If $\dim(A_p) = 3$, and $p \neq m_a$, then A_p is again regular and $\text{depth}(A_p) = 3$. At m_a we have $\dim(A_{m_a}) = 3$, and $\text{depth}(A_{m_a}) \geq 2$, since clearly Y^4, $Z - a$ form an A_{m_a}-regular sequence. Hence (S_2) holds for A.

Actually $\text{depth}(A_{m_a}) = 2$, which gives us an example of a local integral domain which is not a C-M ring, whence A itself is not a C-M ring.

That $\text{depth}(A_{m_a}) = 2$ is proved as follows. One can take n=0. Let $A' = \mathbb{C}[X^4, X^3Y, XY^3, Y^4]$. Then $A/ZA \cong A'$. Let m' be the maximal ideal of A' corresponding to the origin of Spec(A'). We know from above that $\text{depth}(A'_{m'}) \leq 1$, and $\text{depth}(A_{m_0}) \geq 2$. Furthermore we have

$$A'_{m'} = A_{m_0}/ZA_{m_0}$$

and since Z is A_{m_0}-regular, $1 \geq \text{depth}(A'_{m'}) = \text{depth}(A_{m_0}) - 1$, whence $\text{depth}(A_{m_0}) \leq 2$. We are done.

It is a rewarding exercise for the reader to check that the kernel $\mathcal{O}\mathcal{C}$ of the homomorphism $\varphi: \mathbb{C}[T_1, T_2, T_3, T_4] \to \mathbb{C}[X^4, X^3Y, XY^3, Y^4]$ - defined by $\varphi(T_1) = X^4$, $\varphi(T_2) = X^3Y$, $\varphi(T_3) = XY^3$, $\varphi(T_4) = Y^4$ is generated by $T_1^2 T_3 - T_2^3$, $T_2 T_4^2 - T_3^3$, $T_1, T_4^3 - T_3^4$, and that no two of the above three polynomials generate $\mathcal{O}\mathcal{C}$.

§5. BEHAVIOR UNDER LOCAL HOMOMORPHISM

In this section we let A, B be local rings, unless otherwise specified, with unique maximal ideals m, n respectively.

We recall that a homomorphism $\varphi: A \to B$ is called <u>local</u> if

$\varphi(m) \subset n$, or equivalently, $\varphi^{-1}(n) = m$. Geometrically this means that, in the associated continuous map $^a\varphi$:Spec(B) → Spec(A), the unique closed point of Spec(B) maps into the unique closed point of Spec(A).

As an example of a non local homomorphism we consider the inclusion of an integral local ring A into its field of fraction B. Here the unique closed point (in fact the only point) of Spec(B) maps into the generic point of Spec(A), as far from the closed point as one can get!

5A. We study here the behavior of dimension under a local homomorphism.

Let φ:A → B be a local homomorphism, and let X = Spec(B), Y = Spec(A), whence $^a\varphi$:X → Y. Let $^a\varphi$ = f. The inverse image $f^{-1}(y)$ of the unique closed point y of Y contains the unique closed point x of X, and perhaps something more. In any event, $f^{-1}(y)$ consists of all those prime ideals p of B such that $\varphi^{-1}(p) = m$, i.e. those prime ideals which contain mB (we consider B as an algebra over $\varphi(A)$, and write m for $\varphi(m)$). So $f^{-1}(y)$ consists of the prime ideals of the ring $B/mB = A/m \otimes_A B$. We have shown $f^{-1}(y) = $ Spec(B/mB). In the sequel we shall denote by k the residue field A/m.

Optimally one would hope that dim X–dim($f^{-1}(y)$) = dim Y. However, as we shall see, this is not always true. We begin examining the situation with the following

Proposition 5.1. dim(B) $\leq$ dim(A) + dim(k $\otimes_A$ B)

Proof: Note that, with the identification k $\otimes_A$ B = B/mB one easily sees that k $\otimes_A$ B is a local ring with maximal ideal

n B/mB. Hence dim(k $\otimes_A$ B) $< + \infty$.

Let dim(A) = m, and let $s_1,\ldots,s_m$ be a system of parameters
of A. Let $\alpha = s_1$ A +...+ s_m A. By definition A/α is artinian,
whence m/α is nilpotent in A/α, i.e. a sufficiently high
power of every element of m is in α. Since an element of mB
is a linear combination of a finite number of elements of m with
coefficients in B, a sufficiently high power of every element of
mB is in αB, i.e. mB/αB is nilpotent in B/αB. The nil-
radical w of B/αB contains mB/αB, whence

$$\text{dim}(B/m B) = \text{dim}([B/\alpha B]/(m B/\alpha B)) =$$

$$\text{dim}[(B/\alpha B)/w] = \text{dim}(B/\alpha B)$$

clearly B/αB = B/s_1 B +...+ s_m B. Let dim(B/αB) = n, and
let $\bar{t}_1,\ldots,\bar{t}_n$ be a system of parameters of B/αB. Let $t_i \in$ B,
i = 1,...,n be such that $\bar{t}_i = t_i + \alpha$ B. We have that
C = (B/αB)/$\bar{t}_1$(B/αB) +...+ $\bar{t}_n$(B/αB) is artinian, and clearly
C = B/(t_1 B +...+ t_n B + s_1 B +...+ s_m B), i.e. $t_1,\ldots,t_n$,
$\varphi(s_1),\ldots,\varphi(s_m)$ generate an ideal primary for n . Then
dim(B) = s(B) $\leq$ m + n, and the proposition is proved, since
n = dim(B/α B) = dim(B/mB).

