_i$ in its field of fractions. By corollary 6.1, B_i is a local ring, $i = 1,\ldots,s$, and by 3)

$$\prod_{j=1}^{t} \widehat{A'_{\mathcal{m}_j}} = \widehat{A'} = \prod_{i=1}^{s} B_i$$

Therefore $s = t$, and up to a reordering $\widehat{A'_{\mathcal{m}_j}} = (\widehat{A/\mathcal{q}_j})'$. The theorem is proved.

We complete this work with a definition and a theorem of Grothendieck, which we shall leave unproved.

Definition 6.4. A noetherian local ring A is said to be excellent if

      i) A has geometrically regular formal fibers

      ii) Every finitely generated A algebra is catenary

          (i..e. A is universally catenary)

Theorem 6.6. (Grothendieck) Let A be an excellent local ring. Then every localization of a finitely generated algebra over A is excellent. (E.G.A., IV, 7.4.4).