Remark. It is possible that inequality hold in the state-
ment of proposition 5.1. In fact one can take B = A/m = k,
where dim(A) $\geq$ 1. A more difficult example can be given, where
dim(A) = 2, B = C_{mA} where C is a finite algebra over A and
dim(B) = 1. Clearly 1 < 2, whence the inequality.

As a consequence of Theorem 5.1 below we shall see that,
when B is A-flat, equality in proposition 5.1 does hold.

Flatness, however, is a stronger requirement than needed. In fact, the conclusion of the following lemma is sufficient, as we shall see, to guarantee equality in proposition 5.1.

Lemma 5.1.   Let $\varphi : A \to B$ be a homomorphism of (not necessarily local or noetherian) rings and let B be   A-flat. Let X = Spec(B), Y = Spec(A), $^a\varphi : X \to Y$. Let V be an irreducible closed subset of Y.   Then the generic points of all the irreducible components of $^a\varphi^{-1}(V)$ are mapped into the generic point of V.

Proof: Let V = Spec(A/p), where p denotes the generic point of V.   Let $^a\varphi$ = f.   Then $f^{-1}(V)$ = Spec(B/pB) = Spec(A/$p \otimes_A B$).   Since B is A-flat, B/pB is A/p -flat.   In fact, if

$$0 \to M \to N$$

is an exact sequence of A/p -modules, it is also an exact sequence of A-modules and, since B is A-flat

$$0 \to M \otimes_A B \to N \otimes_A B \text{ is exact.}$$

But

$$M \otimes_A B = M \otimes_{A/p} (A/p \otimes_A B)$$
$$N \otimes_A B = N \otimes_{A/p} (A/p \otimes_A B)$$

whence (A/p) $\otimes_A B$ is A/p -flat.   The homomorphism φ induces a canonical homomorphism A/p $\to$ B/pB, i.e. we may assume V = Y, and hence $f^{-1}(V)$ = X.   We denote by O_X and O_Y the sheaves of local rings of X and Y respectively.   (See the introduction)

Let T be an irreducible component of X, with x as generic point.   Let f(x) = y.   We have to show that y is the generic

point of Y. Since flatness is preserved under localization,

O_x is O_y-flat. In fact it is faithfully flat, i.e.

$m_y \cdot O_x \neq O_x$ where m_y denotes the unique maximal ideal of O_y.

To see this we observe that, if $m_y \cdot O_x = O_x$ then, by

Nakayama's lemma $O_x = 0$, a contradiction. Since O_x is faith-

fully flat over O_y, by proposition 8 of B.C.A., I, §3, no. 5,

we have that the homomorphism $\tilde{\varphi}_x : O_y \to O_x$ is injective, and that

$\operatorname{Spec}(O_x) \to \operatorname{Spec}(O_y)$ is surjective. Let y' be the generic point

of Y. Then $j_{y'} \subset j_y$, whence $j_{y'} \cdot O_y \in \operatorname{Spec}(O_y)$ and there exists

a prime ideal $p \in \operatorname{Spec}(O_x)$ such that $\tilde{\varphi}_x^{-1}(p) = j_{y'} \cdot O_y$.

$O_x = B_{j_x}$ and j_x is minimal, whence $p = j_x \cdot O_x$. Then y' = y,

$$\text{Q.E.D.}$$

Note. Lemma 5.1 shows that the projection indicated in the
figure is not a flat morphism.
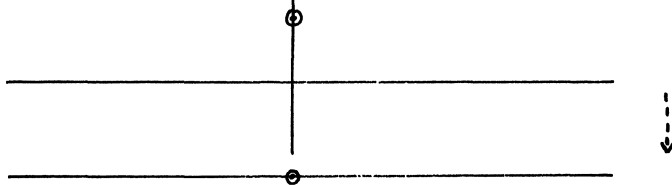
We return now to discussing when equality holds in

Proposition 5.1.

Theorem 5.1. Let A, B be local, noetherian rings,

$\varphi : A \to B$ be a local homomorphism, X = Spec(B), Y = Spec(A),

$^a\varphi : X \to Y$ the associated morphism. We assume the following

condition:

(*) For every closed irreducible subset V of Y,

$V \neq \{m\}$,   none of the irreducible components of $^a\varphi^{-1}(V)$ are

contained in $^a\varphi^{-1}(\mathcal{m})$. Then

$$\dim(B) = \dim(A) + \dim(k \otimes_A B).$$

Remark. By lemma 5.1 (*) clearly holds if B is A-flat, since $\{\mathcal{m}\}$ is not the generic point of V. This justifies the remark made after proposition 5.1.

Proof: We proceed by induction on $n = \dim(A)$. $n = 0$. Then $\text{Spec}(A) = \{\mathcal{m}\}$ and $\mathcal{m}$ is nilpotent. Hence $\mathcal{m}B$ is contained in the nilradical of B, whence $\dim(B/\mathcal{m}B) = \dim(B)$, and the theorem holds in this case. Assume $n > 0$, let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the minimal primes of B, $p_i = \varphi^{-1}(\mathcal{O}_i)$, $i = 1, \ldots, r$. Assume $p_i = \mathcal{m}$ for some i, $1 \leq i \leq r$. Since $\dim(A) > 0$, there exists a prime $p \in \text{Spec } A$ with $p \subsetneq \mathcal{m}$. Then clearly $\mathcal{m} = p_i = \varphi^{-1}(\mathcal{O}_i)$ implies $\mathcal{O}_i \supset pB$. Now $V(p) \neq \{\mathcal{m}\}$, and $\mathcal{O}_i \supset pB$ $\implies \mathcal{O}_i \in {}^a\varphi^{-1}(V(p))$, whence $\mathcal{O}_i$ is the generic point of an irreducible component T of $^a\varphi^{-1}(V(p))$. From $\mathcal{m} = \varphi^{-1}(\mathcal{O}_i)$ we see $^a\varphi(\mathcal{O}_i) = \mathcal{m}$, whence $T \subset {}^a\varphi^{-1}(\mathcal{m})$, contrary to assumption (*). Therefore $\mathcal{m} \neq p_i$, $i = 1, \ldots, r$.

Let now $p'_1, \ldots, p'_s$ be the minimal primes of A. Since $\dim(A) > 0$, $\mathcal{m} \neq p'_j$, $j = 1, \ldots, s$. Hence

$$\mathcal{m} \not\subset (\bigcup_{i=1}^{r} p_i) \cup (\bigcup_{j=1}^{s} p'_j) = E.$$

Let $x \in \mathcal{m}$, $x \notin E$. By proposition 2.6, since $x \notin p'_j$, $j = 1, \ldots, s$, and $\varphi(x) \notin \mathcal{O}_i$, $i = 1, \ldots, r$

$$\dim(A/xA) = n - 1$$

$$\dim(B/xB) = \dim B - 1$$

Furthermore since Spec(A/xA) $\subset$ Spec(A), Spec(B/xB) $\subset$ Spec(B), and $\mathfrak{m}$(A/xA) is the closed point of Spec(A/xA), (*) holds for A/xA and B/xB. Hence we can apply the induction assumption, whence, letting A' = A/xA, B' = B/xB, dim(B) - 1 = dim(B/xB) = dim(A/xA) + dim(A'/$_{\mathfrak{m}'}$ $\otimes$ $_{A'}$B') where $\mathfrak{m}'$ denotes the unique maximal ideal $\mathfrak{m}$A' of A'.

Now

$$A'/_{\mathfrak{m}'} = A/\mathfrak{m} \text{ and } A/\mathfrak{m} \otimes_{A/xA} B/xB =$$

$$A/\mathfrak{m} \otimes_{A/xA} (A/xA \otimes_A B) = A/\mathfrak{m} \otimes_A B$$

and finally

$$\dim(B) - 1 = \dim(A) - 1 + \dim(k \otimes_A B)$$

and the theorem is proved.

We may ask if, when equality holds in proposition 5.1, B is A-flat. The answer is yes, but under fairly strong conditions on A and B. Namely

<u>Proposition 5.2.</u> Let φ:A $\to$ B be a local homomorphism. Assume furthermore that

    1) A is regular

    2) B is C-M

    3) dim(B) = dim(A) + dim(k $\otimes_A$ B)

Then B is A-flat.

<u>Proof:</u> We proceed by induction on n = dim(A). n = 0 implies A is a field (since A is regular), and any vector space over A is flat.

Let n > 0. Since A is regular, there exists x $\in \mathcal{m}$, x $\notin \mathcal{m}^2$. Since A is an integral domain x is A-regular. Let A' = A/xA. Then A' is also regular, by corollary 4.2, and dim(A') = dim(A) - 1 by proposition 3.1.

Let B' = B/xB. By proposition 5.1 we have

$$\dim(B') \leq \dim(A') + \dim(A'/_{\mathcal{m}'} \otimes_{A'} B')$$

where $\mathcal{m}'$ denotes the unique maximal ideal $\mathcal{m}A'$ of A'.

Now $A'/_{\mathcal{m}'} = A/_{\mathcal{m}} = k$, and $A/_{\mathcal{m}} \otimes_{A/xA} B/xB \simeq A/_{\mathcal{m}} \otimes_A B$. From the Hauptidealsatz we have

$$\dim(B) - 1 \leq \dim(B')$$

whence

$$\dim(B) - 1 \leq \dim(B') \leq \dim(A) - 1 + \dim(k \otimes_A B) = \dim(B) - 1.$$

Therefore dim(B') = dim(B) - 1 whence (since B is C-M), x is B-regular by proposition 3.2, whence B' is C-M.

Hence 1), 2), 3) of the statement of our proposition hold for A' and B', whence, by the induction assumption, B' is A'-flat. Now the canonical homomorphism

$$xA \otimes_A B \rightarrow xB$$

is clearly surjective and, since x is B-regular, it is also injective. Hence, by(iii) of theorem 1 of B.C.A., III, §5, no. 2, B is A-flat and the proposition is proved.

Remark. The following examples show that there is no hope of improving proposition 5.2.

Example 1. Take A' = $\mathbb{C}[T]$, B' = $\mathbb{C}[X,Y]/[(X-Y)^2(X+Y),$ $(X-Y)(X+Y)^2]$ then let

$$f:A' \to B'$$

be defined by $f(T)$ = the class of $(X+Y)(X-Y)$, and let A,B be the localizations of A',B' at T, (X,Y) respectively. Then we have

      1) B is not C-M

      2) B is not A-flat

    <u>Example 2</u>. Let $A' = \mathbb{C}[X^2,XY,Y^2]$, $B' = \mathbb{C}[X,Y]$, $f:A' \to B'$ the inclusion, A = the localization of A' at (X^2,XY,Y^2), B = the localization of B' at (X,Y). Then we have

      1) A is normal and C-M

      2) B is regular

      3) B is not A-flat

    <u>5B</u>. We now study the behavior of the notion of depth under local homomorphisms.

    Once again, with the same notations as in section 5A, we wish to relate the depths of the three rings A, B, $B/\mathfrak{m}B$. More specifically, we shall investigate under what conditions we have

$$\text{depth}(B) = \text{depth}(A) + \text{depth}(k \otimes_A B)$$

    Unfortunately here we have no parallel to proposition 5.1, as the following two examples show:

    1. Let $t \in A$ be A-regular, $B = A/tA$. Then, by theorem 3.1,

$$\text{depth } B = \text{depth}(A) - 1 < \text{depth}(A)$$

whence we get $\text{depth}(B) < \text{depth}(A) + \text{depth}(k \otimes_A B)$.

2. Let A be a non C-M ring with nilradical $\mathcal{N} \neq 0$. Let B = A/$\mathcal{N}$. If dim(A) = 1 we have dim(B) = 1, and since A is not C-M depth(A) = 0, depth(B/mB) = 0 (since A $\rightarrow$ B $\rightarrow$ 0 is exact, mB is a maximal ideal and B/mB is a field). But depth(B) = 1. To see this, let $p_1, \ldots, p_t$ be the minimal primes of A. Since dim(A) = 1, $m \neq p_i$, i = 1,...,t, whence $m \not\subset \bigcup_{i=1}^{t} p_i$. Let $x \in m$, $x \notin \bigcup_{i=1}^{t} p_i$, and let $\bar{x} = x + \mathcal{N} \in$ B. Since $\mathcal{N} = \bigcap_{i=1}^{t} p_i$, we see that $\bar{x}$ is not a zero divisor in B.

Even though, in general, depth has an irregular behavior under local homomorphisms, it does behave nicely under <u>flat,</u> local homomorphisms. In fact we have

<u>Theorem 5.2.</u> Let φ:A $\rightarrow$ B be a local homomorphism and assume that B is A-flat. Then

$$\text{depth(B)} = \text{depth(A)} + \text{depth}(k \otimes_A B)$$

<u>Proof</u>: We proceed by induction on n = depth(A) + depth$(k \otimes_A B)$.

1) n = 0. Then depth(A) = depth$(k \otimes_A B)$ = 0. Hence $m \in$ Ass(A) and nB/mB $\in$ Ass(B/mB), by theorem 3.1. Now, by Theorem 2 of B.C.A., IV, §2, no. 6, we have Ass(B) = $\bigcup_{p \in \text{Ass}(A)}$ Ass(B/pB). Since $m \in$ Ass(A), Ass(B) $\supset$ Ass(B/mB), whence n(B/mB) $\in$ Ass(B/mB) implies $n \in$ Ass(B). Therefore depth(B) = 0 by theorem 3.1.

2) Assume n > 0. We proceed in two steps.

<u>Case 1.</u> depth(A) > 0. Then there exists $x \in m$ such that x is A-regular.

Let $A' = A/xA$, $B' = B/xB$. Then

$$(A'/mA') \otimes_{A'} B' \quad = \quad (A/m) \otimes_A (A' \otimes_A B) \quad = \quad (A/m) \otimes_A B.$$

Since B is A-flat, the exact sequence

$$0 \to A \overset{\Psi}{\to} A \qquad \Psi \text{ is multiplication by x}$$

gives an exact sequence

$$0 \to A \otimes_A B \to A \otimes_A B$$

whence x is B-regular. Hence $\mathrm{depth}(A') = \mathrm{depth}(A) - 1$, $\mathrm{depth}(B') = \mathrm{depth}(B) - 1$. Furthermore, B' is A' - flat (see proof of Lemma 5.1 or Corollary 2 of B.C.A., I, §2).

We can hence apply the induction assumption. Since

$$\mathrm{depth}(A') + \mathrm{depth}((A'/mA') \otimes_{A'} B') =$$

$$\mathrm{depth}(A) - 1 + \mathrm{depth}(k \otimes_A B)$$

we have

$$\mathrm{depth}(B') = \mathrm{depth}(A') + \mathrm{depth}(k \otimes_A B)$$

whence the theorem, in this case.

Case 2. $\mathrm{depth}(B/mB) > 0$. Then there exists a $\overline{y} \in n\,B/mB$ which is B/mB-regular. Let $y \in n$ be such that $\overline{y} = y + mB$. The rest of the proof is based upon the following

Theorem 5.3. Let A, B be noetherian local rings, m, n their respective maximal ideals. Let $k = A/m$ and let $\varphi: A \to B$ be a local homomorphism. Let M, N be two finitely generated B-modules, and $u: M \to N$ a B-homomorphism, whence

$$u \otimes 1 : M \otimes_A k \to N \otimes_A k$$

is a $B \otimes_A k$-homomorphism. Assume that N is A-flat. Then the following two conditions are equivalent:

    1) u is injective, and coker (u) is A-flat

    2) $u \otimes 1$ is injective.

    <u>Proof</u>: We write gr(M) for $gr_{\mathcal{m}}(M)$ and similarly for N.
Note that

$$M \otimes_A k = gr_0(M)$$

$$N \otimes_A k = gr_0(N)$$

$$k = gr_0(A).$$

1)$\Longrightarrow$2). From the exact sequence

$$(*) \quad 0 \to M \overset{u}{\to} N \to coker(u) \to 0$$

and from Grothendieck's E.G.A., 0, 6.1.2 we see that M is A-flat, and that $u \otimes 1$ is injective. (Tensor (*) with k.)

2) $\Longrightarrow$ 1). We have $gr_0(u) : gr_0(M) \to gr_0(N)$ is injective.
Since N is A-flat, by theorem 1, B.C.A., III, 5, 2, the
canonical homomorphism $\varphi : gr(A) \otimes_{gr_0(A)} gr_0(N) \to gr(N)$ is

bijective. Hence we can apply proposition 9, B.C.A., III 2, 8, and thereby obtain that gr(u) is injective, and that coker(u) satisfies (iv) of theorem 1, B.C.A., III, 5, 2. (with M = coker(u), $\mathcal{J} = \mathcal{m}$). Since gr(u) is injective, and since the $\mathcal{n}$-adic topology on M is Hausdorf (B is local and M is finitely generated) from Corollary 1, B.C.A., III, 2, 8, we obtain that

u is injective. Furthermore, since M and N are finitely
generated B-modules, so is coker(u), and since B is noetherian,
coker(u) is "idéalement séparé" for n. (See Definition 1, B.C.A.
III, 5.1, and example 1 thereafter.)   Since φ is a local
homomorphism, it follows that coker(u) is   "idéalement séparé"
for m. Hence condition (iv) of theorem 1, B.C.A., III, 5, 2,
implies condition (i) of the same theorem, i.e. Coker(u) is
A-flat (we use here the noetherianity of A).   Q.E.D.

We return to the proof of Case 2, Theorem 5.2. We had
depth(B/mB) > 0, and we had $\bar{y}$ ∈ nB/mB, $\bar{y}$ was B/mB-regular,
and y ∈ B such that $\bar{y}$ = y + mB. Apply Theorem 5.3 to the
B-homomorphism u:B → B defined by u(b) = y·b. Since $\bar{y}$ is
B ⊗ $_A$k-regular,

$$u \otimes 1 : B \otimes {}_A k \to B \otimes {}_A k$$

is injective, whence u is injective and coker(u) is A-flat, i.e.
y is B-regular and B' = B/yB = coker(u) is A-flat. (One can
easily show that, conversely, if y is B-regular then B' is
A-flat.)

By Theorem 3.1, we have

$$depth(B') = depth(B) - 1.$$

Now B' ⊗ $_A$k = (B/yB) ⊗ $_A$k = (B ⊗ $_A$k)/$\bar{y}$(B ⊗ $_A$k). Therefore,
again by Theorem 3.1, depth(B' ⊗ $_A$k) = depth(B ⊗ $_A$k) - 1.

Finally φ':A → B' is again local and B' is A-flat. We can
therefore apply the induction assumption (since depth(A) +
depth(B' ⊗ $_A$k) = [depth(A) + depth(B ⊗ $_A$k)] - 1) and we get

depth(B) = depth(B') + 1 = depth(A) + depth(B' $\otimes_A$ k) + 1 =

depth(A) + depth(B $\otimes_A$ k)

and Theorem 5.2 is proved.

Corollary 5.1. Under the same assumptions as in Theorem 5.2, B is a C-M ring, if, and only if, A and B/mB are C-M rings.

Proof: From Theorems 5.1 and 5.2 we have

(a) dim(B) = dim(A) + dim(k $\otimes_A$ B)

(b) depth(B) = depth(A) + depth(k $\otimes_A$ B).

Therefore, if A and B/mB are C-M rings, so, trivially is B.

Conversely, let B be a C-M ring. We have:

$$\dim(A) \geq \text{depth}(A)$$

$$\dim(k \otimes_A B) \geq \text{depth}(k \otimes_A B)$$

and

$$\dim(A) + \dim(k \otimes_A B) = \dim(B) =$$

$$\text{depth}(B) = \text{depth}(A) + \text{depth}(k \otimes_A B).$$

Therefore dim(A) = depth(A) and dim(k $\otimes_A$ B) = depth(k $\otimes_A$ B), i.e. A and B/mB are C-M rings. The corollary is proved.

Theorem 5.2 and Corollary 5.1 are local in nature. We are now going to examine some of the global consequences of flatness.

As usual we let A, B be two rings, X = Spec(B), Y = Spec(A), O_X = the sheaf of local rings B_p of X, O_Y = the sheaf of local rings A_q of Y. If $\varphi: A \to B$ is a given

homomorphism the subschemes $^a\varphi^{-1}(y)$ of X are called the fibres of X over Y.

   We recall that to say that X satisfies (S_k) is to say that, for all $x \in X$

$$\text{depth}(O_{x,X}) \geq \min[k, \dim(O_{x,X})]$$

(See definition 4.6). We also remark that, if A is a C-M ring then X satisfies (S_k), by (3.5).

   Theorem 5.4. Let $\varphi : A \to B$ be a homomorphism of (not necessarily local) rings. Let X = Spec(B), Y = Spec(A), and assume $^a\varphi : X \to Y$ is a flat morphism (i.e. B is a flat A-module under φ). Then

   1) If X satisfies (S_k) so does Y

   2) If Y and every fiber of X over Y satisfy (S_k) so
       does X.

   Proof: 1. Let $y \in Y$, x the generic point of an irreducible component of $^a\varphi^{-1}(y)$. By lemma 5.1 $^a\varphi(x) = y$ and theorem 5.1 applies. We have to show that $\text{depth}(O_{y,Y}) \geq \min[k, \dim O_{y,Y}]$. If $O_{y,Y} = A_\mathcal{q}$, $O_{x,X} = B_p$ we have $k(y) = A_\mathcal{q} / \mathcal{q} A_\mathcal{q}$, $k(x) = B_p / p B_p$ and $\varphi^{-1}(p) = \mathcal{q}$. Furthermore B_p is $A_\mathcal{q}$-flat. Since x is the generic point of an irreducible component of $^a\varphi^{-1}(y)$, we have $\dim(B_p / \mathcal{q} B_p) = 0$, whence $\text{depth}(B_p / \mathcal{q} B_p) = 0$. Therefore

$$0 = \dim(O_x \otimes_{O_y} k(y)) = \text{depth}(O_x \otimes_{O_y} k(y))$$

By theorems 5.1 and 5.2 we obtain

$$\dim(O_x) = \dim(O_y)$$

$$\text{depth}(0_x) = \text{depth}(0_y)$$

and since 0_x satisfies the condition (S_k), so does 0_y,

Q.E.D.

2) Let $x \in X$, $y = {}^a\varphi(x)$. From theorems 5.1, 5.2 we have

$$\dim(0_x) = \dim(0_y) + \dim(0_x \otimes_{0_y} k(y))$$

$$\text{depth}(0_x) = \text{depth}(0_y) + \text{depth}(0_x \otimes_{0_y} k(y))$$

By assumption both 0_y and $0_x \otimes_{0_y} k(y)$ satisfy the condition of

(S_k). Hence so does 0_x, Q.E.D.

(Note that here we do not know that x is the generic point of
an irreducible component of ${}^a\varphi^{-1}(y)$!).

The answer to the following question is at the moment
unkown: Let A, B be local rings $\varphi: A \to B$ a local flat morphism.
If A and B/mB satisfy (S_k), does B satisfy (S_k)? The
crucial difference between the situation here and the one in
theorem 5.4 is that here we assume (S_k) only for the fiber of
Spec(B) over the closed point of Spec(A), while in 2) of
theorem 5.4 (S_k) is assumed for all fibers.

The previous theorem dealt with the behavior of the
condition (S_k) under global flat morphism. We now examine the
behavior of the notion of regularity in the local case.

Theorem 5.5. Let A, B be noetherian, local rings, $\varphi: A \to B$
a local morphism and let B be a A-flat. Then

1) If B is regular, so is A

2) If A and B/mB are regular, so is B.

Proof: 1) Since B is A-flat, the same argument as in the proof of Corollary 4.3 (replacing A_p with B) shows that

$$\text{coh. dim}(B) \leq \text{coh. dim}(A).$$

1) is therefore a trivial consequence of the Hilbert-Serre theorem (theorem 4.2).

2) Let $\dim(A) = m$, and let $x_1, \ldots, x_m$ be a regular system of parameters of A. Since B is A-flat, $\varphi(x_1), \ldots, \varphi(x_m)$ are B-regular. (Tensor the exact sequence $0 \to A/x_1 A + \ldots + x_{i-1} A \to A/x_1 A + \ldots + x_{i-1} A$ with B.) Now by assumption $B/\mathcal{m}B = B/\varphi(x_1) B + \ldots + \varphi(x_m) B$ is regular. Therefore by proposition 4.1, B is regular. (Replace A with B and $\mathcal{J}$ with $\mathcal{m}B$ in the proposition.)

Corollary 5.2. Let A be a ring, $T_1, \ldots, T_n$ independent transcendentals over A. Then:

i) If A is regular, so is $A[T_1, \ldots, T_n]$ (in particular if k is a field, $k[T_1, \ldots, T_n]$ is regular).

ii) If A is C-M, so is $A[T_1, \ldots, T_n]$ (in particular, if k is a field, $k[T_1, \ldots, T_n]$ is C-M).

Proof: Clearly it suffices to prove i) and ii) when $n = 1$, the general case following by induction. Let now $B = A[T]$. Since B is A-free, it is A-flat. Let $\mathcal{m}$ be a maximal ideal of B, $\mathcal{n}$ the prime ideal of A given by $\mathcal{n} = \mathcal{m} \cap A$. Then $B_{\mathcal{m}}$ is $A_{\mathcal{n}}$-flat and, by theorem 5.5, to prove i) and ii) it suffices to show that $A_{\mathcal{n}}$ and $B_{\mathcal{m}}/\mathcal{n} B_{\mathcal{m}}$ are regular, and C-M, respectively, under the corresponding assumptions for A. That $A_{\mathcal{n}}$ is regular when A is regular follows from Corollary 4.3 to Hilbert-Serre

theorem (theorem 4.2). If A is C-M, then so is $A_{\mathcal{n}}$ by proposition 3.5.

Now

$$B_{\mathcal{m}}/\mathcal{n}B_{\mathcal{m}} = (B/\mathcal{n}B)_{\mathcal{m}(B/\mathcal{n}B)} =$$

$$((A/\mathcal{n}) \otimes_A B)_{\mathcal{m}(B/\mathcal{n}B)} = k[T]_{\mathcal{m}'}$$

where $k = A_{\mathcal{n}}/\mathcal{n}A_{\mathcal{n}}$ and $\mathcal{m}'$ is the canonical image of $\mathcal{m}(B/\mathcal{n}B)$ in $k[T]$. $k[T]$ is a principal ideal domain, $k[T]_{\mathcal{m}'}$ is a discrete valuation ring, hence regular and, a fortiori, C-M (Corollary 4.1). The theorem is proved.

Having examined the behavior of dimension, depth, (S_k), and regularity under flat morphisms, we complete the analysis with the study of the behavior of condition (R_k).

Let as usual $\varphi:A \to B$ be a flat morphism, and let $X = \text{Spec}(B)$, $Y = \text{Spec}(A)$, $f = {}^a\varphi:X \to Y$. We say that X satisfies (R_k) if the ring B does, and similarly for the spectrum of any ring. We remark that to say X satisfies (R_k) is equivalent to saying that, when $\dim O_x \leq k$, O_x is regular. (see definition 4.6) Now:

Theorem 5.6. Let $\varphi:A \to B$ be a flat morphism. Then:

1) If X satisfies (R_k) so does Y
2) If Y and $f^{-1}(y)$ satisfy (R_k), for all $y \in Y$, so does X.

Proof: Let $y \in Y$, $x \in f^{-1}(y)$. Since B is A-flat we have that O_x is O_y-flat, whence, by Theorem 5.1

$$(*) \qquad \dim(O_x) = \dim(O_y) + \dim(O_x \otimes_{O_y} k(y))$$

where $k(y) = 0_y/\mathcal{m}_y$.

1) We have to prove that 0_y is regular if $\dim(0_y) \leq k$. If we choose x to be the <u>generic</u> point of an irreducible component of $f^{-1}(y)$, we have $\dim(0_x \otimes_{0_y} k(y)) = 0$ (since $B_{j_x}/j_y B_{j_x}$ is artinian), whence

$$\dim 0_x = \dim 0_y \leq k$$

and therefore 0_x is regular. Then, by theorem 5.5, 0_y is regular and 1) is proved.

2) Let now $x \in X$ be arbitrary, $\dim 0_x \leq k$, and let $y = f(x)$. We have to show that 0_x is regular. From equation (*) above we have $\dim(0_y) \leq k$ and $\dim(0_x \otimes_{0_y} k(y)) \leq k$. By assumption (R_k) holds for Y and $f^{-1}(y) = \text{Spec}(B \otimes_A k(y))$. Hence 0_y and $0_x \otimes_{0_y} k(y)$ are regular (note that $0_x \otimes_{0_y} k(y)$ is the <u>local ring of x in</u> $f^{-1}(y)$!), and by theorem 5.5, 0_x is regular,   Q.E.D.

A quick comparison shows that theorems 5.4 and 5.6 are identical if one replaces (S_k) by (R_k). It is then natural to ask the same question about (R_k) that was asked about (S_k) after the end of the proof of the theorem namely: Let A, B be local rings, $\varphi:A \to B$ a local flat morphism. If A and $B/\mathcal{m}B$ satisfy (R_k), does B satisfy (R_k)?

As with (S_k), the crucial difference between the situation here and the one in theorem 5.6 is that here we assume (R_k) only for the fiber of Spec(B) over the closed point of Spec(A), while in theorem 5.6 we assume (R_k) for all fibers. Here the answer is known, in the negative. As usual the counter example

is due to Nagata.

The following theorem is an immediate application of theorems 5.4 and 5.6, coupled with the characterization of reduced (normal) rings given in propositions 4.5 and 4.6.

<u>Theorem 5.7</u>. Let $\varphi:A \to B$ be a flat homomorphism of (not necessarily local) noetherian rings. Then:

1) If B is reduced (normal), so is A

2) If, for every $\mathfrak{p} \in \mathrm{Spec}(A)$, A and B/ B are reduced (normal), so is B.

<u>Proof</u>: Obvious.

We complete this section with a few remarks concerning the following situation.

A field k, a noetherian overring A of k, and a field $k' \supset k$ are given. The ring $A' = A \otimes_k k'$ is an overring of k'. We leave to the reader the verification of the following statements:

<u>Proposition 5.3</u>.

1) A' is noetherian if $[k':k] < \infty$ (A' need not be noetherian in general).

2) If A is a local ring, A' is semi-local.

3) A' is a flat A-module.

4) If $x' \in \mathrm{Spec}(A')$, x = the image of x', then $\dim A_x = \dim A'_{x'}$.

5) Under the same assumption as in 4), $\mathrm{depth}(A_x) = \mathrm{depth}(A'_{x'})$.

6) Under the same assumption as in 4), A_x is C-M if, and only if, $A'_{x'}$ is C-M.

7) A satisfies (S_k) if, and only if A' satisfies (S_k).

1, 2) and 3) have easy proofs. To prove 4), 5), 6),
7) apply theorems 5.1, 5.2, corollary 5.1, and
theorem 5.4.

Theorem 5.8. Let k be a field, A an overring of k, k' a
field containing k, $A' = A \otimes_k k'$. If A' is, respectively,
regular, (R_k), normal, reduced, then A is regular (R_k),
normal, reduced.

Proof: Follows directly from the previous results of this
section.

In general, however, A' need not be regular if A is, as
the following example shows:

Let k be a non perfect field $k \neq k^p$, $p > 2$ and let
$a \in k$, $a \notin k^p$. Let

$$A = k[X, Y]/(Y^2 - X^p + a)$$

The Jacobian criterion (Proposition 4.3) tells us that A is
regular. Now let $k' = k(a^{1/p})$. Then one easily verifies,
from $X^p - a = (X - a^{1/p})^p$ that

$$A' \simeq k'[X, Y]/(Y^2 - X^p)$$

and again proposition 4.3 tells us that A' is not regular.

We leave as an exercise to the reader the proof of the
following

Theorem 5.9. Under the same assumption as in theorem 5.8,
if A is regular and k' is a separable extension of k, then A'
is regular.

Theorem 5.9 prompts us to introduce the following

Definition 5.1. Let k be a field, A an overring of k. The ring A is said to be geometrically regular if, for all finite field extensions k' of k, the ring $A' = A \otimes_k k'$ is regular.

Corollary 5.3. a) Every regular overring of a perfect field is geometrically regular. .

b) Every regular overring of an algebraically closed field is geometrically regular.

Remark. Let again $A' = A \otimes_k k'$. Some of the properties of A' can be deduced from those of A and of the field extension k' of k. This process of deduction is known as ascent. Conversely, some of the properties of A can be deduced from those of A'. This latter process of deduction is known as descent.

§6. COMPLETION AND NORMALIZATION

6A. Completion. Let A be a noetherian local ring, $\mathcal{m}$ its maximal ideal. It is well known(see Corollary after Proposition 5 in B.C.A., III, §3, no. 2) that $\cap \, \mathcal{m}^n = (0)$. This implies that the collection $\{\mathcal{m}^n\}$ can be taken as the basis of a filter of neighborhoods of 0 in a (unique) Hausdorff topology which is consistent with the ring structure of A (i.e. A is a Hausdorff topological ring).

The set $\hat{A}$ of (equivalence classes of) Cauchy sequences of elements of A can be given a topological ring structure which is obviously complete (i.e. every Cauchy sequence in $\hat{A}$ is convergent). We refer the reader to the third chapter of B.C.A. for the proof of the above statements, as well as for the